JAMES RUSE AHS
MATH. EXT 1 TRIAL, 2008

Question 1. **Marks**

(a) Find $\lim_{x \to 0} \frac{3x}{\tan 5x}$. **2**

(b) Find the obtuse angle between the lines $x - y - 1 = 0$ and $2x + y - 1 = 0$. **2**

(c) Find the general solution to $\sin\theta = \frac{\sqrt{3}}{2}$. **2**

(d) When the polynomial function $f(x)$ is divided by $x^2 - 16$, the remainder is $3x - 1$. What is the remainder when $f(x)$ is divided by $x - 4$? **2**

(e) Solve for x: $\frac{1-2x}{1+x} \geq 1$. **3**

(f) Find a primitive of $\frac{1}{\sqrt{x^2 - 9}}$. **1**

Question 2. **[START A NEW PAGE]**

(a) Given the function $g(x) = \sqrt{x+2}$ and that $g^{-1}(x)$ is the inverse function of $g(x)$, find $g^{-1}(5)$. **2**

(b) (i) Show that: $\frac{2\tan x}{1+\tan^2 x} = \sin 2x$. **1**

(ii) Hence, or otherwise, find $\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\tan^2 x}\,dx$. **2**

(c) Using the substitution $u = \sqrt{1+x}$, evaluate $\int_0^3 \frac{5x^2 + 10x}{\sqrt{1+x}}\,dx$. **4**

(d) Sketch the graph of the curve: $y = 2\cos^{-1}(x) - 1$, showing all essential information. **3**

Question 3. **[START A NEW PAGE]** **Marks**

(a) Find the exact value of $\tan\left(2\cos^{-1}\frac{12}{13}\right)$. **2**

(b) Let point $P(4p, 2p^2)$ be an arbitrary point on the parabola $x^2 = 8y$ with parameter p.

(i) Show that the equation of the tangent at P is $y = px - 2p^2$. **1**

(ii) The tangent intersects the y-axis at C.
The point Q divides CP, internally, in the ratio $1:3$.
Find the locus of all the Q points as parameter p varies. **3**

(c) The velocity v ms^{-1} of a particle moving in a straight line at position x at time t seconds is given by: $v = x^3 - x$.
Find the acceleration of the particle at any position. **2**

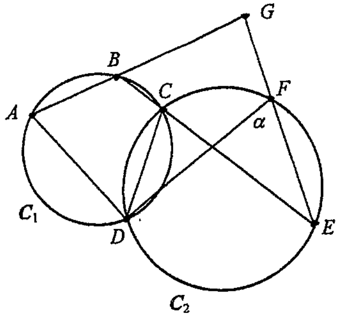
(d) The numbers 1447, 1005 and 1231 all have something in common.
Each is a four-digit number beginning with 1 that has exactly two identical digits
How many such four-digit numbers exist? **2**

(e) Find $\int \cos^2\left(\frac{x}{2}\right) dx$. **2**

Question 4. **[START A NEW PAGE]** **Marks**

(a) Find the term independent of x in the expansion of $\left(2x^2 - \frac{3}{x}\right)^9$. **2**

(b) **3**



Two circles C_1 and C_2 intersect at C and D.
BC produced meets circle C_2 at E.
AB produced meets EF produced at G

Let $\angle DFE = \alpha$.

Copy or trace the diagram onto your writing booklet and prove that $ADFG$ is a cyclic quadrilateral.

(c) A bag contains eleven balls, numbered 1, 2, 3, … and 11.
If six balls are drawn simultaneously at random,

(i) How many ways can the sum of the numbers on the balls drawn be odd? **2**

(ii) What is the probability that the sum of the numbers on the balls drawn is odd? **1**

(d) When Farmer Browne retired he decided to invest \$2 000 in a fund which paid interest of 8% *pa*, compounded annually.
From this fund he decided to donate a yearly prize of \$200 to be awarded to the Dux of Agriculture in Year 12. The prize money being withdrawn from this fund after the year's interest had been added.

(i) Show that the balance $\$B_n$ remaining after n prizes have been awarded will be: $B_n = 500\left(5 - 1\cdot08^n\right)$ **3**

(ii) Calculate the number of years that the \$200 prize can be awarded. **1**

Question 5. **[START A NEW PAGE]** **Marks**

(a) Considering the expansion:

$$(9+5x)^{29} = p_0 + p_1x + p_2x^2 + \ldots + p_kx^k + \ldots + p_nx^n.$$

(i) Use the Binomial theorem to write the expression for p_k. **1**

(ii) Show that: $\frac{p_{k+1}}{p_k} = \frac{5(29-k)}{9(k+1)}$. **2**

(ii) Hence, or otherwise, find the largest coefficient in the expansion. **2**
[you may leave your answer in the form: $\binom{29}{r}3^a5^b$].

