JAMES RUSE
AGRICULTURAL HIGH SCHOOL
4 Unit Mathematics
1999 Trial HSC Examination

QUESTION 1

(a) Find $\int x\sqrt{x^2+16}\,dx$

(b) Find $\int \frac{x}{x+1}\,dx$

(c) Find $\int \frac{dx}{x^2+4x+13}$

(d) Using the substitution $u = \cos x$, or otherwise, find $\int \frac{\sin^3 x}{\cos^2 x}\,dx$

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos x-\sin x}$.

QUESTION 2

(a) If $z = 2+3i$ and $w = 1-i$ express in the form $a+ib$

(i) $\overline{z}$ (ii) zw (iii) $\frac{z}{w}$

(b) (i) Express $1+i$ in mod/arg form.

(ii) Hence write $(1+i)^5$ in the form $x+iy$ where x and y are real.

(c) (i) Find both square roots of $-3+4i$

(ii) Hence solve $z^2-5z+(7-i)=0$ giving your answers in the form $z=p+iq$ where p and q are real.

QUESTION 3

(a) (i) Prove that $\sin(A+B)+\sin(A-B) = 2\sin A\sin B$.

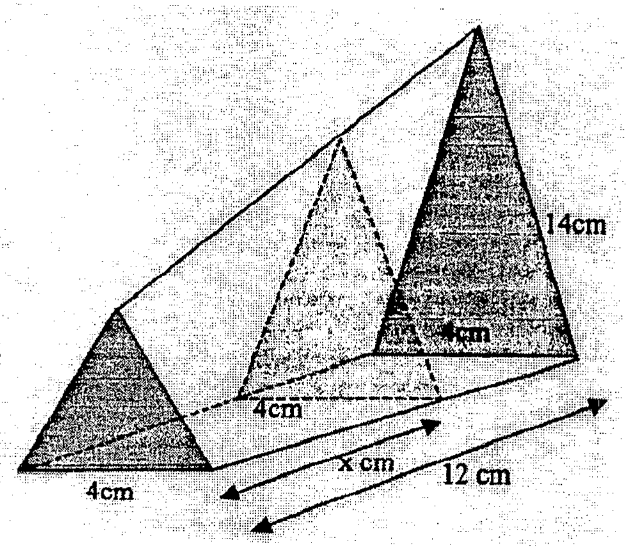
wrong (ii) Hence solve $\sin 3\theta + \sin\theta = \sin 2\theta$ for $0 \le \theta \le \pi$.

(b) Find the volume of the solid formed when the region bounded by $y = \cos x$ and $y = \sin x$ for $0 \le x \le \frac{\pi}{4}$ is rotated one revolution about the x-axis.

(c) The front face of a solid is an equilateral triangle with sides 4 cm and the end face is an isosceles triangle with base 4 cm and equal sides 14 cm. The solid is 12 cm long and cross-sections parallel to the front face are isosceles triangles with base 4 cm. (See diagram).

(i) Show that the height (h cm) of a triangular cross-section x cm from the front face is given by $h = \frac{\sqrt{3}}{2}(x+4)$.

(ii) Hence find the volume of the solid.



QUESTION 4

(a) (i) Prove that $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx$.

(ii) Using the result of (i) and the definition of odd and even functions prove

(α) if $f(x)$ is even then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(β) if $f(x)$ is odd then $\int_{-a}^{a} f(x)\,dx = 0$.

(iii) Hence evaluate $\int_{-\pi}^{\pi} x\cos x\,dx$.

(b) A sequence is defined by the formula $a_n = 3 + 33 + 333 + \cdots + \overbrace{333\ldots3}^{n \text{ digits}}$ where the last term contains n 3's. Use the principle of mathematical induction to prove that $a_n = \frac{1}{27}(10^{n+1} - 9n - 10)$ for integer $n \geq 1$.

QUESTION 5

(a) The point $T(ct, \frac{c}{t})$ lies on the hypebola $xy = c^2$. The tangent at T meets the x-axis at P and the y-axis at Q. The normal at T meets the line $y = x$ at R.

(i) Prove that the tangent at T has equation $x + t^2y = 2ct$.

(ii) Find the co-ordinates of P and Q.

(iii) Write down the equation of the normal at T.

