TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 1999

MATHEMATICS

3 UNIT / 4 UNIT COMMON PAPER

Time Allowed – Two Hours
(Plus 5 minutes reading time)

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

Standard integral tables are printed at the end of the examination paper and may be removed for your convenience. Approved silent calculators may be used.

The answers to the seven questions are to be returned in separate bundles clearly labelled Question 1, Question 2, etc. Each bundle must show your Candidate's Number.

QUESTION 1: (Start a new page)

(a) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{2x}$

(b) Express $\log\left(\frac{x^3 y^2}{\sqrt{z}}\right)$ in terms of a, b and c if $\log x = a$, $\log y = b$, and $\log z = c$.

(c) For the function $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ state the:

(i) Domain

(ii) Range

(iii) Draw a neat sketch of the curve $y = 2\sin^{-1}\left(\frac{x}{3}\right)$

(d) Find all values of θ (in radians) if $\sqrt{3}\sin\theta = \cos\theta$

QUESTION 2: (Start a new page)

(a) If $y = 10^x$, find $\frac{dy}{dx}$ when $x = 1$

(b) Evaluate in terms of π $\int_0^1 \frac{dx}{x^2+3}$

(c) From eight teachers and six pupils a committee of seven is to be formed. How many committees can be selected if both teachers and pupils are represented and the teachers are in the majority?

(d) Given that $y = \sin^{-1}(x^2)$, find $\frac{d^2y}{dx^2}$.

QUESTION 3: (Start a new page)

(a) A manufacturer produces computer components of which 85% are found to be satisfactory. From a sample of 10 components:

(i) Find the probability that at most 1 fails to meet the specification.

(ii) At least 2 are unsatisfactory.

Give your answers to part (i) and (ii) correct to 2 decimal places.

(b) A vessel is being filled at a variable rate $\frac{dV}{dt} = k(A - V)$ where k and A are constants.

(i) Show that $V = A\left(1 - e^{-kt}\right)$ is a solution of the differential equation above.

(ii) Find the capacity of the vessel.

(iii) Find the value of k if $\frac{1}{8}$ of the vessel is filled in 6 minutes.

(iv) Find the fraction of the vessel filled in the next 6 minutes.

QUESTION 4: (Start a new page)

(a) The velocity $\left(Vms^{-1}\right)$ of a body moving in a straight line is given by $V = e^{t} - e^{-t}$ where t is the time in seconds. If its initial position is at the origin:

(i) Find the equation relating x (the displacement from O) and t.

(ii) Find the initial acceleration.

(iii) Show that the body does not have a maximum velocity.

(iv) Find the time taken to reach a point 3m to the right of the origin. Give your answer correct to 1 decimal place.

(b) The region enclosed by the curve $y = \tan x$, the x axis and the ordinate $x = \frac{\pi}{4}$ is rotated about the x-axis. Using Simpson's Rule with 5 function values, find an approximate value (to 1 dec. pl.) for the volume of the solid formed.

QUESTION 5: (Start a new page)

(a) A spherical balloon is being inflated. When the radius of the balloon is 6cms its volume is increasing at the rate of $100\text{cm}^3/\text{sec}$. Find the rate at which its surface area is then increasing.

(b) (i) Find the equation of the tangent to the curve $y = \dfrac{x+1}{x^2+3}$ at the point where the curve cuts the x axis.

(ii) Show that the tangent meets the curve again at a point where the function has a stationary point.

QUESTION 6: (Start a new page)

(a) Use the substitution $u = \sin x$ to evaluate the integral $\int \cos^3 x \, dx$.

(b) When $(3+2x)^n$ is expanded as a polynomial in x, the coefficients of x^5 and x^6 have the same value. Find the value of n.

(c) Prove by induction that $\cos(x + n\pi) = (-1)^n \cos x$ for integer $n \geq 1$.

QUESTION 7: (Start a new page)

(a) Ron put \$500 savings into a Bank for 2 years, where it earned interest at 6% p.a., paid twice a year. He then changed to a Credit Union and his money earned 8% p.a., paid quarterly. If he withdrew all his savings, and had \$633.75, how long was the money kept in the Credit Union?

