TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2002

# MATHEMATICS

*Time Allowed – 3 Hours*
*(Plus 5 minutes reading time)*

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

**Standard integral tables are included with the examination paper.**
**Approved silent calculators may be used.**

**The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2 , etc. Each bundle must show your candidate number.**

**QUESTION 1**

(a) Find the exact value of $\sin\left(\frac{\pi}{4}\right)$. 1

(b) 1500 identical sheets of paper are laid on top of each other to form a pile of sheets 12cm high. Find the thickness of an individual sheet of paper. Give your answer in millimeters. 2

(c) Factorise $2p^2+p-6$. 2

(d) Solve $2\cos\theta=1$ for $0^\circ \le \theta \le 360^\circ$. 2

(e) Simplify $\frac{3}{x+1}-\frac{2}{x^2-1}$. 2

(f) Evaluate $\int_1^4 y\sqrt{y}\,dy$ 3

**QUESTION 2 (START A NEW PAGE)**

(a) Differentiate with respect to $x$:

(i) $x^4-\frac{3}{x}$, 2

(ii) $\log_e(7-3x)$. 1

(b) Find the equation of the tangent to $y=\sqrt{x+3}$ at the point (1,2). 3

(c) (i) The lines $y=2x$ and $x+2y=20$ meet at point $A$. Find the coordinates of $A$. 2

(ii) The line $x+2y=20$ meets the *x-axis* at the point $B$ and $M$ is the midpoint of $AB$. Find the coordinates of the points $B$ and $M$. 2

(iii) Given that $O$ is the origin, show that $\Delta OAM$ is isosceles. 2

**QUESTION 3 (START A NEW PAGE)**

(a) The gradient of a curve $y = f(x)$ is given by $f'(x) = \dfrac{2x^2+1}{x}$. Find the equation of the curve if it passes through the point (1,3). 3

(b) An arc $PQ$ has length 12cm and is drawn on the circumference of a circle with radius 8cm. Find

(i) the size of the angle subtended at the center of the circle by the arc $PQ$, 1

(ii) the length of the chord $PQ$, correct to 2 decimal places. 2

(c) (i) Sketch the parabola $y = x^2 - 4x - 12$, clearly showing its intercepts with the coordinate axes and the coordinates of its vertex. 4

(ii) Hence, or otherwise, solve $x^2 - 4x - 12 \geq 0$. 2

**QUESTION 4 (START A NEW PAGE)**

(a) Given the parabola $y = \frac{1}{2}x^2 - 3x + 1$,

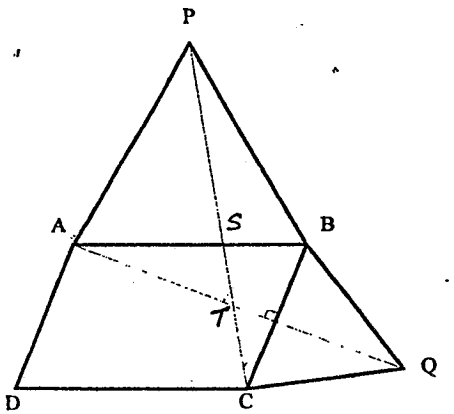
(i) Express the equation in the form $(x-h)^2 = 4a(y-k)$, where $a$, $h$ and $k$ are constants. 3

(ii) Write down the coordinates of the focus of this parabola. 2

(b) $ABCD$ is a parallelogram. $\Delta APB$ and $\Delta BQC$ are equilateral. (see diagram)

(i) Prove that $\Delta ABQ \equiv \Delta PBC$. 4

(ii) Find the size of the acute angle between $AQ$ and $PC$. (Give reasons) 3




**QUESTION 5 (START A NEW PAGE)**

(a) The 4th term of an Arithmetic Progression is 30 and its 10th term is 54. Find

(i) the common difference and the first term,

(ii) the sum of the first 20 terms.

(b) An object, initially at rest at the origin, moves in a straight line with velocity $v\ ms^{-1}$ so that $v = 4t(5-t)$ where $t$ is the time elapsed in seconds. Find

(i) the acceleration of the object at the end of the third second,

(ii) an expression for the displacement $x$ metres of the particle in terms of $t$,

(iii) the position of the particle when it again comes to rest.