(iv) Show that the x co-ordinate of R is $x = \frac{c}{t}(t^2 + 1)$.

(v) Prove that $\triangle PQR$ is isosceles.

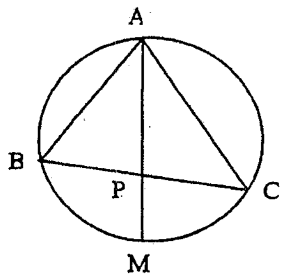
(b) A circle is drawn to pass through the vertices of $\triangle ABC$. AM bisects $\angle BAC$ and meets BC at P. (see diagram)

(i) Prove that $\triangle ABM$ and $\triangle ACP$ are similar.

(ii) Prove that $AB.AC = AP.AM$.

(iii) Explain why $BP.PC = AP.PM$.

(iv) Hence prove that $AB.AC - BP.PC = AP^2$.



QUESTION 6

(a) (i) Sketch $y = \sin(x^2)$ for $-\sqrt{2\pi} \leq x \leq \sqrt{2\pi}$ showing all intercepts with the co-ordinate axes and turning points.

(ii) The region bounded by $y = \sin(x^2)$ and the x-axis for $0 \leq x \leq \sqrt{\pi}$ is rotated one revolution about the y-axis. Find the volume of the solid.

(b) A conical pendulum is constructed using a 200 gram mass attached to a light string. When the mass moves in a horizontal circle with speed 1.5 m.s^{-1} the string makes an angle of magnitude $\tan^{-1}\frac{3}{4}$ with the vertical. Taking $g = 10\ m.s^{-1}$, find

(i) the tension in the string

(ii) the length of the string.

(c) The locus of a point is defined by the equation $|z - 2| = 2\Re(z - \frac{1}{2})$.

(i) If $z = x + iy$ explain why $x \geq \frac{1}{2}$.

(ii) Show that the locus is a branch of the hyperbola $3x^2 - y^2 = 3$.

(iii) Sketch the locus showing its asymptotes and vertex.

(iv) Find the largest set of possible values for each of $|z|$ and $\arg z$.

QUESTION 7

(a) A plane of mass M kg on landing experiences a variable resistive force (due to air resistance) of magnitude Bv^2 Newtons, where v is the speed of the plane, i.e., $M\ddot{x} = -Bv^2$. After the brakes are applied the plane experiences a constant resistive force A Newtons (due to the brakes) as well as the variable resistive force Bv^2, i.e., $M\ddot{x} = -(A + Bv^2)$.

(i) Show that the distance (D_1) travelled in slowing from speed V to speed U under the effect of air resistance only is given by: $D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$.

(ii) After the breaks are applied with the plane travelling at speed U, show that the distance (D_2) required to come to rest is given by: $D_2 = \frac{M}{2B}\ln\left(1 + \frac{B}{A}U^2\right)$.

(iii) Use the above information to estimate the stopping distance for a 100 tonne plane if it slows from 90 m.s^{-1} to 60 m.s^{-1} under a resistive force of magnitude $125v^2$ Newtons and is finally brought to rest with the assistance of constant braking force of magnitude 75000 Newtons.

(b) (i) Prove that $\frac{x^2}{(x^2+1)^{n+1}} = \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$

(ii) Given $I_n = \int_0^1 \frac{1}{(x^2+1)^n}\,dx$, prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$.

(iii) Hence evaluate $\int_0^1 \frac{1}{(x^2+1)^3}\,dx$.

QUESTION 8

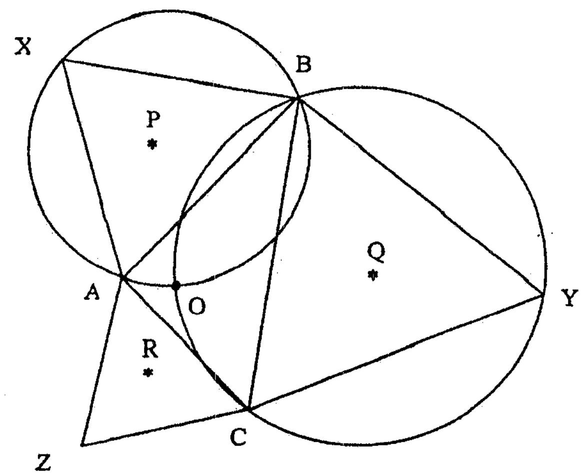
(a) (i) Use the substitution $u = 1 + x$ to evaluate $\int_0^1 x(1+x)^n \, dx$.