(b) A boy throws a ball vertically and it just reaches a height of 40 metres. What is the greatest distance that he is able to throw it on a horizontal plane? (Let $g = 10\text{ms}^{-2}$)

END OF PAPER

Q1

(a) $\lim_{x \to 0} \frac{\sin 3x}{2x}$

$= \lim_{x\to 0} \frac{3}{2} \frac{\sin 3x}{3x}$

$= \frac{3}{2}$.

(b) $\log\left(\frac{x^3 y^2}{\sqrt{z}}\right)$

$= \log x^3 + \log y^2 - \log \sqrt{z}$

$= 3\log x + 2\log y - \frac{1}{2}\log z$

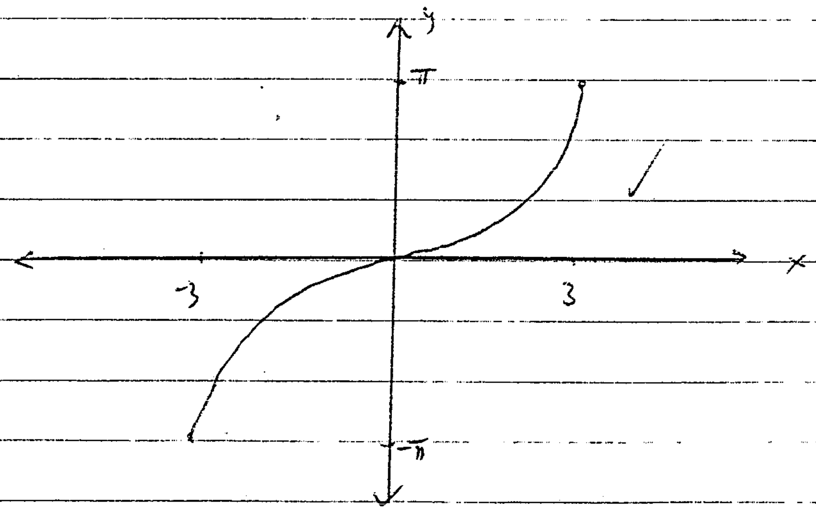
$= 3a + 2b - \frac{c}{2}$

(c) $y = 2\sin^{-1}\left(\frac{x}{3}\right)$

(i) Domain $-3 \le x \le 3$.

(ii) Range $-\pi \le y \le \pi$

(iii)



(d) $\sqrt{3}\sin\theta = \cos\theta$

$\therefore \tan\theta = \frac{1}{\sqrt{3}}$

$\theta = n\pi + \tan^{-1} b$.

$= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= n\pi + \frac{\pi}{6}$

Q2

(a) $y = 10^x$

$\therefore \ln y = x \ln 10$

$\therefore \frac{dx}{dy} = \frac{1}{\ln 10}\left[\frac{1}{y}\right]$

$\therefore \frac{dx}{dy} = \frac{1}{y \ln 10}$

$\frac{dy}{dx} = y \ln 10$

$\frac{dy}{dx} = 10^x \ln 10.$

$\therefore$ when $x = 1$ $\therefore \frac{dy}{dx} = 10 \ln 10.$

(b) $\int_0^1 \frac{dx}{x^2+3}$

$\therefore \left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\right]_0^1$

$\frac{1}{\sqrt{3}}\left[\left(\tan^{-1}\frac{1}{\sqrt{3}}\right) - \left(\tan^{-1} 0\right)\right]$

$\frac{1}{\sqrt{3}} \cdot \frac{\pi}{6}$

$\frac{\pi\sqrt{3}}{18}$

(c) 4T, 3P $\therefore \left({}^8C_4 \times {}^6C_3\right) + \left({}^8C_5 \times {}^6C_2\right) + \left({}^8C_6 \times {}^6C_1\right)$

5T, 2P $= 1400 + 840 + 168$

6T, 1P $= 2408.$

d) $y = \sin^{-1}(x^2).$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(f(x))^2}} \times f'(x)$

$= \frac{2x}{\sqrt{1-x^4}}$

$\therefore$ $u = 2x$ $\quad v = (1-x^4)^{\frac{1}{2}}$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = \frac{1}{2}(1-x^4)^{-\frac{1}{2}} \cdot -4x^3$

$$\therefore \frac{d^2y}{dx^2} = \frac{vu' - uv'}{v^2}$$

$$= \frac{2\sqrt{1-x^4} \quad \dfrac{+\overset{4}{\cancel{8}}x^4}{\sqrt{1-x^4}}}{1-x^4}$$

$$= \frac{2(1-x^4) \quad +\overset{4}{\cancel{8}}x^4}{\sqrt{(1-x^4)^3}}$$

$$= \frac{2\cancel{6}x^4 + 2}{\sqrt{(1-x^4)^3}} = \frac{2(x^4+1)}{\sqrt{(1-x^4)^3}}$$

Q3.

