YEAR 11 HALF YEARLY
PRELIMINARY EXAMINATION 2004

MATHEMATICS
EXTENSION 1

Time Allowed – 85 minutes
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate's Number.

Question 1. **[START A NEW PAGE]** **Marks**

(a) The point T divides the interval AB, where $A(-2, 1)$ and $B(1, -5)$ externally, in the ratio $2:1$. Find the coordinates of T. **2**

(b) Find the gradient of the normal to the curve $y = \sec x$ at $x = \frac{\pi}{6}$. **3**

(c) Find the least positive integer n such that $\left(\frac{4}{5}\right)^n < 0 \cdot 01$. **2**

(d) Simplify fully $\frac{1}{x+1} - \frac{3}{x^3+1}$. **3**

(e) Evaluate $\underset{\theta \to 0}{Lim} \frac{\sin 2\theta \tan 3\theta}{\theta^2}$. **2**

(f) Use the definition: $f'(x) = \underset{h \to 0}{Lim}\left[\frac{f(x+h) - f(x)}{h}\right]$ **3**

to differentiate the function $f(x) = \frac{2}{\sqrt{x}}$ for $x > 0$ with respect to x, from first principles.

Question 2. **[START A NEW PAGE]** **Marks**

(a) Find $\frac{dy}{dx}$ in the following.

(i) $y = \ln\left[\frac{x+3}{(x-1)^2}\right]$. 2

(ii) $y = \frac{1}{(1-7x^2)^5}$. 2

(b) Show that $y = \frac{3x+1}{x}$ is always a decreasing function for all x except $x \neq 0$. 2

(c) (i) Prove that $a^x = e^{x\log_e a}$, for $a > 0$. 1

(ii) Hence, or otherwise find $\frac{d}{dx}[(\sin x)^x]$. 3

(d) On the same set of axes sketch $y = e^{-x}$ and $y = \frac{1}{x}$. 3

Hence sketch (without using calculus) the curve

$$y = e^{-x} + \frac{1}{x} \text{ for } x \neq 0,$$

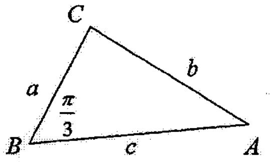
given that the point $(-k, 0)$ (where $k > 0$) lies on the curve.
Show all essential detail.

(e) Given that $\frac{d^2y}{dx^2} = \ln x + 1$, for the function $y = f(x)$, 2

find for what value(s) of x, this function is concave downwards?

Question 3. [START A NEW PAGE] **Marks**

(a) Given the triangle ABC where angle B is $\frac{\pi}{3}$,



(i) Find the expression for b^2 in terms of a and c. **1**

(ii) Hence show that the area of triangle ABC is given by: **3**

$$\frac{\sqrt{3}}{4}[b^2-(a-c)^2] \text{ square units.}$$

(b) (i) State the expansion for $\sin(A-B)$. **1**

(ii) Hence, or otherwise, show that $\frac{\sin 3x}{\sin x}-\frac{\cos 3x}{\cos x}=2$. **2**

(c) Suppose $P(x)$ is the polynomial given by:

$$P(x)=(x-a)(x-b)(x-c)-(b+c)(c+a)(a+b).$$

(i) Show that $x=a+b+c$ is a zero of $P(x)$. **1**

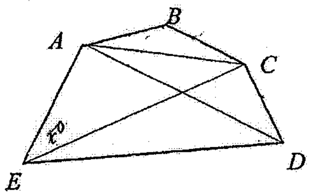
(ii) Hence, or otherwise, factorise the polynomial: **3**

$$P(x)=(x-2)(x+3)(x+1)-4.$$

(d) In the pentagonal figure $ABCDE$ below, **4**

$AB=BC=CD, \angle BAE+\angle BAC=\angle BCD$

and $\angle ACE=\angle AEC=x^0$.



Copy the diagram onto your writing paper and
Prove that $\Delta EAB \equiv \Delta ACD$.

Question 4. [START A NEW PAGE] **Marks**

(a) Consider the function: $f(x) = (x-1)^3(8-3x)+8$,
where $f(0) = f(3) = 0$ and $f''(x) = 6(x-1)(11-6x)$.