(ii) Use the binomial theorem to write an expansion of $x(1+x)^n$.

(iii) Prove that $\sum\limits_{r=0}^{r=n} \frac{1}{r+2} \cdot {}^nC_r = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$.

(iv) Find the largest integer value of n such that $\sum\limits_{r=0}^{r=n} \frac{1}{r+2} \cdot {}^nC_r < 50$.

(b) ABC is any triangle. Equilateral triangles ABX, BCY and ACZ are constructed on the sides of $\triangle ABC$. Circles with centres P and Q are drawn to pass through the vertices of $\triangle ABX$ and $\triangle BCY$. The circles meet at B and O. (see diagram)



(i) Find the size of $\angle AOB, \angle BOC$ and $\angle AOC$, giving reasons.

(ii) Prove that $AOCZ$ is a cyclic quadrilateral.

(iii) If R is the centre of the circle $AOCZ$, prove that $\triangle PRQ$ is equilateral.

1.(a). $\int x\sqrt{x^2+16}\,dx$

$= \frac{1}{2}\int 2x\sqrt{x^2+16}\,dx$

$= \frac{1}{2}\left[\frac{(x^2+16)^{3/2}}{3/2}\right] + c.$

$= \frac{1}{3}(x^2+16)\sqrt{x^2+16} + c.$

(b). $\int \frac{x}{x+1}\,dx$

$= \int \frac{x+1}{x+1} - \frac{1}{x+1}$

$= \int dx - \int \frac{dx}{x+1}$

$= [x - \ln|x+1|] + c.$

(c). $\int \frac{dx}{x^2+4x+13}$

$\int \frac{dx}{x^2+4x+4+9}$

$= \int \frac{dx}{(x+2)^2+9}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c.$

(d). $\int \frac{\sin^3 x}{\cos^2 x}\,dx$

~~Let u = cos x~~

$\cancel{\frac{du}{dx} = -\sin x \qquad u^{-2}}$

$\cancel{I = \int \frac{-\sin^3 x\,dx}{\cos^2 x}}$

$\cancel{= -\int \frac{du \cdot \sin^2 x}{\cos^2 x}}$

$=$

$\int \tan^2 x \cdot \sin x\,dx$

$\int (\sec^2 x - 1)\sin x\,dx$

Let $u = \cos x$ $\quad du = -\sin x\,dx$

$I = \int 1 - \sec^2 x\,du$

$= \int 1 - \frac{1}{u^2}\,du$

$= u + u^{-1}$

$= \cos x + \frac{1}{\cos x} + c$

$= \cos x + \sec x + c.$

(e). $\int_0^{\pi/3} \frac{dx}{1+\cos x-\sin x}$

Let $t = \tan x/2$

$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$

$= \frac{1}{2}(\tan^2 x + 1)$

$\frac{2dt}{(t^2+1)} = dx$

At $x = \pi/3$, $t = \frac{1}{\sqrt{3}}$

$x = 0$, $t = 0$.

$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$I = \int_0^{1/\sqrt{3}} \frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} \times \frac{2dt}{(t^2+1)}$

$= \int_0^{1/\sqrt{3}} \frac{1}{1+t^2+1-t^2-2t} \times 2dt.$

$= \int_0^{1/\sqrt{3}} \frac{dt}{1-t}$

$= \left[\ln|1-t|\right]_0^{1/\sqrt{3}}$

$= \ln\left|1-\frac{1}{\sqrt{3}}\right|$

2.(a). $z = 2+3i$, $w = 1-i$.