**QUESTION 6 (START A NEW PAGE)**

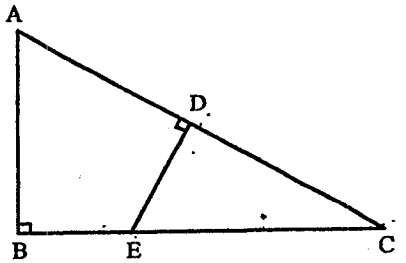
(a) $\Delta ABC$ is right-angled at $B$ and $DE$ is perpendicular to $AC$ (see diagram)

(i) Prove that $\Delta ABC$ and $\Delta CDE$ are similar.

(ii) Prove that $BC \times CE = AC \times CD$.

(iii) Prove that:

$DE^2 = AD \times DC - BE \times EC$




(b) (i) Sketch the parabola $y = x^2 - 4x$ and the line $y = 2x$. Clearly show their points of intersection.

(ii) Find the area bounded by the above curves.

(a) Water flows into and out of a tank at a rate (in litres/hour) given by $R = 2t \quad \pi t$. If the tank is initially empty at 10am, find:

(i) The first time (after 10am) when the tank is filling at its greatest rate. 2

(ii) An expression for the volume ($V$ litres) of water in the tank after $t$ hours. 2

(iii) The maximum volume of water in the tank. 1

(b) Given the curve $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, $x > 0$.

(i) Find the coordinates of any stationary points and determine their nature. 3

(ii) Find the coordinates of any points of inflexion. 2

(iii) Sketch the curve showing all stationary points and inflexion points. 2

**QUESTION 8 (START A NEW PAGE)**

(a) (i) Prove that the line with equation $y = px + (1 - 2p^2)$ is a tangent to the parabola $x^2 = 8(y-1)$ for all values of $p$. 3

(ii) Find the angle between the tangents drawn to $x^2 = 8(y-1)$ from the point $(0,-7)$. 3

(b) (i) If $f(x) = \sin^2 x$, find $f'(x)$. 1

(ii) The area bounded by the curve $y = \sin x + \cos x$ and the *x-axis* for $0 \le x \le 2\pi$ is rotated one revolution about the *x-axis*. Find the volume of the solid formed. 5

(a) In a hat are six red and four green discs. Two discs are chosen at random from the hat one before the other without replacing the first disc that was chosen.

(i) Draw a probability tree diagram for the above information. 2

Find the probability that:

(ii) two red discs are chosen, 1

(iii) the second disc chosen is green, 1

(iv) at least one green is chosen. 1

(b) Figure 1 shows the end view of a small rectangular aquarium filled with water to a depth of 1 metre. The end of the tank has dimensions 2m by 1.5m. Figure 2 shows the same aquarium with the base tilted 30° to the horizontal.

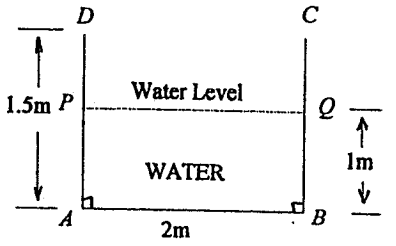



Figure 1

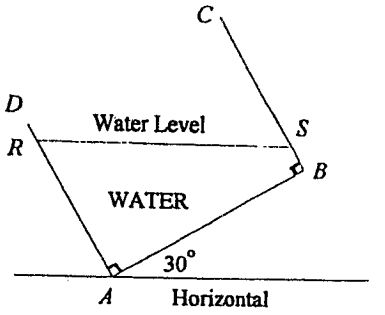



Figure 2

(i) Show that the length $SB = \frac{\sqrt{3}-1}{\sqrt{3}}$. 3

(ii) Find the height of $R$ above the horizontal. 4

**QUESTION 10 (START A NEW PAGE)**

(a) The number ($N$) of bacteria in a colony after $t$ minutes is given by the formula $N = 200e^{kt}$. If the population grows to 5000 in 40 minutes, find

(i) the exact value of $k$, 2

(ii) the number of bacteria in the colony at the end of 1 hour. 2

(b) The velocity of a train increases from 0 to $V$ at a constant rate $a$. The velocity then remains constant at $V$ for a certain time. After this time the velocity decreases to 0 at a constant rate $b$. Given that the total distance travelled by the train is $s$ and the time for the journey is $T$,

(i) Draw a velocity-time graph for the above information, 2

(ii) Show that the time ($T$) for the journey is given by $T = \frac{s}{V} + \frac{1}{2}V\left(\frac{1}{a} + \frac{1}{b}\right)$. 3

(iii) When $a$, $b$ and $s$ are fixed, find the speed that will minimize the time for the journey. 3