(a) (i) $p(0.85)$

$q(0.15)$

$n = 10$

$\therefore$ 0 or 1 failure.

$\therefore\ {}^{10}C_0 p^{10} q^0 + {}^{10}C_1 p^9 q$

$= (0.85)^{10} + 10 \cdot (0.85)^9 (0.15)$

$\doteqdot 0.54$ ✓

(ii) $\therefore P$ 2, 3, 4, 5, 6, 7, 8, 9, 10

$\therefore P \doteqdot 1 - P(0,1)$

$\doteqdot 1 - 0.54$

$\doteqdot 0.46$ ✓

(b). $\frac{dV}{dt} = K(A - V)$

(i) $V = A - Ae^{-kt}$

$\therefore \frac{dV}{dt} = KAe^{-kt}$

$= K(A - V)$ ✓

$\therefore V = A(1 - e^{-kt})$ is a solution.

(ii) $V = A - Ae^{-kt}$.

$\therefore$ as $t \to \infty$, $V \to A$ since $Ae^{-kt} \to 0$.

$\therefore$ Max capacity $= A$. ✓

(iii) when $t = 6$ $V = \frac{A}{8}$

$\therefore \frac{A}{8} = A(1 - e^{-kt})$

$\frac{1}{8} = 1 - e^{-6k}$

$e^{-6k} = \frac{7}{8}$

$-6k = \ln \frac{7}{8}$

$k = \frac{\ln(\frac{7}{8})}{-6}$ ✓

$k \doteqdot 0.022255$

(iv) $e^{-6k} = \frac{7}{8}$

$(e^{-6k})^2 = (\frac{7}{8})^2$

$e^{-12k} = \frac{49}{64}$ ✓

$\therefore$ fraction filled after 12 min $= 1 - \frac{49}{64} = \frac{15}{64}$

$\therefore$ fraction filled in the next 6 min

$= \frac{15}{64} - \frac{1}{8}$

$= \frac{7}{64}$. ✓

Q4.

(a) $V = e^{t} - e^{-t}$

(i) $\frac{dx}{dt} = e^{t} - e^{-t}$

$\therefore x = \int (e^{t} - e^{-t})\,dt$

$= e^{t} - (-e^{-t}) + c$

$x = e^{t} + e^{-t} + c.$

$\therefore$ when $t = 0$, $x = 0$

$\therefore 0 = 1 + 1 + c$

$-2 = c$

$\therefore x = e^{t} + e^{-t} - 2.$

(ii). $v = e^{t} - e^{-t}$

$\frac{dv}{dt} = e^{t} + e^{-t}$

when $t = 0$

$\ddot{x} = e^{0} + e^{-0}$

$\ddot{x} = 2\ ms^{-2}$

(iii) $V = e^{t} - e^{-t}.$

$\frac{dv}{dt} = e^{t} + e^{-t}.$

$\therefore$ Max when $\frac{dv}{dx} = 0$

$\therefore 0 = e^{t} + e^{-t}$

but $e^{t} + e^{-t} \neq 0$

$\therefore$ has no max velocity.

(iv) $x = e^{t} + e^{-t} - 2.$

$\therefore 3 = e^{t} + e^{-t} - 2$

$\therefore e^{t} + \frac{1}{e^{t}} - 5 = 0$

let $u = e^{t}$ $\therefore u + \frac{1}{u} - 5 = 0$

$u^{2} - 5u + 1 = 0$

$\therefore u = \frac{5 \pm \sqrt{25 - 4}}{2}$

$u = \frac{5 \pm \sqrt{21}}{2}$

$\therefore e^{t} = \frac{5 \pm \sqrt{21}}{2}$ $\therefore t = \ln\left(\frac{5 + \sqrt{21}}{2}\right)$

$= 1.6$ sec.