(i) $\bar{z} = 2-3i$

(ii) $zw = (2+3i)(1-i)$

$= 5+i$

(iii) $\frac{z}{w} = \frac{2+3i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{-1+5i}{2}.$

$= -\frac{1}{2} + \frac{5}{2}i$

(b)(i) $1+i$

$z = \sqrt{2}\,\text{cis}\,\pi/4.$

(ii) $z^5 = 4\sqrt{2}\,\text{cis}\,\frac{5\pi}{4}$

$= 4\sqrt{2}\,\text{cis}\,\frac{-3\pi}{4}.$

$= 4\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$

$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= -4 - 4i$

(c). $\sqrt{-3+4i} = (a+ib)$

$-3+4i = (a+ib)^2$

$-3+4i = a^2-b^2+2iab$

$-3 = a^2-b^2 \qquad 4 = 2ab$

$\frac{2}{a} = b.$

$-3 = a^2 - \frac{4}{a^2}$

$-3a^2 = a^4 - 4.$

$a^4 + 3a^2 - 4 = 0.$

$a^2 = 1,\ a^2 = -4.$ No sol

$\therefore a = \pm 1$

$\therefore b = \pm 2.$

$\therefore z = \pm(1+2i)$

(ii). $z^2 - 5z + (7-i) = 0.$

$z = \frac{5 \pm \sqrt{25-4(7-i)}}{2}$

$= \frac{5 \pm \sqrt{-3+4i}}{2}$

$= \frac{5 \pm (1+2i)}{2}$

$\Rightarrow z = 3+i$ or $z = 2-i$

3 (a). $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$= \sin A\cos B + \cancel{\sin B\cos A} + \sin A\cos B - \cancel{\sin B\cos A}$

$= 2\sin A\cos B$

Question is WRONG

perhaps: $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$= \cos A\cos B + \sin A\sin B - \cos A\cos B + \sin A\sin B$

$= 2\sin A\sin B$

$= RHS.$

(ii). $\sin 3\theta + \sin\theta = \sin 2\theta$

perhaps: $\cos 3\theta + \cos 2\theta = \sin 2\theta$

Let $B = A$.

To solve $\underbrace{\sin 3\theta + \sin\theta}$ $= \sin 2\theta$

$A+B = 3\theta$

$A-B=\theta$

$2A = 4\theta$

$2\sin 2\theta\cos\theta - \sin 2\theta = 0$

$A = 2\theta \Rightarrow B = \theta$

$\therefore \sin 2\theta\,(2\cos\theta - 1) = 0$

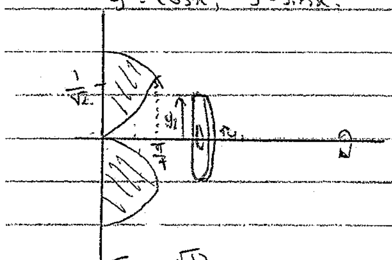
$\sin 2\theta = 0 \Rightarrow 2\theta = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{2}, \pi$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$.

(b). $y = \cos x$, $y = \sin x$.



$V = \pi(y_1^2 - y_2^2)\,\delta x$

$= \pi(\cos^2 x - \sin^2 x)\,dx$

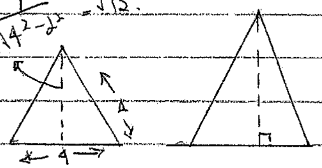
$= \pi\int_0^{\pi/4} \cos^2 x - \sin^2 x\,dx$

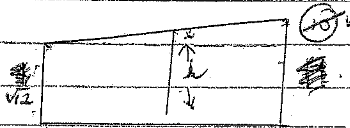
$= \pi\int_0^{\pi/4} \cos 2x\,dx$

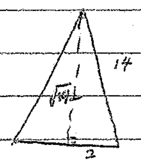
$= \pi\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \frac{\pi}{2}$.

(c) $\sqrt{4^2-2^2} = \sqrt{12}$.







$* \quad h = \frac{x(\sqrt{192}-\sqrt{12})}{12} + \sqrt{12}$

$\therefore \quad = \frac{x(8\sqrt{3}-2\sqrt{3})}{12} + \sqrt{12}$

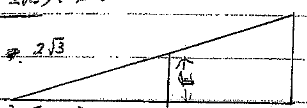
$= \frac{6\sqrt{3}\,x}{12} + 2\sqrt{3}$

$= \frac{\sqrt{3}}{2}(x+4)$

$\frac{x}{12} = \frac{h}{\sqrt{192}-\sqrt{12}} = \frac{l}{8\sqrt{3}-\sqrt{12}}$

$\frac{x(\sqrt{192}-\sqrt{12})}{12} = h$ $\therefore$

$\therefore h = l + 4$



$\frac{10}{12} = \frac{H}{x}$

$10x = 12H$

$\frac{5}{6}x = l$

$\therefore l = 4 + L$

$= 4 + \frac{5}{6}x$

$V = \int_0^{12} \frac{1}{2}\times 4\times\frac{\sqrt{3}}{2}(x+4)\,dx$

$= \sqrt{3}\left[\frac{x^2}{2} + 4x\right]^{12} = 120\sqrt{3}\ u^3$

4. (a) (i) $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x) + f(-x))\,dx$.