(b) An ice cube tray is filled with water which is at a temperature of $20^0\,C$ and placed in a freezer that is at a constant temperature of -15^0C.
The cooling rate of the water is proportional to the difference between the temperature of the water W^0C, a and the freezer temperature so that W satisfies the rate equation:
$\frac{dW}{dt} = -k(W+15)$, where k is the rate constant of proportionality.

(i) Show that: $\frac{d}{dt}\left(We^{kt}\right) = -15ke^{kt}$. **2**

(ii) Hence, show that: $W = 35e^{-kt} - 15$. **2**

(iii) After 5 minutes in the freezer, the temperature of the water cubes is 6^0C.

1. Find the rate of cooling at this time (correct to 1 decimal place) **2**

2. Find the time for the water cubes to reach -10^0C (correct to the nearest minute). **1**

Question 6. **[START A NEW PAGE]** **Marks**

(a) A ball is projected from a point O on horizontal ground in a room of length $2R$ metres with an initial speed of U ms^{-1} at an angle of projection of α. There is no air resistance and the acceleration due to gravity is g ms^{-2}.

(i) Assuming after t seconds the ball's horizontal distance x metres, is given by: $x = Ut\cos\alpha$, and the vertical component of motion is $\ddot{y} = -g$, show that the vertical displacement y of the ball is given by: **2**

$$y = Ut\sin\alpha - \frac{1}{2}gt^2.$$

(ii) Hence show that the range R metres for this ball is given by: **2**

$$R = \frac{U^2\sin 2\alpha}{g}.$$

(iii) Suppose that the room has a height of $3\cdot5$ metres and the angle of projection is fixed for $0 < \alpha < \frac{\pi}{2}$ but the speed of projection U varies.

Prove that:

(α) the maximum range will occur when $U^2 = 7g\,\mathrm{cosec}^2\alpha$. **2**

(β) the maximum range would be $14\cot\alpha$. **1**

(b) Given the polynomial function:

$$f_n(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)\ldots(x+n-1)}{n!}, \text{ for } n = 1, 2, 3, \ldots$$

where for $n = 1$: $f_1(x) = 1 + \frac{x}{1!} = x + 1$ which has a zero at -1.

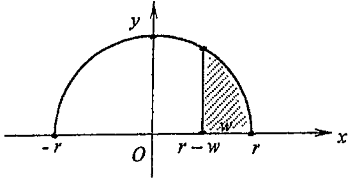
(i) Show that for $n = 2$: $f_2(x) = \frac{1}{2!}(x+1)(x+2)$ and state the zeros of $f_2(x)$. **2**

(ii) Hence **complete** the proof by mathematical induction that the zeros of the polynomial function $f_n(x)$ are $-1, -2, -3, \ldots$ and $-n$ for $n = 1, 2, 3, \ldots$, that is **3**

prove that: $f_n(x) = \frac{1}{n!}(x+1)(x+2)(x+3)\ldots(x+n)$, for $n = 1, 2, 3, \ldots$.

Question 7. **[START A NEW PAGE]** **Marks**

(a) Given the semi-circle equation: $y = \sqrt{r^2 - x^2}$, **2**

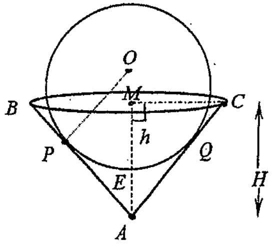


The shaded area of thickness w is rotated about the x-axis to form the volume of a 'cap'.

Show that the volume of the solid of revolution V is given by:

$$V = \frac{\pi}{3}(3r - w)w^2.$$

(b) An inverted cone ABC of height H units with a base radius of R units is filled with water.
A sphere of radius r units is inserted into the inverted cone so as to touch the inner walls of the cone at P & Q to a depth of h units, as shown below.



Not to scale

Given:
$MB = MC = R$, $MA = H$, $AC = L$,
$OP = r$ and $ME = h$.