**THIS IS THE END OF THE EXAMINATION PAPER**

# JRAHS TRIAL HSC MATHS (2U) 2007

## QUESTION 1

(a) $\frac{1}{\sqrt{2}}$

(b) thickness $= \frac{120}{1500}$ mm

$= 0{\cdot}08$ mm

(c) $(2p-3)(p+2)$

(d) $\cos\theta = \frac{1}{2}$

$\theta = 60°, 300°$

(e) $\frac{3(x-1)-2}{(x-1)(x+1)} = \frac{3x-5}{x^2-1}$

(f) $\int_1^4 y^{\frac{1}{2}}\,dy = \left[\frac{y}{5}y^{\frac{5}{2}}\right]_1^4$

$= \frac{2}{5}\left(4^{\frac{5}{2}} - 1^{\frac{5}{2}}\right)$

$= \frac{2}{5}(32-1)$

$= 62/5$

## QUESTION 2

(a)(i) $y = x^4 - 3x^{-1}$

$y' = 4x^3 + 3x^{-2}$

$= 4x^3 + \frac{3}{x^2}$

(ii) $y' = \frac{-3}{7-3x}$

(b) $y = (x+3)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x+3)^{-\frac{1}{2}} \cdot 1$

$= \frac{1}{2\sqrt{x+3}}$

when $x = 1$, $y' = \frac{1}{2\sqrt{4}}$

$= \frac{1}{4}$

Tangent: $y - 2 = \frac{1}{4}(x-1)$

$4y - 8 = x - 1$

$x - 4y + 7 = 0$

(c) (i) $y = 2x$ —①

$x + 2y = 20$ —②

sub. ① into ②

$5x = 20$

$x = 4$

$\therefore y = 8$

$A$ is $(4,8)$

(ii) at $B$, $y = 0$ $\therefore x = 20$

$B$ is $(20,0)$

$M\left(\frac{20+4}{2}, \frac{0+8}{2}\right) = M(12,4)$

(iii) $OA = \sqrt{4^2+8^2}$

$= \sqrt{80}$

$AM = \sqrt{(4-8)^2+4^2}$

$= \sqrt{80}$

$\therefore OA = AM$ $\therefore \Delta$ is isosceles (two equal sides).

## QUESTION 3

(a) $f'(x) = 2x + \frac{1}{x}$

$f(x) = x^2 + \ln x + c$

at $(1,3)$ $3 = 1 + \ln 1 + c$

$c = 2$

$f(x) = x^2 + \ln x + 2$

(b) (i) $l = r\theta$

$12 = 8\theta$

$\theta = 1\frac{1}{2}$

(ii) $PQ^2 = 8^2 + 8^2 - 2(8)(8)\cos 1{\cdot}5$

$= 128 - 128\cos 1{\cdot}5$

$PQ = 10{\cdot}91$ (to 2 dp)

(c) $y = (x-6)(x+2)$

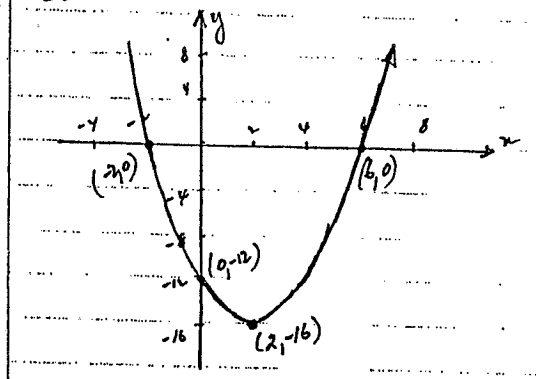
$x$-int: $(-2,0)$, $(6,0)$

$y$-int: $(0,-6)$

axis $x = 2$

vertex $(2,-16)$

Q3 (cont)




(ii) $x \le -2$ or $x \ge 6$

## QUESTION 4

(a) (i) $x^2 - 6x = 2y - 2$

$x^2 - 6x + 9 = 2y - 7$

$(x-3)^2 = 2(y-3\frac{1}{2})$

$(x-3)^2 = 4(\frac{1}{2})(y-3\frac{1}{2})$

(ii) focus $(3, 4)$

(b) (i) In $\Delta ABQ$ & $\Delta PBC$

$AB = PB$ (equal sides of equilateral $\Delta APB$)

$BQ = BC$ (equal sides of equilateral $\Delta BQC$)

$A\hat{B}Q = P\hat{B}C$ (both $60° + A\hat{B}C$, all angles of equilateral triangle are $60°$)

$\therefore \Delta ABQ \equiv \Delta PBC$ (SAS)