RHS $= \int_0^a f(x)\,dx + f(-x)\,dx$

Let $u = -x$

$du = -dx$

$= \int_0^a f(x) - \int_0^{-a} f(u)\,du$

$= \int_{-a}^{a} f(x)\,dx$

$= LHS.$

(α). $\int_{-a}^{a} f(x)\,dx$

If even then $f(-x) = f(x)$.

$\therefore \int_{-a}^{a} f(x) = \int_0^a f(x) + f(x)$

$= \int_0^a 2f(x).$

(β) If odd $f(-x) = -f(x)$

$\therefore \int_{-a}^{a} f(x) = \int_0^a f(x) - f(x)$

$= 0.$

(iii). $\int_{-\pi}^{\pi} x\cos x\,dx$

$f(x) = x\cos x$

$f(-x) = (-x)\cos(-x)$

NB: $\cos(-x) = \cos x$ (ASTC)

$f(-x) = -x\cos x$

$= -f(x)$

$\therefore$ Odd.

$\therefore \int_{-\pi}^{\pi} x\cos x = 0.$

(b). $a_n = 3 + 33 + 333 + \ldots \underbrace{333\ldots3}_{n}$

$a_n = \frac{1}{27}(10^{n+1} - 9n - 10)$

Let $n=1$

LHS $= 3$

RHS $= \frac{1}{27}(10^2 - 9 - 10)$

$= \frac{81}{27}$

$= 3$

$=$ LHS, True.

Let $n = k$

$3 + 33 + 333 + \ldots \overbrace{33\ldots33}^{k} = \frac{1}{27}(10^{k+1} - 9k - 10)$

Let $n = k+1$

$3 + 33 + 333 + \ldots \overbrace{33\ldots33}^{k} + \overbrace{33\ldots33}^{k+1} = \frac{1}{27}(10^{k+2} - 9(k+1) - 10)$

$= \frac{1}{27}(10^{k+1} - 9k - 10) + \underbrace{33\ldots33}_{k+1}$

$= \frac{1}{27}(10^{k+1} - 9k - 10) + \cancel{3+30+3+300+30+3}.$

$3 + 30 + 300 + 3000 + \ldots \overbrace{30000}^{k+1}$

$= \frac{1}{27}(10^{k+1} - 9k - 10) + 3(n+1) + \cancel{30n} + \cancel{300(n+1)} + \ldots$ $\quad 3 + 30 + 300 + 3000 + \ldots$

$= \frac{1}{27}(10^{k+1} - 9k - 10) + S_{k+1} \qquad a = 3, r = 10.$

$+ \frac{3(10^{k+1} - 1)}{10 - 1}$

$= \frac{1}{27}(10^{k+1} - 9k - 10) + \frac{(10^{k+1} - 1)}{3}.$

$= \frac{1}{27}(10^{k+1} - 9k - 10 + 9 \times 10^{k+1} - 9)$

$= \frac{1}{27}(10^{k+1}(1+9) - 9(k+1) - 10)$

$= \frac{1}{27}(10^{k+2} - 9(k+1) - 10)$

= RHS.

$\therefore$ True if $n=k$, and shown for $n=k+1$, by the principles of Mathematical Induction, true for all n.

5. (a)(i) $T(ct, \frac{c}{t})$

$xy = c^2$

$y = \frac{c^2}{x}$

$= c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

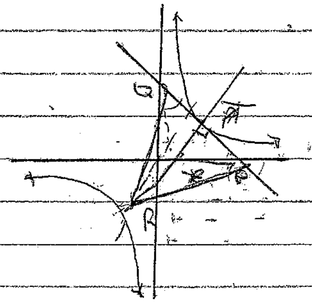
At $x = ct$, $m = \frac{-c^2}{c^2 t^2}$

$= -\frac{1}{t^2}$

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$\underline{x + t^2 y = 2ct.}$



(ii). x axis at P, ie $y = 0$.