(i) Show that: $r = \frac{(H-h)R}{L-R}$, where $L = \sqrt{H^2 + R^2}$. **2**

(ii) Hence show that the volume of water V cubic units displaced by the sphere is given by: **1**

$$V = \frac{\pi}{3(L-R)}\left[3RHh^2 - (L+2R)h^3\right].$$

(iii) Hence, or otherwise find the radius of the sphere that displaced the maximum volume of water under the above conditions. **4**

(c) (i) Write down the binomial expansion of $(1-x)^{2n}$ in ascending powers of x. **1**

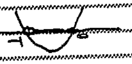
(ii) Hence show that: **2**

$$\binom{2n}{1} + 3\binom{2n}{3} + \ldots + (2n-1)\binom{2n}{2n-1} = 2\binom{2n}{2} + 4\binom{2n}{4} + \ldots + 2n\binom{2n}{2n}.$$

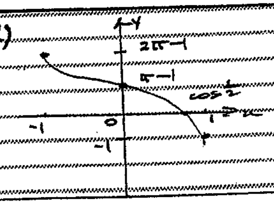
THE END ☺ 😐 ☹ 👍

MATHS EXTENSION 1 TRIAL, 2008

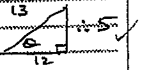
MATHEMATICS Extension 1 : Question 1

Suggested Solutions	Marks	Marker's Comments
Q1(a) $\lim_{x\to0}\frac{3x}{\tan 5x} = \lim_{x\to0}\frac{3x(5x)}{\tan(5x)\times5}$ ✓ $= \frac{3}{5}\lim_{x\to0}\frac{5x}{\tan 5x}$ $= \frac{3}{5}\times1$ ✓ $\boxed{2}$ $= \frac{3}{5}$	1 1	
(b) $x-y-1=0$ $m_1=1$ $2x+y-1=0$ $m_2=-2$ $\boxed{2}$ $\tan\theta = \frac{-2-1}{1+-2\times1} = \frac{-3}{1-2} = \frac{-3}{-1}$ ∴ $\tan\theta = 3$ ∴ obtuse angle $= 180^\circ - \tan^{-1}3$ ✓ $= 108^\circ26'$	1 1	or $\tan^{-1}(3)$
(c) $\sin\theta = \frac{\sqrt{3}}{2}$ $\boxed{2}$ $\theta = n\pi + (-1)^n\sin^{-1}\frac{\sqrt{3}}{2}$ ✓ $\theta = n\pi+(-1)^n\frac{\pi}{3}$ where ✓ $n\in\mathbb{Z}$	1 1	or $\theta = \begin{cases}\frac{\pi}{3}+2n\pi\\ \pi-\frac{\pi}{3}+2n\pi\end{cases}$ Acc $\theta = 180^\circ n + (-1)^n.60^\circ$
(d) $f(x) = (x^2-16)Q(x) + 3x-1$ $\boxed{2}$ Rem $= f(4) = 0 + 3\times4-1$ $= 11$ ✓	1 1	
(e) $\frac{1-2x}{1+x}\geq1$ $\boxed{3}$ $\frac{1-2x-1(1+x)}{1+x}\geq0$ $\frac{-3x}{1+x}\geq0$ $\frac{3x}{1+x}\leq0$ Now $x\neq-1$ so $3x(1+x)\leq0$  ⇒ $-1<x\leq0$	1 1+1	
(f) Primitive $\ln[x+\sqrt{x^2-9}]$ $(+c)$ ✓ $\boxed{1}$		1 For $\ln[x+\sqrt{x^2-9}]$

