James Ruse AHS
3U Trial '01

Question 1:

(a)(i) Find the derivative of $x^2 \cos x$. 2

(ii) Evaluate $\int_1^6 \frac{x}{x^2+4}\,dx$. 2

(b)(i) Sketch $y = |x+1|$. 2

(ii) Hence or otherwise solve $|x+1| = 3x$. 1

(c) If $f(x) = 2\sin^{-1}(3x)$, find

(i) the domain and range of $f(x)$, 2

(ii) $f\left(\frac{1}{6}\right)$, 1

(iii) $f'\left(\frac{1}{6}\right)$. 2

QUESTION 2: **(START A NEW PAGE)**

(a) P(-7,3), Q(9,15) and B(14,0) are three points and A divides the interval PQ in the ratio 3:1. Prove that PQ is perpendicular to AB. 3

(b) By using the substitution $u^2 = x+1$ evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$. 3

(c) Water flows from a hole in the base of a cylindrical vessel at a rate given by 6

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is a constant and h mm is the depth of water at time t minutes.
If the depth of water falls from 2500mm to 900mm in 5 minutes, find how much longer it will take to empty the vessel.

QUESTION 3: **(START A NEW PAGE)**

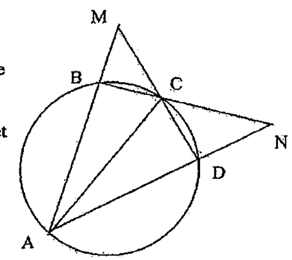
(a) Find the value of the constant term in the expansion of $\left(3x + \frac{2}{\sqrt{x}}\right)^6$. 3

(b) Three boys (Adam, Bruce, Chris) and three girls (Debra, Emma, Fay) form a single queue at random in front of the school canteen window. Find the probability that:

(i) the first two to be served are Emma and Adam in that order, 2

(ii) a boy is at each end of the queue, 1

(iii) no two girls stand next to each other. 1



(c) In the figure ABM, DCM, BCN and ADN are straight lines and $\angle AMD = \angle BNA$.

(i) Copy the diagram onto your answer sheet and prove that $\angle ABC = \angle ADC$. 3

(ii) Hence prove that AC is a diameter. 2

QUESTION 4: **(START A NEW PAGE)**

(a)(i) Given that $\sin^2 A + \cos^2 A = 1$, prove that $\tan^2 A = \sec^2 A - 1$. 2

(ii) Sketch the curve $y = 4\tan^{-1} x$ clearly showing its range. 2

(iii) Find the volume of the solid formed when the area bounded by the curve $y = 4\tan^{-1} x$, the y-axis and the line $y = \pi$ is rotated one revolution about the y-axis. 2

(b)(i) An object has velocity $v\ ms^{-1}$ and acceleration $\ddot{x}\ ms^{-2}$ at position $x\ m$ from the origin, show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$. 2

(ii) The acceleration (in ms^{-2}) of an object is given by $\ddot{x} = 2x^3 + 4x$.

(α) If the object is initially 2 m to the right of the origin traveling with velocity 6 ms^{-1}, find an expression for v^2 (the square of its velocity) in terms of x. 2

(β) What is the minimum speed of the object? (Give a reason for your answer) 2

QUESTION 5: **(START A NEW PAGE)**

(a) The curves $y = e^{-2x}$ and $y = 3x + 1$ meet on the y-axis. Find the size of the acute angle between these curves at the point where they meet. 3

(b)(i) Sketch the function $y = f(x)$ where $f(x) = (x-1)^2 - 4$ clearly showing all intercepts with the co-ordinate axes. (Use the same scale on both axes) 2

(ii) What is the largest positive domain of f for which $f(x)$ has an inverse $f^{-1}(x)$? 1

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same axes as (i). 1

(c) In tennis a player is allowed a maximum of two serves when attempting to win a point. If the first serve is not legal it is called a fault and the server is allowed a second serve. If the second serve is also illegal then it is called a double fault and the server loses the point. The probability that Pat Smash's first serve will be legal is 0.4. If Pat Smash needs to make a second serve then the probability that it will be legal is 0.7.

