QUESTION 1 [12 Marks]

Marks

(a) Differentiate the following:

(i) $f(x)=\cos^{-1}2x$ — 1

(ii) $y=\ln(\tan^{-1}x)$ — 2

(b) Find $\int \cos^2 2x\,dx$ — 3

(c) Find $\lim_{x\to 0}\dfrac{x^2}{2-2\cos 2x}$ — 3

(d) $\int_0^1 \dfrac{dx}{x^2+3}=a\pi$ Find the exact value of a — 3

QUESTION 2 [12 Marks]

Marks

(a) (i) Graph accurately the curve $y=\dfrac{2}{x-1}$ — 3

(ii) Hence, solve $\dfrac{2}{x-1}\geq -1$ — 2

(b) The interval PQ has endpoints P(2,3) and Q(-3,5). Find the coordinates of the point T, which divides the interval PQ externally in the ration 3:1. — 2

(c) Find the general solution of $\tan 3\vartheta=1$ — 2

(d) A particle is moving in simple harmonic motion. Its displacement x at time t is given by $x=3\sin(2t-\frac{\pi}{4})$.

(i) Find the period of the motion. — 1

(ii) Find the velocity of the particle when t=0 — 2

QUESTION 3 [12 Marks]

Marks

(a) A particle is moving along the x-axis. Its velocity, v m/s at position x metres is given by — 2

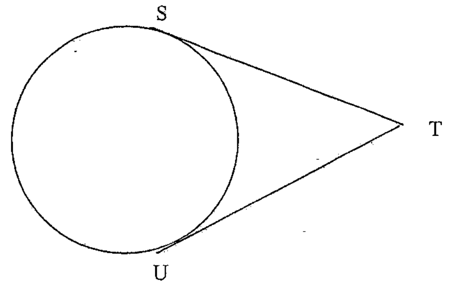
$$v=\sqrt{3x-x^2}$$

Find the acceleration of the particle when $x=5$

(b) $Q(x)=x^3+ax^2+2x+b$. Given that Q(x) has a factor of (x+3) and when Q(x) is divided by (x-1) the remainder is 4. — 3

Find the values of a and b.

(c)



S and U are points on a circle. The tangents to the circle at S and U meet at T. R are a point on the circle so that the chord SR is parallel to UT.

(i) Draw a neat sketch showing the given information. — 1

(ii) Prove that SU=UR — 3

(d) Find the ratio of the 5[th] term to the 8[th] term in the expansion $(2x+3)^{10}$ when $x=\dfrac{1}{2}$ — 3

QUESTION 4 [12 Marks]

Marks

(a) Using the substitution $x=1-u^2$, find $\int\dfrac{x\,dx}{\sqrt{1-x}}$ — 3

(b) Consider the function $f(x)=\dfrac{1}{2}\sin^{-1}x$.

(i) State the domain and range of the function. — 2

(ii) Find the area of the region bounded by the curve, the x-axis and the line x=1. — 3

(c) Show that the constant term in the expansion $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^9$ is $\dfrac{^9C_6}{6^3}$ — 4

QUESTION 5 [12Marks]

Marks

(a) Solve for $0\leq\theta\leq\pi$, $\cos\theta+3\sin\dfrac{\theta}{2}-2=0$ — 3

(b) Homer Simpson borrows \$15 000 at 11.95% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 monthly instalments of \$333.30 at the end of the month.

The amount A_n, of the loan remaining after n months is given by

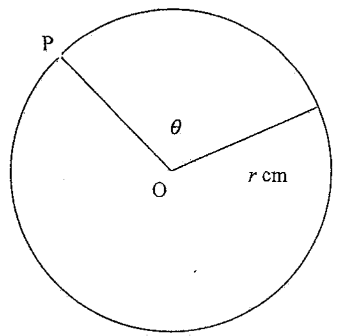
$A_n=MR^n-\$333.30(\dfrac{R^n-1}{R-1})$, where M is the principle amount borrowed.

(i) Find the exact value of R. — 1

(ii) After 2 years, Homer inherits \$1500 and wishes to pay this towards his loan. By how many months is the term of his loan reduced, by paying this extra amount? — 3

QUESTION 5 continued

(c)



A sector with centre O and radius r cm, is bounded by radii OP and OQ and arc PQ. $\angle POQ$ is θ radians.