 $h = \frac{\frac{\pi}{4}}{4}$

1	2	3	4	5
0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$

(b) $V = \pi \int_a^b y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx.$

$V = \pi \left\{ \frac{h}{3} \left[f(a) + f(b) + 4\left(f\Sigma(\text{even})\right) + 2\left(\Sigma f(\text{odd})\right) \right] \right\}$

$V = \pi \left\{ \frac{\pi}{48} \left[\tan^2 0 + \tan^2 \frac{\pi}{4} + 4\left(\tan^2 \frac{\pi}{16} + \tan^2 \frac{3\pi}{16}\right) + 2\left(\tan^2 \frac{\pi}{8}\right) \right] \right\}$ ✓

$\doteqdot \frac{\pi^2}{48} \left[1 + 4\left((0.039566129) + (0.446462692)\right) + 2\left(\cancel{0.414213562}\ 0.171572875\right) \right]$

$\doteqdot \cancel{0.77561794}\ 0.696$

$\doteqdot \cancel{0.8}\ 0.7$ ✓

Q5

(a) $\frac{dv}{dt} = 100 \text{ cm}^3 s^{-1}$

$v = \frac{4}{3}\pi r^3$ $\quad$ $s = 4\pi r^2$

$\frac{dv}{dr} = 4\pi r^2$ $\quad$ $\frac{ds}{dr} = 8\pi r$

$\frac{dr}{dv} = \frac{1}{4\pi r^2}$

$\therefore \frac{dr}{dv} \times \frac{ds}{dr} \times \frac{dv}{dt} = \frac{ds}{dt}$

$\therefore \frac{ds}{dt} = \frac{1}{4\pi r^2} \times 8\pi r \times 100$

$= \frac{200}{r}$

$\therefore$ when $r = 6$

$\frac{ds}{dr} = 33\frac{1}{3} \text{ cm}^2 s^{-1}$.

(b) $y = \frac{x+1}{x^2+3}$ $\quad$ $y = 0$ at x int

$\therefore x+1 = 0$

$x = -1$

$\frac{dy}{dx} = \frac{(x^2+3) - (x+1)2x}{(x^2+3)^2}$

$= \frac{3-2x-x^2}{(x^2+3)^2}$

$\therefore$ when $x = -1$ $M = \frac{3+2-1}{16}$

$= \frac{1}{4}$

$\therefore y - 0 = \frac{1}{4}(x+1)$

$y = \frac{1}{4}(x+1)$.

(ii) Stationary when $y' = 0$

$$\therefore \; x^2 + 2x - 3 = 0$$

$$\therefore (x+3)(x-1) = 0$$

$$\therefore \; x = 1, -3. \qquad (1, \tfrac{1}{2}) \; (-3, -\tfrac{1}{6})$$

$\therefore$ tangent meets curve when

$$\frac{1}{4}(x+1) = \frac{(x+1)}{(x^2+3)}$$

$$(x^2+3)(x+1) = 4(x+1)$$

$$(x^2+3) = 4.$$

$$x^2 = 1$$

$$x = \pm 1.$$

$$\therefore \; (1, \tfrac{1}{2})$$

$\therefore$ tangent meets stationary point.

Q6

(a) $\int \cos^3 x \, dx$ $\quad u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$\therefore \int \cos^2 x . \cos x \, dx$

$\int (1 - \sin^2 x) \cos x \, dx$

$\int (1 - u^2) \, du$

$u - \frac{u^3}{3} + c$

$= \sin x - \frac{\sin^3 x}{3} + c.$

(b) $(3 + 2x)^n$

$\therefore \; {}^nC_0 3^n 2x^0 + {}^nC_1 3^{n-1} 2x^1 + {}^nC_2 3^{n-2} 2x^2 + {}^nC_3 3^{n-3} 2x^3 + {}^nC_4 3^{n-4} 2x^4 + {}^nC_5 3^{n-5} 2x^5 + {}^nC_6 3^{n-6} 2x^6 + \ldots$

$\ldots + {}^nC_n 3^0 2x^n$

$\therefore \ldots + {}^nC_5 3^{n-5} 2x^5 + {}^nC_6 3^{n-6} 2x^6 + \ldots$

$\therefore \; {}^nC_5 3^{n-5} 2^5 = {}^nC_6 3^{n-6} 2^6$

${}^nC_5 \, 3 = 2 \, {}^nC_6$

$\frac{3}{2} = \frac{n!}{(n-6)! \, 6!} \div \frac{n!}{(n-5)! \, 5!}$

$\therefore \frac{3}{2} = \frac{(n-5)! \, 5!}{(n-6)! \, 6!}$

$\frac{3}{2} = \frac{(n-5)\cancel{(n-6)!} \, \cancel{5!}}{\cancel{(n-6)!} \, 6 . \cancel{5!}}$

$9 = n - 5$

$14 = n.$

(i) $\cos(x + n\pi) = (-1)^n \cos x$. for $n \geq 1$.

prove true for $n=1$

$\therefore \cos(\pi + x) = (-1)\cos x$.