$x = 2ct$

$\therefore \underline{P(2ct, 0)}$

Y axis at Q, ie $x = 0$.

$t^2 y = 2ct$

$y = \frac{2c}{t}$

$\therefore \underline{Q(0, \frac{2c}{t})}$

(iii). $m_2 = t^2$

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3 x - ct^4$

$\underline{ct^4 - c = t^3 x - ty}$

(iv). $y = x$

$ct^4 - c = t^3 x - tx$

$ct^4 - c = t^3 x - ty$

Sub $y = x$

$ct^4 - c = x(t^3 - t)$

$x = \frac{c(t^4 - 1)}{t(t^2 - 1)}$

$= \frac{c(t^2 - 1)(t^2 + 1)}{t(t^2 - 1)}$

$= \frac{c(t^2+1)}{t}. \qquad y = \frac{c(t^2+1)}{t}$

(v). $m_{PQ} = \frac{\left(\frac{2c}{t}\right)}{-2ct}$

$= \frac{-2c}{2ct^2}$

$= -\frac{1}{t^2}$

$m_{PR} = \frac{\left(\frac{c(t^2+1)}{t}\right)}{\frac{c(t^2+1)}{t} - 2ct}$

$= \frac{c(t^2+1)}{ct^2 + 1 - 2ct^2}$

$= \frac{c(t^2+1)}{1 - ct^2}$

$\therefore \tan Q\hat{P}S = \left| \frac{\frac{c(t^2+1)}{1 - ct^2} - \frac{1}{t^2}}{\frac{-c(t^2+1)}{t^2(1 - ct^2)}} \right|$

$m_{QR} = \frac{\left(\frac{c(t^2+1)}{t} - \frac{2c}{t}\right)}{\frac{c(t^2+1)}{t}} = \frac{ct^2 + 1 - 2c}{ct^2 + 1}$

$= \frac{ct^2 - 2c + 1}{ct^2 + 1}$

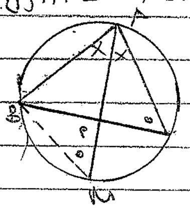
$\therefore \tan R\hat{Q}P = \left| \frac{\frac{ct^2 - 2c + 1}{ct^2 + 1} - \frac{1}{t^2}}{\frac{(ct^2 - 2c + 1)}{ct^2 + 1}} \right|$

$= \tan Q\hat{P}S$

$\therefore Q\hat{P}S = R\hat{Q}P$

$\therefore$ Isosceles Δ.

(b). (i) In Δ's ABM & APC



$A\hat{C}B = A\hat{M}B$ ($\angle$ subtended on the same arc).

$B\hat{A}M = M\hat{A}C$ (given)

$\therefore \Delta ABM \mathbin{|||} \Delta APC$ (equiangular).

(ii) $\therefore \frac{AC}{AM} = \frac{AP}{AB} = \frac{PC}{BM}$ (corresponding sides of $|||\Delta$'s).

$= AC \cdot \cancel{A}C = AP \cdot AM$ — (1)

(iii). $\underline{AP \cdot PM = BP \cdot PC}$ (intersecting chord theorem). — (2)

(iv). $BC = BP + PC$.

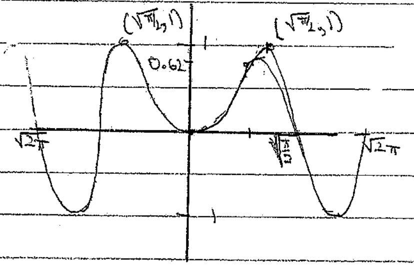
(1) − (2)

$AP \cdot AM - AP \cdot PM = AC \cdot \cancel{AB} - BP \cdot PC$

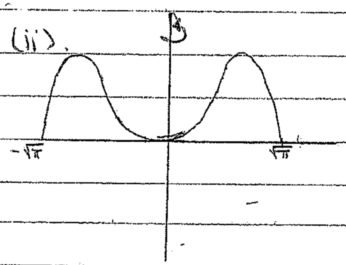
since $AM - PM = AP$ (diagram) $\Longleftarrow$ $AP(AM - PM) = AB \cdot AC - BP \cdot PC$

$\underline{AP(AP) = AB \cdot AC - BP \cdot PC}$

6.(a).(i) $y = \sin(x^2)$



(ii).