(i) Show that $f'(x) = 3(x-1)^2(9-4x)$. **2**

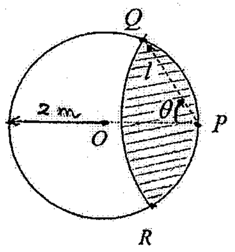
(ii) Find the stationary points and determine their nature. [express the y–values to one decimal place]. **4**

(iii) Locate all the points of inflection. **2**

(iv) Sketch the curve. **3**

(b) In the diagram below, O is the centre of a circular field of radius 2 metres. A sheep named Dolly is tethered to the circular fence at P by a rope of length l metres ($l < 4$) and Dolly is able to graze over the shaded area which is half the area of the field.

Not to scale



(i) Given angle $OPQ = \theta$, show that $\sin 2\theta = \frac{\pi}{2} + 2\theta \cos 2\theta$. **3**

(ii) Verify that $\theta = 54^{0}36'$ is the solution to the equation. **1**

THE END

Y11 MATH. EXT 1 SOLUTIONS
T 2, 2004.
HY PRELIMINARY

MARKING GUIDELINES

Q1 (a) $A(-2,1)$ $2:-1$ $B(1,-5)$

$T = \left(\frac{2\times1 + -1\times(-2)}{2+-1} , \frac{2\times(-5) + -1\times1}{2+-1} \right)$

$\therefore T = \left(\frac{4}{1} , -\frac{11}{1} \right) = (4, -11)$

1 for 4
1 for −11

②

(b) $y = \sec x$

$\frac{dy}{dx} = \sec x . \tan x.$

1 for $\frac{dy}{dx}$

Grad. of tangent at $x = \frac{\pi}{6}$

$m_T = \sec\frac{\pi}{6} . \tan\frac{\pi}{6} = \frac{2}{\sqrt{2}} . \frac{1}{\sqrt{3}} = \frac{2}{3}.$

1 for $\frac{2}{3}$

$\therefore$ Grad. of normal $m_N = -\frac{3}{2}$

1 for $-\frac{3}{2}$

③

(c) $\left(\frac{4}{5}\right)^n < 0{\cdot}01$

$\log (0{\cdot}8)^n < \log 0{\cdot}01$ METHOD 1.

$n \log (0.8) < \log 0{\cdot}01$

$\therefore n > \frac{\log 0{\cdot}01}{\log 0{\cdot}8}$ as $\log 0{\cdot}8 < 0$

1 for >

i.e $n \geq 20{\cdot}6377\ldots$

1 for $n = 21$

$\therefore$ least integer $n = 21$

$\left[\left(\frac{4}{5}\right)^{20} = 0{\cdot}0115\ldots \left(\frac{4}{5}\right)^{21} = 0{\cdot}00922\ldots \right]$ ②

(d) $\frac{1}{x+1} - \frac{3}{x^3+1} = \frac{1}{x+1} - \frac{3}{(x+1)(x^2-x+1)}$

1

$= \frac{x^2-x+1 -3}{(x+1)(x^2-x+1)}$

$= \frac{x^2 - x - 2}{(x+1)(x^2-x+1)}$

1

$= \frac{(x-2)\cancel{(x+1)}}{\cancel{(x+1)}(x^2-x+1)}$ $x \neq -1.$

$= \frac{x-2}{x^2-x+1}$ only if $x \neq -1$

1

③

Q.1(e) $L = \lim_{\theta\to0} \frac{\sin 2\theta . \tan 3\theta}{\theta^2}$

METHOD 1 $L = \lim_{\theta\to0} 2. \frac{\sin 2\theta}{(2\theta)} \times 3. \frac{\tan 3\theta}{(3\theta)}$

1 for organizing the forms

$= 2.1 \times 3.1$ as $\lim_{x\to0} \begin{cases} \frac{\sin x}{x} = 1 \\ \frac{\tan x}{x} = 1 \end{cases}$

$\therefore L = 6$

1

②

Q1(f) $f'(x) = \lim_{h\to0} \left[\frac{f(x+h) - f(x)}{h} \right]$

$= \lim_{h\to0} \left[\frac{\frac{2}{\sqrt{x+h}} - \frac{2}{\sqrt{x}}}{h} \right]$