$\therefore \cos\pi \cos x - \sin\pi \sin x = -1\cos x$

$-1\cos x = -1 \cos x$

$\therefore$ true for $n=1$.

assume true for $n=k$

$\therefore \cos(x + k\pi) = (-1)^k \cos x$.

prove true for $n = k+1$

$\therefore \cos(x + (k+1)\pi) = (-1)^{k+1} \cos x$.

take LHS.

$\cos x \cos(k\pi + \pi) + \sin x \sin(k\pi + \pi)$.

$= \cos x(\cos k\pi \cos\pi - \sin\pi \sin k\pi) - \sin x(\sin k\pi \cos\pi + \sin\pi \cos$

$= \cos x((-1)^k \cdot -1)$

$= \cos x \; (-1)^{k+1}$

$= (-1)^{k+1} \cos x$

$= \text{RHS}$.

$\therefore$ Since true for $n=1$ it must be true for $n=2$. Since true for $n=2$ it must be true for $n=3$ and so on. Therefore true for all $n \geq 1$.

Q7.

(a) $P = 500$

$I = 6\%$ pa

$= 3\%$ for $\frac{1}{2}$ Year.

$\therefore A_1 = 500\left(1 + \frac{3}{100}\right)^4$

$= \$562.75.$

$\therefore \$633.75 = \$562.75(1 + (0.02))^{4n}$

$\frac{633.75}{562.75} = (1.02)^{4n}$

$\ln\left(\frac{633.75}{562.75}\right) = 4n \ln(1.02)$

$\frac{\ln\left[\frac{633.75}{562.75}\right]}{\ln(1.02)} = 4n$

$n = 1\frac{1}{2}$ year.

(b) $\ddot{y} = -g$

$\therefore \dot{y} = -gt + C$

$\therefore$ when $t = 0$ $u = v$

$\therefore \dot{y} = v - 10t.$

reaches max height when $\dot{y} = 0$

$\therefore 10t = u$

$t = \frac{u}{10}$

$y = \frac{-gt^2}{2} + ut + C_2$

when $t = 0$, $y = 0$

$\therefore C_2 = 0$

$\therefore y = -\frac{gt^2}{2} + ut + 0$

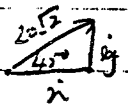
$\therefore$ when $t = \frac{v}{10}$, $y = 40$

$\therefore 40 = -\frac{v^2}{20} + \frac{v^2}{10}$

$v^2 = 800$

$v = 20\sqrt{2}.$

Max range when $\alpha = 45°$



$\therefore \frac{\dot{x}}{20\sqrt{2}} = \cos 45$

$\dot{x} = \frac{1}{\sqrt{2}} \times 20\sqrt{2}$

$= 20$

$\frac{\dot{y}}{20\sqrt{2}} = \sin 45$

$\dot{y} = 20.$

$\therefore \ddot{x} = 0$

$\dot{x} = \int 0 \, dt$

$= c_1$

when $t=0$, $\dot{x} = 20$

$\dot{x} = 20$

$x = \int 20 \, dt$

$= 20t + c_2$

when $t=0$ $x=0$.

$\therefore x = 20t.$

$\ddot{y} = -10$

$\dot{y} = \int -10 \, dt$

$= -10t + c$

when $t=0$ $\dot{y} = 20$

$\therefore \dot{y} = -10t + 20$

$y = \int -10t + 20 \, dt$

$y = -\frac{10t^2}{2} + 20t + c$

when $t=0$, $y=0$

$\therefore y = -5t^2 + 20t.$

$\therefore$ Max range when $y=0$

$\therefore 0 = -5t^2 + 20t$

$5t^2 - 20t = 0$

$5t(t-4) = 0$

$\therefore t = 4$

V. Good!

$\therefore x = 20 \times 4$

$x = 80$ m.