$V = 2\pi \int_0^{\sqrt{\pi}} rh \cdot dx$

$= 2\pi \int_0^{\sqrt{\pi}} xy \cdot dx$

$= 2\pi \int_0^{\sqrt{\pi}} \sin^2(x^2) \cdot dx$

(b) ———

(c)(i) $|z - 2| = 2\mathrm{Re}(z - 1/2)$

~~working~~ since $|z| > 0$.

$2\mathrm{Re}(z - 1/2) > 0$.

$2\mathrm{Re}(x + iy - 1/2) > 0$.

$\therefore \underline{x > 1/2}$.

(ii) $|z - 2| = 2\mathrm{Re}(z - 1/2)$

$|x + iy - 2| = 2(x + iy - 1/2)$

$\sqrt{(x-2)^2 + y^2} = 2(x - 1/2)$

$= 2x - 1$

$(x-2)^2 + y^2 = (2x-1)^2$

$x^2 - \cancel{4x} + 4 + y^2 = 4x^2 - \cancel{4x} + 1$

$= 3x^2 - 3 - y^2 = 0$.

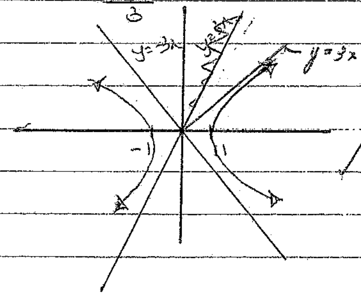
$\underline{3x^2 - y^2 = 3}$.

(iii). $x^2 - \frac{y^2}{3} = 1$.

$a = 1, b = 3$

Asy $= y = \pm b/a\, x$

$= \pm 3x$



(iv).

7.(a). $M\ddot{x} = -Bv^2$

$M\ddot{x} = -(A + Bv^2)$

$M \frac{v\,dv}{dx} = -(A + Bv^2)$

$M \frac{dv}{dx} = \frac{-A - Bv^2}{vM}$

$\int \frac{dx}{dv} = \int \frac{-vM}{A + Bv^2} \cdot dv$

$x = -\frac{M}{2B} \int \frac{2VB}{A+BV^2} \cdot dx$

$x = -\frac{M}{2B} \ln |A+BV^2| + c.$

$M\ddot{x} = -BV^2$

$\ddot{x} = -\frac{B}{M} v^2$

$\frac{v\,dv}{dx} = \frac{-B}{M} v^2$

$\frac{dx}{dv} = -\frac{Mv}{Bv^2}$

$= -\frac{M}{Bv}$

$x = \int -\frac{M}{Bv}\, dv$

$= -\frac{M}{B} \int \frac{1}{v} \cdot dv$

$= -\frac{M}{B} \ln|v| + c.$

$A+$

(b) (i) $\frac{x^2}{(x^2+1)^{n+1}} = \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$

$RHS = \frac{(x^2+1)}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}}$

$= \frac{x^2+1-1}{(x^2+1)^{n+1}}$

$= \frac{x^2}{(x^2+1)^{n+1}}$

$= LHS.$

(ii) $I_n = \int_0^1 \frac{1}{(x^2+1)^n}$

$=$

$= \int_0^1 (x^2+1)^{-n}$

$u = (x^2+1)^{-n}$ $v = x$

$u' = -2xn(x^2+1)^{-(n+1)}$ $v' = 1$

$I = \left[x(x^2+1)^{-n}\right]_0^1 + n\int 2x^2 (x^2+1)^{-n+1}$

$= \left[\frac{x}{(x^2+1)^n}\right]_0^1 + n\int_0^1 \frac{2x^2}{(x^2+1)^{n+1}}$

$= \left[\frac{1}{2^n}\right] + 2n \int \frac{x^2}{(x^2+1)^{n+1}}$

$= \left[\frac{1}{2^n}\right] + 2n\int \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$