(i) Find the probability that Pat Smash will serve a double fault when trying to win a point. 2

(ii) If Pat Smash attempts to win six points, what is the probability that he will serve at least two double faults? (Give answer correct to 2 decimal places) 3

QUESTION 6: **(START A NEW PAGE)**

(a) A spherical bubble is expanding so that its volume is increasing at $10\ cm^3s^{-1}$. Find the rate of increase of its radius when the surface area is $500\ cm^2$. 3

(Volume $= \frac{4}{3}\pi r^3$, Surface area $= 4\pi r^2$)

(b) Prove by Mathematical Induction that: 4

$$2(1!) + 5(2!) + 10(3!) + \ldots + (n^2+1)n! = n(n+1)! \text{ for positive integers } n \geq 1.$$

(c) If $y = \dfrac{\log_e x}{x}$ find $\dfrac{dy}{dx}$ and hence show that $\displaystyle\int_e^{e^2} \frac{1-\log_e x}{x\log_e x}\,dx = \log_e 2 - 1$. 5

QUESTION 7: **(START A NEW PAGE)**

(i) By considering the expansion of $\sin(X+Y) - \sin(X-Y)$ prove that 3

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

(ii) Also given that $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ prove that 2

$\dfrac{\sin A - \sin B}{\cos A - \cos B} = -\cot\left(\dfrac{A+B}{2}\right)$.

(iii) Prove that the position of a projectile t seconds after projection from ground level with initial horizontal and vertical velocity components of $V\cos\alpha$ and $V\sin\alpha$ respectively is given by $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$. (Assume that there is no air resistance) 2

(iv) Two objects P and Q are projected from the same ground position at the same time with initial speed $V\ ms^{-1}$ at angles α and β respectively ($\beta > \alpha$).

(α) If at time t seconds the line joining P and Q makes an acute angle θ with the horizontal prove that $\tan\theta = \left|\dfrac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right|$. 3

(β) Hence show that $\theta = \frac{1}{2}(\pi - \alpha - \beta)$. 2

THIS IS THE END OF THE EXAMINATION PAPER

81/84 Excellent work!

1.(a)(i) $x^2\cos x$

$u = x^2 \qquad v = \cos x$

$u' = 2x \qquad v' = -\sin x$

$y' = -x^2\sin x + 2x\cos x$

$= x[2\cos x - x\sin x]$

(ii) $\int_1^6 \frac{x}{x^2+4}\,dx$

$= \frac{1}{2}\int_1^6 \frac{2x}{x^2+4}$

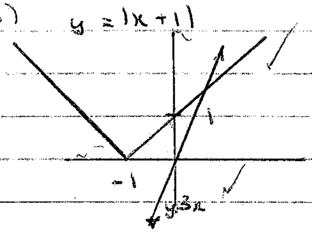
$= \frac{1}{2}\left[\ln(x^2+4)\right]_1^6$

$= \frac{1}{2}\ln\left(\frac{40}{5}\right)$

$= \frac{1}{2}\ln 8$

$= \frac{3}{2}\ln 2.$

(b)(i) $y = |x+1|$



$x+1 = 3x$

$1 = 2x$

$x = \frac{1}{2}.$

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(c) $f(x) = 2\sin^{-1}(3x)$

D: $-1 \le 3x \le 1$ R: $-\pi/2 \le y/2 \le +\pi/2$

$-\frac{1}{3} \le x \le \frac{1}{3}$ $\quad -\pi \le y \le +\pi$

$f(\frac{1}{6}) = 2\sin^{-1}(\frac{1}{2})$

$= \pi/3.$

$f'(\frac{1}{6}) = \frac{6}{\sqrt{1-9(\frac{1}{6})^2}} = \frac{6}{\sqrt{3}/\sqrt{4}}$

$= \frac{6}{\sqrt{1-\frac{9}{36}}} = \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6}{\sqrt{\frac{27}{36}}} = \frac{12}{3\sqrt{3}} = 4\sqrt{3}.$

~~2.(a) P(-7,3), Q(9,15) B(14,0) 3:1~~

~~$\frac{27-7}{4}, \frac{45+3}{4}$~~ $\quad m_{PQ} = \frac{12}{16}$

~~A(5,12)~~ $\quad$ ~~$m_{AB} = -\frac{12}{9}$~~

2(a) P(-7,3), Q(9,15) 3:1

$\frac{27-7}{4}, \frac{45+3}{4}$

A(5, 12)

$m_{PQ} = \frac{12}{16}$

$= 3/4.$

$m_{AB} = \frac{12}{-9}$

$= -4/3$

$\therefore m_{PQ} \times m_{AB} = -1$

(b). $u^2 = x+1$ $\quad \int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$