$= \lim_{h\to0} \frac{2(\sqrt{x} - \sqrt{x+h})}{h\sqrt{x}.\sqrt{x+h}}$

1

$= \lim_{h\to0} \frac{2.(\sqrt{x} - \sqrt{x+h})}{h\sqrt{x^2+hx}} \times \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$

$= \lim_{h\to0} \frac{2(x - (x+h))}{h\sqrt{x^2+hx}\,(\sqrt{x}+\sqrt{x+h})}$

1

$= \lim_{h\to0} \frac{2(-\cancel{h})}{\cancel{h}\sqrt{x^2+hx}\,(\sqrt{x}+\sqrt{x+h})}$ as $h \neq 0$, $h \to 0$

$= \lim_{h\to0} \frac{-2}{\sqrt{x^2+hx}\,(\sqrt{x}+\sqrt{x+h})}$

1

$= \frac{-2}{\sqrt{x^2+0}\,(\sqrt{x}+\sqrt{x+0})}$

$= \frac{-2}{x.(2\sqrt{x})}$

$\therefore f'(x) = \frac{-1}{x\sqrt{x}}, \quad x > 0.$

③

Q2 (a) (i) $y = \ln\left[\frac{x+3}{(x-1)^2}\right] = \ln(x+3) - 2\ln(x-1)$ 1

$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2}{x-1}$ 1

$= \frac{-(2x+7)}{(x+3)(x-1)}$ ②

(ii) $y = \frac{1}{(1-7x^2)^5} = (1-7x^2)^{-5}$

$\frac{dy}{dx} = -5(1-7x^2)^{-6} \times -14x$ 1

$= \frac{70x}{(1-7x^2)^6}$ 1 ②

(b) $y = \frac{3x+1}{x} = 3 + \frac{1}{x}$

$\frac{dy}{dx} = -\frac{1}{x^2}$ 1

For an decreasing function $\frac{dy}{dx} < 0$

as $x^2 > 0$ for all x except $x = 0$

$\therefore -\frac{1}{x^2} < 0$ " " x except $x \neq 0$ 1 for explanation

i.e. $\frac{dy}{dx} < 0$

$\therefore y = \frac{3x+1}{x}$ is always decreasing ②

(c) (i) Let $N = a^x$

$\therefore \log_e N = \log_e a^x = x \log_e a$ 1 for a correct method.

so $N = e^{x\log_e a}$

i.e $a^x = e^{x\log_e a}$. ①

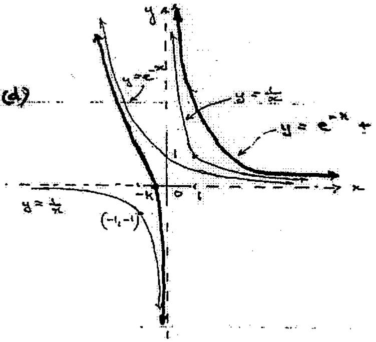
(ii) $\frac{d}{dx}[(\sin x)^x] = \frac{d}{dx}[e^{x \ln \sin x}]$

$= e^{x.\ln \sin x} \times [1.\ln \sin x + x.\frac{\cos x}{\sin x}]$ (1+1)

$= [\ln \sin x + x \cot x](\sin x)^x$ 1

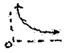
③

Q2(d)



③

1 for e^{-x} + $\frac{1}{x}$

1 for 

1 for 

(e) For a function to be concave downwards

$f''(x) < 0$.

$\therefore \ln x + 1 < 0$

$\ln x < -1$

$x < e^{-1}$ 1

but from $\ln x \Rightarrow x > 0$

$\therefore$ $\underline{0 < x < e^{-1}}$. 1

②

Q3(a) (i) $b^2 = a^2 + c^2 - 2.a.c.\cos\frac{\pi}{3}$

$\therefore\ \underline{b^2 = a^2 + c^2 - ac}$ $\quad \cos\frac{\pi}{3} = \frac{1}{2}$ 1

①

(ii) Area $\Delta ABC = \frac{1}{2}ac.\sin\frac{\pi}{3}$

$= \frac{1}{2}.ac.\frac{\sqrt{3}}{2}$

$= \frac{ac\sqrt{3}}{4}$. 1 1

Method 1. $ac = a^2 + c^2 - b^2$ from (i)

$\therefore\ ac - 2ac = a^2 - 2ac + c^2 - b^2$ c.t.sq. 1

$-ac = (a-c)^2 - b^2$

i.e. $ac = b^2 - (a-c)^2$ 1

so $\frac{ac\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[b^2 - (a-c)^2]$

i.e. Area $= \frac{\sqrt{3}}{4}[b^2 - (a-c)^2]$.