(i) Given that r and θ vary in such a way that the area of the sector POQ is always equal to 50 cm^2, show that $\theta = \dfrac{100}{r^2}$. **2**

(ii) Given also that the radius is increasing at a constant rate of 0.5cm/s, finds the rate at which the angle POQ is decreasing when r=10 cm. **3**

QUESTION 6 [12 Marks]

Marks

(a) (i) Find the equation of the tangent to the parabola $x = 2at$, $y = at^2$ at the point P where $t = p$. **1**

(ii) If the point Q is the point where $t = q$, and O is the origin, show that if OQ is parallel to the tangent, then $q = 2p$. **1**

(iii) If M is the midpoint of PQ, find the equation of the locus of M as P and Q vary long the parabola such that OQ remains parallel to the tangent at P. **4**

(b) Using the principles of mathematical induction, prove that $\ln[(n+2)!] > n+2$, for $n \geq 4$ **4**

(c) At a referendum, 30% of parents were in favour of a new uniform logo. An SRC member approached 8 parents chosen at random. **2**

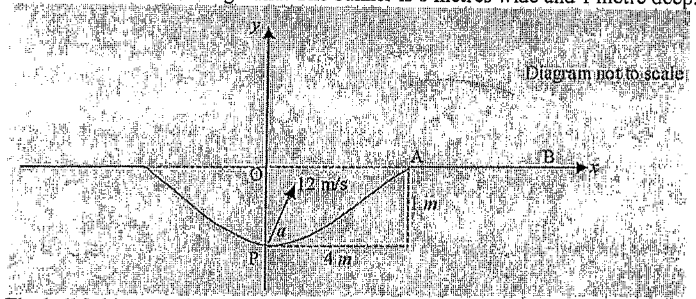
Find the probability that from this group, exactly 3 parents voted in favour of the logo.

Question 7 [12 Marks]

Marks

(a) All the letters of the word **ENCUMBRANCE** are arranged in a line. Find the total number of arrangements, which contain all the vowels in alphabetical order but separated by at least one consonant. **4**

(b) A golf ball is lying at point P, at the middle of a sand bunker, which is surrounded by level ground. The point A is at the edge of the bunker and the line AB lies on the level ground. The bunker is 8 metres wide and 1 metre deep.



The ball is hit towards A with an initial speed of 12m/s and angle of elevation α. (You may assume that the acceleration due to gravity is 10m/s^2)

The golf ball's trajectory at time t seconds after being hit may be defined by the equations $x = (12\cos\alpha)t$ and $y = -5t^2 + (12\sin\alpha)t - 1$ where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O, shown in the diagram.

(i) If $\alpha = 30°$, how far to the right of A will the ball land? (Give your answer correct to 0.1m) **4**

(ii) Find the range of values of α, to the nearest degree, at which the ball must hit so that it will land to the right of A **4**

James Ruse 2003 Danielay

(a)(i) $f(x) = \cos^{-1} 2x$

$f'(x) = \frac{-2}{\sqrt{1-4x^2}}$ ✓

(ii) $f(x) = \ln(\tan^{-1} x)$

$f'(x) = \frac{\frac{1}{1+x^2}}{\tan^{-1} x}$ ✓

$= \frac{1}{\tan^{-1} x(1+x^2)}$ ✓

(b) $\int \cos^2 2x \, dx$

$\cos^2 x = 1 + \cos 2x = \cos 2x$
$2\cos^2 x = 1 + \cos 2x$
$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{1}{2} \int 1 + \cos 4x \, dx$

$= \frac{1}{2}[x + \frac{\sin 4x}{4}] + c$ ✓

$= \frac{x}{2} + \frac{\sin 4x}{8} + c.$

(c). $\lim_{x \to 0} \frac{x^2}{2 - 2\cos 2x} = \frac{x^2}{2(1-\cos 2x)}$ ✓

$\lim_{x \to 0} = \frac{x^2}{2(1-(\cos^2 x - \sin^2 x))}$

$\lim_{x \to 0} = \frac{x^2}{2(1-(1-2\sin^2 x))}$ ✓

$\lim_{x \to 0} = \frac{x^2}{4\sin^2 x}$

$\lim_{x \to 0} = \frac{1}{4} \times \frac{x}{\sin x} \times \frac{x}{\sin x}$ ✓

$= \frac{1}{4}.$

(d). $\int_0^1 \frac{dx}{x^2+3} = a\pi$

$I = \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$ ✓

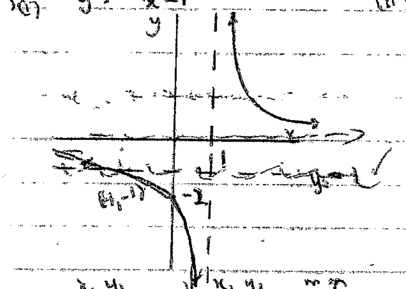
$= \frac{1}{\sqrt{3}} \times \frac{\pi}{6}$