$\frac{dx}{du} = 2u$

At $x = 3, u = 2.$

$x = 0, u = 1$ $\quad I = \int_1^2 \frac{u^2+1}{u}\cdot 2u\,du$

$= 2\left[\frac{u^3}{3} + u\right]_1^2$

$= 2\left[(8/3+2) - (1/3+1)\right]$

$= \frac{20}{3}$

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(c). $\frac{dh}{dt} = -k\sqrt{h}$

$\frac{dh}{dt} = 320\text{mm}/\text{min}$

$\frac{125}{16}$ min

$\frac{dh}{dt} = -k(h)^{1/2}$

$\frac{dt}{dh} = \frac{-1}{k\sqrt{h}}$

$\frac{dt}{dh} = -\frac{1}{k}\cdot(h)^{-1/2}$

$t = -\frac{2}{k}\sqrt{h} + c.$

At $t=0$, $h = 2500$

$0 = -\frac{100}{k} + c.$

$c = 25/k.$

$\therefore t = -\frac{\sqrt{h}}{2k} + \frac{25}{k}.$

At $t = 5$, $h = 900.$

$5 = -\frac{15}{k} + \frac{25}{k}.$

$5 = \frac{10}{k}$

$k = 2.$

$t = -\frac{\sqrt{h}}{2} + \frac{25}{2}.$

$h = 0,\ t = \frac{25}{2}.$

Time further $= t - 5$

$= 7\frac{1}{2}.$

3 (a). $(3x + 2x^{-1/2})^6$

$\binom{6}{r}(3x)^{6-r}(2x^{-1/2})^r$

$x^{6-r} \cdot x^{-1/2 r}$

$x^{6-3/2 r} = x^0$

$6 - 3/2 r = 0.$ ✓

$4 = r.$

$\therefore \binom{6}{4}(3x)^2(2x^{-1/2})^4$

$= \binom{6}{4}(3)^2(2)^4$

$= 2160.$ ✓

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(b) (i) $\frac{4!}{6!} = \frac{1}{30}.$ ✓

(ii). ~~$\frac{{}^3P_1 \times 3! \times {}^3P_2}{6!} = \frac{108}{720} = 3/20$~~

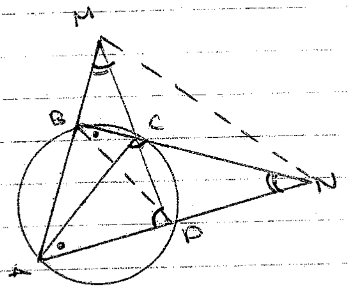
B xxxx B

$\frac{3! \times 2 \times 3}{720}$ ✓

$= \frac{1}{20}.$ ✓

(iii). $\frac{3! \times 3! \times 2}{6!} = \frac{1}{10}.$ ✓

(c).



(i) $A\hat{B}C = A\hat{D}C$ (opp. Angles in cyclic quad)

Let $A\hat{B}N = \beta$, $B\hat{M}C = \alpha$. $\pi - \alpha - \pi + \beta$

$\therefore M\hat{B}C = \pi - \beta$, $\therefore B\hat{C}M = \pi - \alpha - \pi + \beta = \beta - \alpha$

$\therefore D\hat{C}N = \beta - \alpha$ (vert. opp)

$\therefore A\hat{D}C = \beta - \alpha$ (exterior Angle)

$= A\hat{B}C$ ✓

(ii) $A\hat{B}C = A\hat{D}C$ (proven above)

$\therefore A\hat{B}C + A\hat{D}C = \pi$ (cyclic Angles)

$\therefore A\hat{B}C = \pi/2$

$\therefore AC =$ diameter ✓ (Angle in semicircle $= \pi/2$)

4.(a) $\sin^2 A + \cos^2 A = 1$

Divide by $\cos^2 A$ ✓

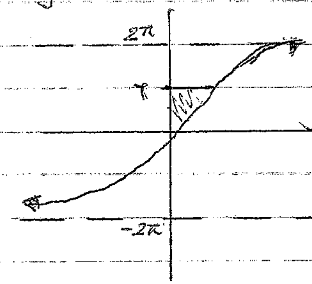
$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$ ✓