$I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$

$2n\, I_{n+1} = \frac{1}{2^n} + (2n-1)\, I_n$

$\underline{\underline{2n\, I_{n+1} = 2^{-n} + (2n-1)\, I_n}}$

(iii) $\int_0^1 \frac{1}{(x^2+1)^3}\, dx$

$\therefore 2(n-1)\, I_n = 2^{-(n+1)} + (2n-3)\, I_{n-1}$

$\therefore 2\times 2 I_3 = 2^{-(4)} + 3 I_2$

$I_3 = \frac{1}{64} + \frac{3}{4} I_2$

$2 \times 1\, I_2 = 2^{-(3)} + I_1$

$I_2 = \frac{1}{16} - \frac{I_1}{2}$

$I_3 = \frac{1}{64} + \frac{3}{4}\left(\frac{1}{16} - \frac{\pi}{8}\right)$ $I_1 = \int_0^1 \frac{1}{x^2+1}\, dx \Rightarrow \left[\tan^{-1}(x)\right]_0^1 \Longrightarrow \pi/4.$

$= \frac{1}{\cdot} - \frac{3\pi}{\cdot} \Longleftarrow$ $\therefore I_2 = \frac{1}{16} - \pi/8.$

8(a)(i) $\int_0^1 x(1+x)^n\,dx$

Let $u = 1+x$

$x = u-1$

At $x=1,\ u=2$.

$x=0,\ u=1$ ✓

$dx = du$.

$I_n = \int_1^2 (u-1)u^n\,du$

$= \int_1^2 u^{n+1} - u^n\,du$

$= \left[\frac{u^{n+2}}{n+2} - \frac{u^{n+1}}{n+1}\right]_1^2$ ✓

$= \left[\frac{(n+1)u^{n+2} - u^{n+1}(n+2)}{(n+2)(n+1)}\right]_1^2$

$= \left[\frac{u^{n+1}[u(n+1) - (n+2)]}{(n+2)(n+1)}\right]_1^2$ ✓

$= \frac{2^{n+1}[2n+2-n-2]}{(n+2)(n+1)} - \frac{[(n+1)-(n+2)]}{(n+2)(n+1)}$

$= \frac{2^{n+1}[n]+1}{(n+2)(n+1)}$

$= \frac{n2^{n+1}+1}{(n+2)(n+1)}$ ✓

(ii) $x(1+x)^n$

$= x[1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\dots\binom{n}{k}x^k+\dots\binom{n}{n}x^n]$ ✓

(iii) $x(1+x)^n = x + \binom{n}{1}x^2+\binom{n}{2}x^3+\binom{n}{3}x^4+\dots\binom{n}{k}x^{k+1}+\dots\binom{n}{n}x^{n+1}$

∴ Integrating

$\int_1^2 x(1+x)^n = \int_1^2 x+\binom{n}{1}x^2+\dots\binom{n}{k}x^{k+1}+\dots\binom{n}{n}x^{n+1}$

$\frac{n2^{n+1}+1}{(n+1)(n+2)} = \left[\frac{x^2}{2}+\frac{\binom{n}{1}x^3}{3}+\dots\frac{\binom{n}{k}x^{k+2}}{k+2}+\frac{\binom{n}{n}x^{n+2}}{n+2}\right]_1^2$ ✓

$= \sum_{r=0} \frac{1}{r+2}\cdot\binom{n}{r}$

(iv). $\sum_0 \frac{1}{r+2}\binom{n}{r} < 50$.

∴ $\frac{n2^{n+1}+1}{(n+1)(n+2)} < 50$.

By Trial & Error ✓

$n=8$.

(b)

(i). $A\hat{X}B = \pi/3$ (equilateral △)

∴ $A\hat{O}B = \pi - \pi/3$ (cyclic quad)

$= 2\pi/3$. ✓

Similar $B\hat{O}C$ & $A\hat{O}C = 2\pi/3$. ✓

(ii). Since △AZC is equilateral

then $\angle AZC = \frac{\pi}{3}$

∴ $\angle AZC + \angle AOC = \frac{\pi}{3}+\frac{2\pi}{3} = \pi$

∴ AOCZ is a cyclic quad (Opp. ∠s are supplementary)

(iii) In cyclic quad AOCZ, OC is the intersecting chord of two circles with centres R and Q and OC ⊥ RQ say S and similarly AO ⊥ PR at say T

∴ ∠OSR = ∠RTO = 90° making TOSR also a cyclic quad.

∴ ∠SRT = 180° − ∠AOC

= 180° − 120°

= 60°

Similarly it can be shown that

∠RPQ = 60° = ∠PQR

∴ △PQR is equilateral.