$= \frac{\pi}{6\sqrt{3}}$

$\therefore a = \frac{1}{6\sqrt{3}}$ ✓

Question 2.

(a)(i) $y = \frac{2}{x-1}$



(ii). $\frac{2}{x-1} = -1$

$2 = -x + 1$ ✓

$x = -1$

$\therefore x \le -1 \cap x > 1$ ✓

(b). $P(3, 3)$ $Q(-3, 5)$ $3:1$ Externally

$x: \frac{-9-2}{2}$ ✓ $y: \frac{9-5}{2}$ ✓

$= -11/2$ ✓ $= 2.$

$T: (-11/2, 2)$

(c). $\tan 3\theta = 1$

$3\theta = -3\pi/4, \frac{\pi}{4}, \frac{5\pi}{4},$ $\Longrightarrow n\pi + \pi/4$ ✓

$\theta = -\frac{3\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12},$

$\theta = \frac{1}{3}[n\pi + \pi/4]$ ✓

(d). $x = 3\sin(2t - \pi/4)$, $n = 2.$

(i) $T = \frac{2\pi}{2}$ ✓

$= \pi$

(ii) ~~$v^2 = n^2(a^2 - x^2)$~~

~~$v^2 = 4(9 - x^2)$~~

~~at $t = 0$, $x = 3\sin(-\pi/4)$~~

~~$= -3/\sqrt{2}$~~

~~at $x = \frac{-3}{\sqrt{2}}$~~

~~$v^2 = 4(9 - \frac{9}{2})$~~

~~$v^2 = 18$~~

~~$v =$~~

$x = 3\sin(2t - \pi/4)$

$\dot{x} = \frac{dx}{dt} = 6\cos(2t - \pi/4)$

at $t = 0$.

$v = 6\cos(-\pi/4)$

$= \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= 3\sqrt{2}$ ✓

Question 3.

i) $v = \sqrt{3x - x^2}$

$v^2 = 3x - x^2$

$\frac{v^2}{2} = \frac{3x - x^2}{2}$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = \ddot{x} = \frac{1}{2}(3 - 2x)$ ✓

at $x = 5$

$\ddot{x} = \frac{-7}{2}$ ✓

ii). $Q(x) = x^3 + ax^2 + 2x + b$.

$Q(-3) = 0. = -27 + 9a - 6 + b = 0.$

$-33 + 9a + b = 0.$ ✓ —— ①

$Q(1) = 1 + a + 2 + b = 4$

$a + b = 1$ —— ②

$b = 1 - a.$ ✓

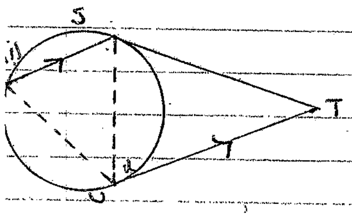
$-33 + 9a + 1 - a = 0.$

$-32 + 8a = 0.$

$8a = 32$

$a = 4.$ ✓

$\therefore b = -3.$



(ii) Let $S\hat{U}T = \alpha$ ✓

$U\hat{S}R = \alpha$ (alternate ∠'s in ∥) ✓

$S\hat{R}U = \alpha$ (∠'s in the alternate segment) ✓

$\therefore U\hat{S}R = S\hat{R}U$. ∴ △RSU is isosceles

$\therefore SU = UR.$ ✓

$(2x+3)^{10}$

$\frac{8^{th}\ \text{Term}}{5^{th}\ \text{Term}} = \frac{T_8}{T_5} = \frac{\binom{10}{7} 2x^3\, 3^{7}}{\binom{10}{4} 2x^{6}\, 3^{4}}$

$= \frac{10! \times 3^3}{7!\,3!} \times \frac{6! \times 4!}{10!\,(2x)^3}$

$=$ when $x = 1/2$

$= \frac{3^3 \times 6! \times 4!}{7!\,3! \times 1}$

$= 15\frac{3}{7} = \frac{7}{108}$

Question 4.