③ RHS → LHS re Hence ! Ans 1

(b) (i) $\sin(A-B) = \sin A\cos B - \sin B\cos A$ ①

or $= \sin A\cos B - \cos A\sin B$. 1.

(ii) LHS $= \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

$= \frac{\sin 3x\cos x - \cos 3x.\sin x}{\sin x\cos x}$ 1

$= \frac{\sin(3x - x)}{\sin x\cos x}$

$= \frac{\sin 2x}{\sin x\cos x} = \frac{2\cancel{\sin x}\cancel{\cos x}}{\cancel{\sin x}\cancel{\cos x}}$ 1 $\sin x, \cos x \neq 0$.

$= 2$

$=$ RHS ②

$\therefore\ \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$.

Q3 (c) $P(x) = (x-a)(x-b)(x-c) - (b+c)(c+a)(a+b)$

(i) Now $P(a+b+c) = (a+b+c-a)(a+c)(a+b) - (b+c)(c+a)(a+b)$

$= (b+c)(a+c)(a+b) - (b+c)(c+a)(a+b)$ 1

$= 0$ ①

$\therefore$ By the Factor thm $x - (a+b+c)$ is a factor or $x = a+b+c$ is a zero of $P(x)$.

(ii) For $P(x) = (x-2)(x+3)(x+1) - 4$

suppose $a = 2$, $b = -3$ and $c = -1$.

Now $a+b+c = 2 + (-3) + (-1) = -2$.

$(b+c)(c+a)(a+b) = (-3+-1)(-1+2)(2+-3)$

$= -4 \times 1 \times (-1) = \underline{4}$ 1 for showing fully that $x = -2$

$\therefore\ x = -2$ is a zero of $P(x)$.

$\therefore\ P(x) = (x+2)\,Q(x)$.

Now $P(x) = (x-2)(x+3)(x+1) - 4$

$= x^3 + 2x^2 - 5x - 6 - 4$

$= x^3 + 2x^2 - 5x - 10$

$$\text{So}\quad x+2\,\overset{\displaystyle x^2 - 5}{\overline{\big)\,x^3 + 2x^2 - 5x - 10}}$$

$x^3 + 2x^2$

$0 \quad 0 \quad -5x - 10$

$-5x - 10$

$0 \quad 0$

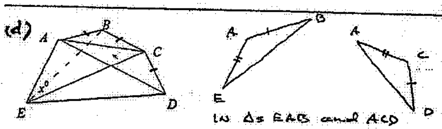
or $x^2(x+2) - 5(x+2)$

$(x+2)(x^2 - 5)$ 1

$\therefore\ P(x) = (x+2)(x^2-5)$

$= (x+2)(x-\sqrt{5})(x+\sqrt{5})$ ③ 1

(d)



In Δs EAB and ACD

S 1. $AB = CD$ (given data) 1

A 2. As $\angle BAC = \angle BCA = y^\circ$ (angles opposite equal sides are equal) 1

and as $\angle BAE + \angle BAC = \angle BCD$

i.e. $\angle BAE + y^\circ = \angle BCA + \angle ACD$

$\angle BAE + y^\circ = y^\circ + \angle ACD$

so $\angle BAE = \angle ACD$ 1

S 3. $AE = AC$ (sides opposite equal angles are equal) 1

$\therefore\ \Delta EAB \equiv \Delta ACD$ (SAS) ④

Q4(a) $f(x) = (x-1)^3(8-3x) + 8$

(i) $f'(x) = 3(x-1)^2(8-3x) + (x-1)^3 \times (-3)$ 1

$= 3(x-1)^2[8-3x-(x-1)]$

$= 3(x-1)^2[8-3x-x+1]$ 1

$\therefore f'(x) = 3(x-1)^2(9-4x)$ ②

(ii) For SPs to occur $f'(x) = 0$

$3(x-1)^2(9-4x) = 0$

$\therefore x = 1$ or 1 or $2\frac{1}{4}$

$\therefore \quad y = 8$ or $10.441...$

$\therefore$ SPs are $(1, 8)$ and $(2\frac{1}{4}, 10.4)$ 1, 1

Test

at $x = 1$ $f''(1) = 6 \times 0 \times 5 = 0$!