(ii) $y = 4\tan^{-1}x.$

Range: $-\pi/2 < y/4 < \pi/2$

$-2\pi < y \leq 2\pi$ ✓



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(iii). $y = 4\tan^{-1}x$

$y/4 = \tan^{-1}x$ ✓

$\tan(y/4) = x$

$V = \pi\int_0^{\pi} x^2 \, dy$

$= \pi\int_0^{\pi} \tan^2(y/4) \, dy$

$= \pi\int_0^{\pi} \sec^2(y/4) - 1 \, dy$

$= \pi\left[4\tan(y/4) - y\right]_0^{\pi}$

$= \pi\left[4 - \pi\right]$ ✓

(b). $\frac{d}{dx}(\frac{1}{2}v^2)$ Let $v^2 = z$

$\frac{dz}{dv} = 2v$

$\frac{d}{dx}(\frac{1}{2}v^2) = \frac{1}{2} \times \frac{dz}{dv} \times \frac{dv}{dx}$ ✗

$= \frac{1}{2} \times 2v \times \frac{dv}{dx}$

$= v\frac{dv}{dx} = \frac{dx}{dt} \times \frac{dv}{dx} = \frac{dv}{dt}$

$= \ddot{x}$

$=$ RHS

(ii). $\ddot{x} = 2x^3 + 4x$

$\frac{d}{dx}(\frac{1}{2}v^2) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$

$v^2 = x^4 + 4x^2 + c$ ✓

At $x = 2$, $v = 6$

$36 = 16 + 16 + c$

$c = 4.$ ✓

$\therefore v^2 = x^4 + 4x^2 + 4.$

$v^2 = (x^2 + 2)^2$

when $\ddot{x} = 0$, $x = 0.$ ✓

$v = \pm 2$ But $v > 2$.

Min speed $= 2$.

5.(a). $y = e^{-2x}$ $y = 3x+1$ $3x+1 = e^{-2x}$

$\frac{dy}{dx} = -2e^{-2x}$ Careless! $m_2 = 3$. $x = 0$.

$m_1 = -2e^{-2x}$, $m_1 = -2$.

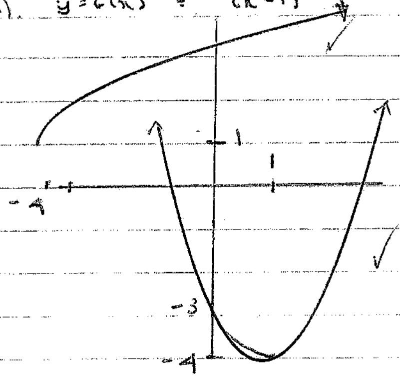
$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\frac{3+2}{1-6}\right|$

$= 1$

$\theta = \frac{\pi}{4}$

(b)(i). $y = f(x) = (x-1)^2 - 4$. (ii). $x \geq 1$



(c)(i) 0.6×0.3

$= 0.18$.

(ii) $p = 0.18$, $q = 0.82$

$n = 6$.

$P(\text{At least } 2) = 1 - P(0) - P(1)$

$= 1 - \binom{6}{0}(0.82)^6 - \binom{6}{1}(0.18)(0.82)^5$

$= 1 - 0.304 - 0.4$

$= 0.29559$

$= 0.30$.

6. (a). $\frac{dV}{dt} = 10$.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{10}{4\pi r^2}$ when $SA = 500 = 4\pi r^2$

$\sqrt{\frac{125}{\pi}} = r$

$= \frac{10}{4\pi \frac{125}{\pi}}$

$= \frac{1}{50}$. cm/sec.

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6(b) $2(1!) + 5(2!) + 10(3!) + \dots (n^2+1)n! = n(n+1)!$

Let $n=1$

LHS $= 2$ RHS $= 1(1+1)!$

$= 2$

LHS = RHS

Assume true for $S(k) = k(k+1)!$

Prove true for $S(k+1) = (k+1)(k+2)!$

LHS $= k(k+1)! + [(k+1)^2+1](k+1)!$

$= (k+1)![k + k^2 + 2k + 2]$

$= (k+1)!(k^2+3k+2)$

$= (k+1)!\,(k+1)(k+2)$

$= (k+2)!\,(k+1)$

$=$ RHS

If true for $n=k$, then shown true for $n=k+1$. By the principles of Mathematical Induction, true for $n \geq 1$ integers.