$\int \frac{x\,dx}{\sqrt{1-x}}$

Let $x = 1 - u^2 \implies u^2 = 1 - x$, $u = \sqrt{1-x}$

$\frac{dx}{du} = -2u$ ✓ $\implies dx = -2u\,du$

~~$\frac{du}{dx} = \frac{1}{-2u} = \frac{-1}{2\sqrt{1-x}}$~~

$I = \int \frac{1 - u^2}{u} \cdot -2u\,du$

$= -2\int 1 - u^2\,du$ ✓

$= -2\left[u - \frac{u^3}{3}\right] + c$

$= -2\sqrt{1-x} - \frac{2(1-x)\sqrt{1-x}}{3} + c$ ✓

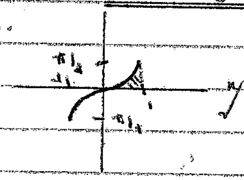
(i) $f(x) = \frac{1}{2}\sin^{-1}x$

D: $-1 \le x \le 1$ ✓

R: $-\pi/2 \le 2y \le \pi/2$

$-\pi/4 \le y \le \pi/4.$ ✓

(ii).



$y = \frac{1}{2}\sin^{-1}x$

$2y = \sin^{-1}x$

$\sin 2y = x$

$A = \pi/4 - \int_0^{\pi/4} \sin 2y\,dy$ ✓

$= \pi/4 + \left[\frac{\cos 2y}{2}\right]_0^{\pi/4}$

$= \pi/4 + (0 - 1/2)$

$= \pi/4 - 1/2.$ ✓

$\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

Powers of x

$T_1 = x^{18}$

$T_2 = x^{16} \cdot x^{-1}$ ✓

$= x^{15}$

$\therefore T_7 = x^{6} \cdot x^{-6}$

$= 0.$

$\therefore T_7 = \binom{9}{6}\left(\frac{3x^2}{2}\right)^3 \left(\frac{1}{3x}\right)^6$ ✓

$= \binom{9}{6} \frac{27x^6}{8} \times \frac{1}{729x^6}$

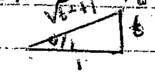
$= \binom{9}{6} \times \frac{1}{216}$

$= \frac{\binom{9}{6}}{6^3}$

Question 5

$3\cos\theta + 3\sin\frac{\theta}{2} - 2 = 0$.

Let $\frac{t}{\sqrt{t^2+1}}$ = $\tan\theta/2$



$\sin\theta/2 = \frac{t}{\sqrt{t^2+1}}$

$\cos\theta = 2\cos^2\theta/2 - 1$

$= 2\left(\frac{1}{t^2+1}\right) - 1$

$= \frac{2}{t^2+1} - 1$

$\therefore \cos\theta + 3\sin\theta/2 - 2$

Using $\cos\theta = \cos(2.\frac{\theta}{2})$

$= 1 - 2\sin^2(\frac{\theta}{2})$

$\therefore 1 - 2\sin^2(\frac{\theta}{2}) + 3\sin(\frac{\theta}{2}) - 2 = 0$

$\therefore 2\sin^2(\frac{\theta}{2}) - 3\sin(\frac{\theta}{2}) + 1 = 0$

$\therefore 2u^2 - 3u + 1 = 0$ where $u = \sin\frac{\theta}{2}$

$\therefore u = \frac{3 \pm \sqrt{9-8}}{4}$

$= \frac{3 \pm 1}{4}$

$\therefore \sin\frac{\theta}{2} = 1$ or $\frac{1}{2}$ but $0 \le \theta \le \pi$

$\therefore 0 \le \frac{\theta}{2} \le \frac{\pi}{2}$

$\therefore \frac{\theta}{2} = \frac{\pi}{2}$ or $\frac{\theta}{2} = \frac{\pi}{6}$

$\theta = \frac{\pi}{4}$ or $\frac{\pi}{3}$

(5)(b)(i) $R = 1 + \frac{1195}{1200}$

$= 1\frac{239}{240}$

(ii) When $n = 24$

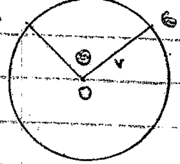
$A_{24} = 15000\left(1\frac{239}{240}\right)$