(ii) Let $B\hat{P}C = \theta°$

$\therefore B\hat{A}Q = \theta°$ (corresponding angles on congruent triangles)

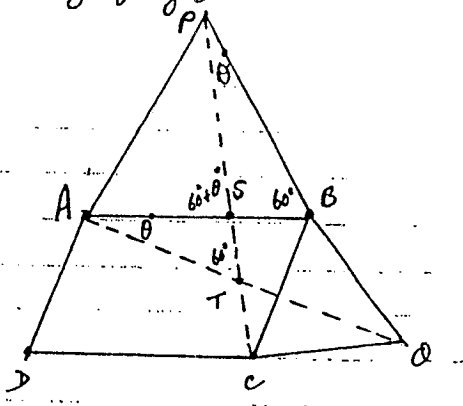
Let $AB$ & $PC$ meet at $S$.

$P\hat{S}A = \theta° + 60°$ (exterior angle of $\Delta PSB$ equals sum of opposite interior angles, $P\hat{B}S = 60°$)

Let $AQ$ & $PC$ meet at $T$

$\therefore P\hat{T}A = 60°$ (exterior angle of $\Delta ATS$ equals sum of opposite interior angles)

$\therefore$ size of angle $= 60°$




## QUESTION 5

(a)(i) $T_4 = a + 3d \Rightarrow a + 3d = 30$ —①

$T_{10} = a + 9d \Rightarrow a + 9d = 54$ —②

② − ①: $6d = 24$

$d = 4$

sub into ① $a = 30 - 3d$

$= 30 - 12$

$= 18$

$\therefore$ common diff $= 4$, first term $= 18$

(ii) $S_{20} = \frac{n}{2}(2a + (n-1)d)$

$= 10(36 + 19(4))$

$= 1120$

(b)(i) $v = 20t - 4t^2$

$a = 20 - 8t$

when $t = 3$, $a = 20 - 24$

$= -4$ ms$^{-2}$

Q5(b) cont'

(iii) $x = 10t^2 - \frac{4}{3}t^3 + C$

when $t=0, x=0 \therefore C=0$

$x = 10t^2 - \frac{4}{3}t^3$

(iii) when $v=0$, $4t(5-t)=0$

$t=0, 5$

when $t=5$, $x = 10(5)^2 - \frac{4}{3}(5)^3$

$= 83\frac{1}{3}$ m.

## QUESTION 6

(a)(i) In $\triangle ABC$ & $\triangle EDC$

$A\hat{B}C = E\hat{D}C$ (both 90°)

$A\hat{C}B = E\hat{C}D$ (common)

$\therefore \triangle ABC \parallel\parallel \triangle EDC$ (equiangular)

(ii) $\frac{BC}{DC} = \frac{AC}{EC}$ (ratio of corresponding sides)

$BC \times EC = AC \times DC$

(iii) $AC = AD + DC$

$BC = BE + EC$

$(BE+EC)EC = (AD+DC)DC$

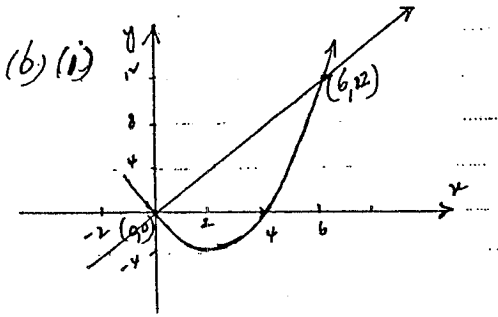
$BE.EC + EC^2 = AD.DC + DC^2$

$EC^2 - DC^2 = AD.DC - BE.EC$

$ED^2 = AD.DC - BE.EC$

(since $EC^2 - DC^2 = ED^2$ by Pythag. Theorem)

(b)(i)




$y = 2x$, $y = x^2 - 4x$

$x^2 - 4x = 2x$

$x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0, 6$

$x=0, y=0$ (0,0)

$x=6, y=12$ (6,12)

(ii) $A = \int_0^6 2x - (x^2 - 4x)\,dx$

$= \int_0^6 6x - x^2\,dx$

$= \left[3x^2 - \frac{1}{3}x^3\right]_0^6$

$= 3(6)^2 - \frac{1}{3}(6)^3 - 0$

$= 36$ u$^2$

## QUESTION 7

(a)(i) $R = 2\pi \sin \pi t$

max $R$ when $\pi t = \pi/2$

$t = \frac{1}{2}$

$\therefore$ time = 10:30 am

(ii) $V = \int 2\pi \sin \pi t\,dt$

$= -2\cos \pi t + c$

$t=0, V=0 \Rightarrow 0 = -2 + c$

$c = 2$

$\therefore V = 2 - 2\cos \pi t$ l

(iii) max vol. $= 4$ l (when $\cos \pi t = -1$)