so

x	0.5	1	1.5
$f'(x)$	5.25 ⊕ /	0 — —	2.25 ⊕ /

3(+)(+) ⊕ 0 3(+)(+) ⊕

since $f(x)$ is cont + diff'ble over the domain $0.5 \le x \le 1.5$

and $f'(x)$ does not change sign

$\therefore$ a H.P.o Inflection exists at $x=1$ 1

at $x = 2\frac{1}{4}$

$f''(2\frac{1}{4}) = 6 \times 1\frac{1}{4} \times (11 - 6 \times 2\frac{1}{4})$

$= -18\frac{3}{4}$

< 0

$\therefore$ at $x = 2\frac{1}{4}$ $f(x)$ is concave downwards

and since it is a SP, ④

$\therefore$ a relative/local min. TP at $x = 2\frac{1}{4}$. 1

(iii) For possible points of inflection $f''(x) = 0$

$6(x-1)(11-6x) = 0$

$\therefore x = 1$ or $1\frac{5}{6}$

$y = 8$ or $8\frac{25}{432} \doteq 8.1$

We know at $(1, 8)$ there is a HPoI

$\therefore$ need to test $(1\frac{5}{6}, 8.1)$ 1

x	1.5	$1\frac{5}{6}$	2
$f''(x)$	6 ⊕	0	−6 ⊖

6(+)(+) ⊕ 0 6(+)(−) ⊖ 1

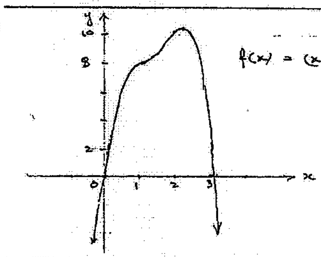
{since $f(x)$ is cont + diff'ble over the domain $1.5 \le x \le 2$.}

and $f''(x)$ changes sign

$\therefore$ there is a change in concavity

$\therefore$ a P.O.I. exists at $x = 1\frac{5}{6}$ ②

(iv)



$f(x) = (x-1)^3(8-3x) + 8$.

1 for $f(0) = f(3) = 0$

$\frac{1}{2}$ for HPOI

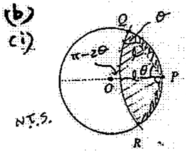
$\frac{1}{2}$ for max TP

1 for shape

③

(b)

(i)



Note: $\angle PQO = \theta$

$\therefore \angle POQ = \pi - 2\theta$ 1

Area grazed

$= 2$ sectors $+ 2$ segments

$= 2 \times \frac{1}{2} \times l^2 \times \theta + 2 \times \frac{1}{2} \times 2^2[\pi - 2\theta - \sin(\pi - 2\theta)]$ 1

$= l^2\theta + 4(\pi - 2\theta - \sin 2\theta)$

but Area $= \frac{1}{2} \times \pi \times 2^2 = 2\pi$

$\therefore l^2\theta + 4\pi - 8\theta - 4\sin 2\theta = 2\pi$

i.e. $l^2\theta + 2\pi - 8\theta - 4\sin 2\theta = 0$

but $l^2 = 2^2 + 2^2 - 2.2.2.\cos(\pi - 2\theta)$

$= 8 + 8\cos 2\theta$ 1

$\therefore \cancel{8\theta} + 8\theta\cos 2\theta + 2\pi - \cancel{8\theta} - 4\sin 2\theta = 0$

$4\sin 2\theta = 2\pi + 8\theta\cos 2\theta$

$\therefore \sin 2\theta = \frac{\pi}{2} + 2\theta\cos 2\theta$ ③