(5)(b)(i) $R = 1 + \frac{0.1195}{12}$

$= 1.0099583333$

Questions 5 (c)

(i)



Area of sector $= \frac{1}{2} r^2 \theta = 50$. ✓

$r^2\theta = 100$

$\theta = \frac{100}{r^2}$ ✓ $\Rightarrow 100 r^{-2}$

(ii). $\frac{dr}{dt} = 0.5$

$\frac{d\theta}{dr} = -2 \times 100 r^{-3}$

$= \frac{-200}{r^3}$

$\frac{d\theta}{dt} = \frac{dr}{dt} \times \frac{d\theta}{dr}$

$= \frac{1}{2} \times -\frac{200}{r^3}$

$= -\frac{100}{r^3}$ ✓

$= -\frac{100}{r^3}$

At $r = 10$

$\frac{d\theta}{dt} = \frac{-100}{1000}$ ✓

$= \frac{-1}{10}$

∴ Decreasing at a rate of $\frac{1}{10}$ degrees per second.

Question 6.

(a) $x = 2at,\ y = at^2$, at $t = p$, $x = 2ap$, $y = ap^2$

$\frac{x}{2a} = t$

$y = a\left(\frac{x}{2a}\right)^2$

$= \frac{ax^2}{4a^2}$

$y = \frac{x^2}{4a}$ ✓

$\frac{dy}{dx} = \frac{2x}{4a}$

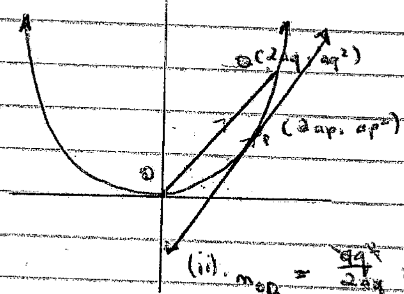
at $x = 2ap$.

$m = \frac{2ap}{2a} = p$. ✓

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$ ✓

$px - y + ap^2 = 0$.



(ii). $m_{OQ} = \frac{aq^2}{2aq} = p$ (since $OQ \parallel t$)

$\frac{q}{2} = p$

$q = 2p$. ✓

(iii). ~~Gradient of OQ = p.~~

~~Eqn: $y - 0 = p(x - 0)$~~ ~~$y - aq^2 = p(x - 2aq)$~~

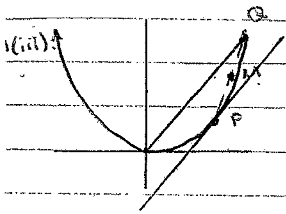
~~$y = px$~~ ~~$y - aq^2 = px - 2apq$~~

~~but $y = aq^2$, $x = 2ap$~~ ~~$y - px = 2apq + aq^2$~~

~~$ap^2 - px + ap^2 = 0$.~~ ~~$px + ap^2 - px = 2apq + aq^2$~~

~~$ap^2 = 0$.~~ ~~$p^2 - q^2 + 2pq = 0$.~~

(iii)



Midpoint PQ: $x = \frac{2ap + 2aq}{2}$ $\quad y = \frac{ap^2 + aq^2}{2}$

$= a(p+q)$ $\quad = \frac{a}{2}[p^2 + q^2]$ ✓

But $q = 2p$

$x = a(p + 2p)$

$x = 3ap$ ✓

$\frac{x}{3a} = p$.

$y = \frac{a}{2}[4p^2 + p^2]$

$= \frac{a}{2}(5p^2)$

$= \frac{5a}{2}\left(\frac{x}{3a}\right)^2$

$= \frac{5}{2} \times \frac{x^2}{9a}$

$y = \frac{5x^2}{18a}$ ✓

$18ay = 5x^2$

$5x^2 - 18ay = 0$. ✓

~~(b). $\ln[(n+2)!] > n+2$, for $n \geq 4$.~~

~~Test for $n = 4$~~

~~LHS $= \ln[6!]$ RHS $= 6$.~~

~~$= 6.579$~~

~~∴ LHS > RHS, ∴ True for $n = 4$.~~

~~Assume true for $n = k$~~

~~$\ln[(k+2)!] - k - 2 > 0$.~~

~~Prove true for $n = k+1$~~

~~LHS $= \ln[(k+3)!] - k - 3 > 0$.~~

(b) $(n+2)! > e^{n+2}$ for $n \geq 4$. ✓

Test for $n = 4$.