(b)(i) $y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

for stat pt $y' = 0$

$\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$

Q7 (cont)

$2x\sqrt{x} = 2\sqrt{x}$

$2\sqrt{x}(x-1) = 0$

$x = 1$ $(x>0)$

when $x=1$, $y=2$ stat pt $(1,2)$

$y'' = -\frac{1}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

when $x=1$, $y'' = -\frac{1}{4} + \frac{3}{4}$

$> 0$

$\therefore$ concave up $\therefore$ local min tp.

(ii) $y'' = 0$

$\frac{-1}{4x\sqrt{x}} + \frac{3}{4x^2\sqrt{x}} = 0$

$x^2\sqrt{x} - 3x\sqrt{x} = 0$

$x\sqrt{x}(x-3) = 0$

$x = 3$ $(x>0)$

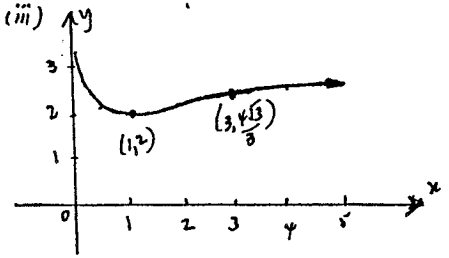
| $x$ | 1 | 3 | 4 |
|---|---|---|---|
| $y''$ | $\frac{1}{4} > 0$ | 0 | $-\frac{1}{8} < 0$ |

since curve is cts for $1<x<4$ & concavity changes then there is an inflexion pt when $x=3$

$x=3$, $y = \sqrt{3} + \frac{1}{\sqrt{3}}$

$= \frac{4}{3}\sqrt{3}$

(iii)




## QUESTION 8

(a)(i) $x^2 = 8y - 8$ — ①

$y = px + (1-2p^2)$ — ②

sub ② into ①

$x^2 = 8px + 8(1-2p^2) - 8$

$= 8px - 16p^2$

$x^2 - 8px + 16p^2 = 0$

quadratic has only one solution if $\Delta = 0$ & $\therefore$ line will be a tangent to parabola

$\Delta = (-8p)^2 - 4(16p^2)$

$= 64p^2 - 64p^2$

$= 0$ for all $p$

$\therefore$ line is tangent to parabola

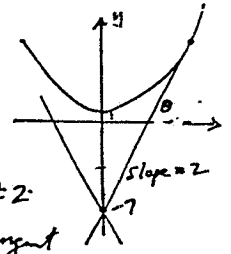
(ii) if tangents pass thro' $(0,-7)$

$-7 = 1 - 2p^2$

$2p^2 = 8$

$p^2 = 4$

$p = \pm 2$




$\therefore$ slope of tangents are $\pm 2$

$\therefore$ angle ($\theta$) between tangent an x-axis is given by $\tan\theta = 2$

$\therefore \theta = 63°26'$

$\therefore$ angle between y-axis & tangent $= 90 - 63°26'$

$= 26°34'$

$\therefore$ angle between tangents $= 53°08'$

(b) (i) $f'(x) = 2\sin x \cos x$

(ii) $V = \pi \int_0^{2\pi} (\sin x + \cos x)^2\,dx$

$= \pi \int_0^{2\pi} \sin^2 x + \cos^2 x + 2\sin x \cos x$

6 (cont)

$$= \pi \int_0^{2\pi} 1 + 2\sin x \cos x \, dx$$

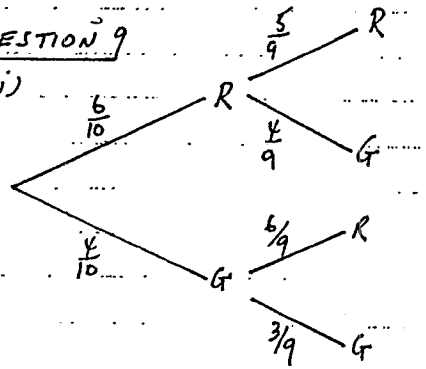
$$= \pi \left[ x + \sin^2 x \right]_0^{2\pi}$$

$$= \pi \{ (2\pi + 0) - (0 + 0) \}$$

$$= 2\pi^2 \ u^3$$

## QUESTION 9

a) (i)