(c). $y = \frac{\ln x}{x}$ $u = \ln x$, $v = x$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ $u' = \frac{1}{x}$ $v' = 1$

$= \frac{1 - \ln x}{x^2}$

$\int \frac{1-\ln x}{x^2} = \frac{\ln x}{x} + c$.

$I = \int_e^{e^2} \frac{1-\ln x}{x\ln x} = \int_e^{e^2} \frac{\frac{1-\ln x}{x^2}}{\frac{\ln x}{x}}$

$= \left[\frac{\ln x}{x}\right]_e^{e^2}$

$= \int_e^{e^2} \frac{\frac{1-\ln x}{x^2}}{\frac{\ln x}{x}}$

$= \left(\ln\left[\frac{\ln x}{x}\right]\right)_e^{e^2}$

$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$

$= \ln\left(\frac{2}{e^2}\right) - \ln\left(\frac{1}{e}\right)$

$= \ln\left(\frac{\frac{2}{e^2}}{\frac{1}{e}}\right)$

$= \ln\left(\frac{2}{e}\right)$

$= \ln 2 - \ln e$

$= \ln 2 - 1$

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7 (a)(i) $\sin(x+y) - \sin(x-y)$

Let $x+y = A$, $x-y = B$ $\quad \underline{A+B = 2x} \quad \underline{A-B = 2y}$

$\sin(x+y) - \sin(x-y)$ ✓

$= \sin x\cos y + \sin y\cos x - \sin x\cos y + \sin y\cos x$

$= 2\sin y\cos x$ ✓

$= \underline{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}.$

(ii). $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ ✓

$\frac{\sin A - \sin B}{\cos A - \cos B} = \frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$

$= \frac{\cancel{\sin\left(\frac{A-B}{2}\right)}\cos\left(\frac{A+B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right)\cancel{\sin\left(\frac{A-B}{2}\right)}}$

$= \frac{-\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}$ ✓

$= \underline{-\cot\left(\frac{A+B}{2}\right)}$

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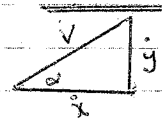
(iii). $\ddot{x} = 0$



$\dot{x} = \int 0 \, dt$

$= c$

At $t = 0$, $c = V\cos\alpha$ ✓

$\dot{x} = V\cos\alpha$

$x = \int V\cos\alpha \, dt$

$= Vt\cos\alpha + c_2$

At $x = 0$, $t = 0$ ✓

$\underline{x = Vt\cos\alpha}.$

$\ddot{y} = -g$

$\dot{y} = \int -g \, dt$

$\dot{y} = -gt + c_3$ ✓

At $\dot{y} = 0$, $c = V\sin\alpha$

$\dot{y} = -gt + V\sin\alpha$

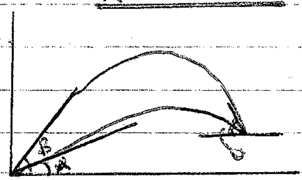
$y = \int \dot{y}\, dt$ ✓

$= \underline{\frac{-gt^2}{2} + Vt\sin\alpha + c}$ ✓

At $y = 0$, $t = 0$.

$\underline{y = -\frac{gt^2}{2} + Vt\sin\alpha}$

(iv)



(α). $x_P = Vt\cos\alpha$, $y_P = \frac{-gt^2}{2} + Vt\sin\alpha$

$x_Q = Vt\cos\beta$, $y_Q = \frac{-gt^2}{2} + Vt\sin\beta$ ✓

$\tan\theta = \left|\frac{x_P - x_Q}{y_P - y_Q}\right|$

$= \left|\frac{-\frac{1}{2}gt^2 + Vt\sin\beta + \frac{1}{2}gt^2 - Vt\sin\alpha}{Vt\cos\beta - Vt\cos\alpha}\right|$

$= \left|\frac{\cancel{Vt}(\sin\beta - \sin\alpha)}{\cancel{Vt}(\cos\beta - \cos\alpha)}\right|$ ✓

(β). $\tan\theta = \left|\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right|$ ✓ Careless!

$= \left|-\cot\left(\frac{\beta+\alpha}{2}\right)\right|$ ✓

$\tan\theta = \tan\left(\pi/2 - \frac{\alpha+\beta}{2}\right)$

$\theta = \pi/2 - \frac{\alpha}{2} - \beta/2$

$= \frac{1}{2}[\pi - \alpha - \beta]$ ✓