$(n+2)! - e^{n+2} > 0$.

$6! - e^6 > 0$ True for $n = 4$.

Assume true for $n = k$

$(k+2)! - e^{k+2} > 0$.

Prove true for $n = k+1$

$(k+3)! - e^{k+3} > 0$.

$= (k+2)!(k+3) - e^{k+2} \cdot e^1 > 0$.

But $(k+2)! - e^{k+2} > 0$ from assumption

∴ Need to prove $(k+3) - e > 0$.

$k + 3 - e > 0$.

✓ $k > 3 - e$

$k > 0.28$

∴ Since true for $n = 4$, then it true for $n = k$ and if true for $n = k+1$, then by the Principles of Mathematical Induction, true for $n \geq 4$.

Question 6.

i): 30% YES.

$p = 3/10$, $q = 7/10$

$(3/10 + 7/10)^8$

$T_4 = \binom{8}{3}(\frac{3}{10})^3(\frac{7}{10})^5$

$= 0.25412\%$

Question 7.

i). ENCUMBRANCE 2 C's, 2 N's, 2 E's

$\frac{11!}{2!2!2!} - 7!6$ — All these below.

Case 1:

A x E x E x U x x x x

Ans $= \frac{7!}{2! \times 2!}$

$= 1260.$

Try using its complement i.e. AEEU x x x x x x x

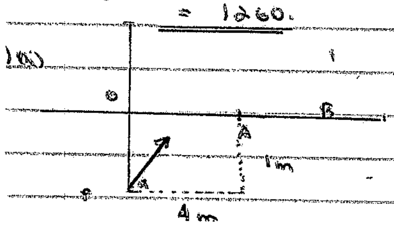
A x EEU x x x x x x or AxExEU... or AxEExU...

AExEU x x x x x x or AExExU...

AEExU x x x x x x

$7! \times 6$

ii)(i)



$x = 12t\cos\alpha$

$\frac{x}{12\cos\alpha} = t$

$y = -5t^2 + (12\sin\alpha)t - 1$

$= -5\left(\frac{x^2}{144\cos^2\alpha}\right) + \frac{x\sin\alpha}{\cos\alpha} - 1$

$y = \frac{-5x^2}{144\cos^2\alpha} + x\tan\alpha - 1$

when $\alpha = 30$ and $y = 0$.

$0 = \frac{-5x^2}{144 \times 3} \times 4 + \frac{x}{\sqrt{3}} - 1$

$0 = \frac{-5x^2}{108} + \frac{x}{\sqrt{3}} - 1$

$-5x^2 + \frac{108}{\sqrt{3}}x - 108 = 0.$

$-5\sqrt{3}x^2 + 108x - 108\sqrt{3} = 0.$

$x = \frac{-108 \pm \sqrt{11664 - 4 \times 108 \times 3 \times 5}}{-10\sqrt{3}}$

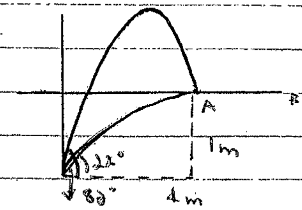
$= \frac{108 \pm 72}{10\sqrt{3}}$

$= 10.3923$ or $2.0784.$

$\therefore$ 6.4m to the right of A

b(ii). $y = \frac{-5x^2}{144}\sec^2\alpha + x\tan\alpha - 1$

to hit A, $y = 0$, $x = 4$.



$0 = \frac{-5}{9}\sec^2\alpha + 4\tan\alpha - 1$

$= \frac{-5}{9}(\tan^2\alpha + 1) + 4\tan\alpha - 1 = 0.$

$= \frac{-5}{9}\tan^2\alpha + 4\tan\alpha - \frac{14}{9} = 0.$

$5\tan^2\alpha - 36\tan\alpha + 14 = 0.$

$\tan\alpha = \frac{36 \pm \sqrt{1296 - 5 \times 14 \times 4}}{10}$

$= 6.787$ or 0.4125

$\therefore \alpha = 81°37'$ or $22°25'$

$= 82°$ or $22°$ to hit A.

$\therefore \alpha$ needs to be such that $22° \leq \alpha \leq 82°$