(ii) $P(RR) = \frac{6}{10} \times \frac{5}{9}$

$= 1/3$

(iii) $P(2 \text{ of } G) = P(RG) + P(GG)$

$= \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{3}{9}$

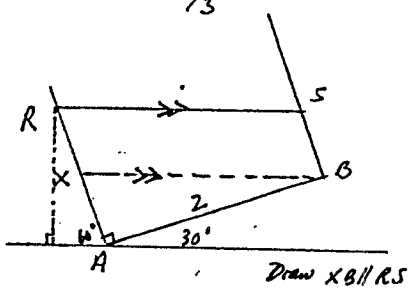
$= 2/5$

(iv) $P(\text{at least 1 G}) = 1 - P(RR)$

$= 1 - 1/3$

$= 2/3$

b)




$\frac{XA}{2} = \tan 30°$

$AX = 2/\sqrt{3}$

Also area $ARSB$ = area $APQB$ = 2

Let $SB = x$ (= RX since $XRSB$ is a parallelogram)

$\therefore \frac{1}{2}(2)\left(x + 2/\sqrt{3} + x\right) = 2$

$2x + 2/\sqrt{3} = 2$

$x = 1 - 1/\sqrt{3}$

$x = \frac{\sqrt{3} - 1}{\sqrt{3}} = SB.$

(ii) $\frac{\text{height}}{RA} = \sin 60°$ $\quad RA = 1 - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}$

$= 1 + \frac{1}{\sqrt{3}}$

$h = \frac{\sqrt{3}+1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}$

Height $= \frac{\sqrt{3}+1}{2}$

## QUESTION 10

(a)(i) $5000 = 200e^{40k}$

$25 = e^{40k}$

$40k = \ln 25$

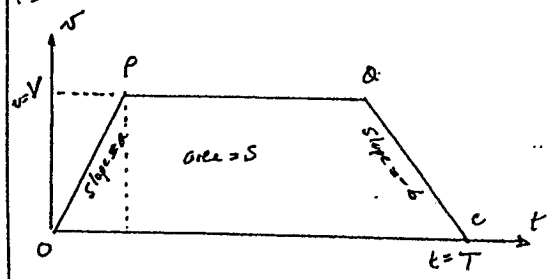
$k = \ln 25 / 40$

(ii) $t = 60$

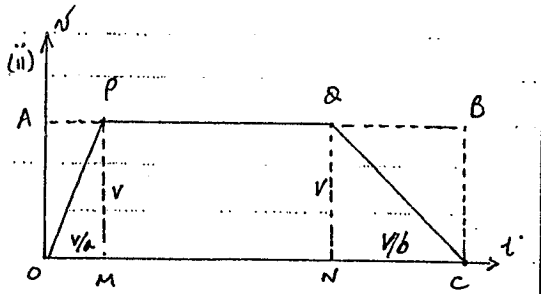
$N = 200e^{60 \times \frac{\ln 25}{40}}$

$= 250\,000$

(b) (i)




(ii)




| | |
|---|---|
| $OA = PM = V$ | $QN = BC = V$ |
| $\therefore \frac{PM}{OM} = a$ | $\frac{QN}{NC} = b$ |
| $OM = V/a$ | $NC = V/b$ |
| $(= AP)$ | $(= QB)$ |

Area of $OABC = VT$

Area of $OABC$ = area $\triangle OPM$ + trapezium $OPQC$ + area $\triangle NQC$

$= \frac{1}{2}V^2/a + S + \frac{1}{2}V^2/b.$

$\therefore VT = \frac{V^2}{2a} + \frac{V^2}{2b} + S$

$T = \frac{S}{V} + \frac{V}{2a} + \frac{V}{2b}$

$T = \frac{S}{V} + \frac{V}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

(iii) $\frac{dT}{dV} = \frac{-S}{V^2} + \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

for max/min $\frac{dV}{dT} = 0$

$\frac{S}{V^2} = \frac{b+a}{2ab}$

$\frac{2abS}{a+b} = V^2$

$V = \sqrt{\frac{2abS}{a+b}}$ (speed $V > 0$)

$\frac{d^2T}{dV^2} = \frac{2S}{V^3}$

$> 0$ since $V > 0$

$\therefore$ concave up $\Rightarrow$ local min tp.

& Since the function of $T$ is cts for $V > 0$ & there is only one $t_p$ which is a local min tp then it is the absolute min tp.

$\therefore$ speed $= \sqrt{\frac{2abS}{a+b}}$