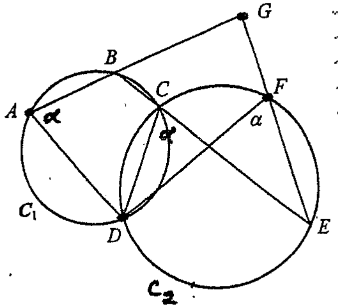
MATHEMATICS Extension 1 : Question 2

Suggested Solutions	Marks	Marker's Comments
Q2(a) $g(x) = \sqrt{x+2}$ $\boxed{2}$ $g^{-1}(5)$ is $g(x) = 5$ ∴ $5 = \sqrt{x+2}$ ∴ $x = 23$	1 1	$g^{-1}(x) = x^2-2$
(b) (i) $\frac{2\tan x}{1+\tan^2x} = \frac{2\frac{\sin x}{\cos x}}{\sec^2x} = \frac{2\sin x\times\cos^2x}{\cos x}$ ✓ $\boxed{1}$ $= 2\sin x\cos x$ $= \sin 2x$ qed.	1	
(ii) $\int_0^{\pi/4}\frac{\tan x}{1+\tan^2x}\,dx = \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx$ $\boxed{2}$ $= -\frac{1}{4}\Big[\cos 2x\Big]_0^{\pi/4}$ $= -\frac{1}{4}\left[\cos\frac{\pi}{2} - \cos 0\right]$ $= -\frac{1}{4}[0-1]$ $= \frac{1}{4}$	1 1	
(c) $I = \int_0^3\frac{5x^2+10x}{\sqrt{1+x}}\,dx$ $\boxed{4}$ $u = \sqrt{1+x}$ $u^2 = 1+x$ $2u\,du = dx$ ✓ $\begin{array}{c\|c} x & u \\ \hline 3 & 2 \\ 0 & 1 \end{array}$ ✓ ∴ $I = 5\int_1^2\frac{(u^2-1)^2+2(u^2-1)}{\cancel{u}}\times2\cancel{u}\,du$ $x = u^2-1$ $= 10\int_1^2 u^4-2u^2+1+2u^2-2\,du$ ✓ $= 10\int_1^2 u^4-1\,du$ $= 10\left[\frac{1}{5}u^5-u\right]_1^2 = 10\left[\left(\frac{32}{5}-2\right)-\left(\frac{1}{5}-1\right)\right]$ $= 10\left(\frac{31}{5}-1\right) = 52$ ✓	1 1 1 1	
(d) NTS  $y = 2\cos^{-1}x - 1$ $\boxed{3}$		1 For x-int $\cos\frac{1}{2} \doteqdot 0.88$ 1 For $2\pi-1$ and/or -1 $\frac{1}{2}$ For $\pi-1$ $\frac{1}{2}$ For shape

MATHEMATICS Extension 1 : Question 3

Suggested Solutions	Marks	Marker's Comments
Q 3(a) $\tan\left(2\cos^{-1}\frac{12}{13}\right)$  Let $\theta = \cos^{-1}\frac{12}{13} \Rightarrow \cos\theta = \frac{12}{13}$ ∴ 5 ✓ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{5}{12}}{1-\frac{25}{144}}$ ✓ $\boxed{2}$ $= \frac{2\times5\times12}{144-25} = \frac{120}{119}$.		$\frac{1}{2}$ For $\cos\theta = \frac{12}{13}$ $\frac{1}{2}$ For $\tan\theta = \frac{5}{12}$ 1 For $\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}$ or $\frac{120}{119}$
(b) (i) $x^2 = 8y$ $\therefore y = \frac{x^2}{8}$ $P(4p, 2p^2)$ ① $\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$ Gradient of tangent at P : $m_T = \frac{4p}{4} = p$ Equ. of tangent at P: $y - 2p^2 = p(x-4p)$ ✓ $y - 2p^2 = px - 4p^2$ $\therefore y = px - 2p^2$.	1	1 For getting to ✓
(ii) $C = (0, -2p^2)$ For Q $(0, -2p^2)$ 1:3 $P(4p, 2p^2)$ $Q = \left(\frac{4p+0}{4}, \frac{2p^2-6p^2}{4}\right) = (p, -p^2)$ ✓ ③ Let $Q(x,y)$ be the general point on the required locus $\therefore x = p$ — (1) $y = -p^2$ — (2) (1) $\Rightarrow p = x$ in (2) $y = -(x)^2$ $\therefore$ locus of Q $x^2 = -y$ ✓	1+1 1	
(c) $v = x^3 - x$ $\boxed{2}$ $\ddot{x} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $\ddot{x} = (x^3-x)(3x^2-1)$	1 1	
(d) For two 1s $1\ \frac{1}{}\ \_\ \_$ Nº of ways $= 3\times9.8$ For not having two 1s Nº of ways $= 9.3\times8$ $1\ \underline{X}\ \underline{X}\ \underline{Y}$ TOTAL or ${}^4C_2\times1\times9.8 = 432$ $\boxed{2}$	$= 216$ $= 216$ 432	1 1
(e) $I = \int\cos^2\frac{x}{2}\,dx = \frac{1}{2}\int(1+\cos x)\,dx$ $= \frac{1}{2}[x + \sin x] + c$ $\boxed{2}$	1 1	1 For $\cos^2 A = \frac{1}{2}(1+\cos 2A)$ or equiv.

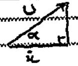
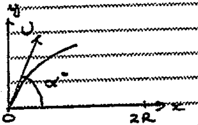
MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments
Q 4(a) $\left(2x^2 - \frac{3}{x}\right)^9$ $\boxed{2}$ General term $T_{r+1} = {}^9C_r(2x^2)^{9-r}\times\left(-\frac{3}{x}\right)^r = Ax^0$ $\therefore {}^9C_r\,2^{9-r}(-3)^r.\,x^{18-2r}\times x^{-r} = Ax^0$. $\Rightarrow 18-3r = 0$ $r = 6$ ✓ $\therefore$ Term is the seventh ✓ $T_7 = {}^9C_6\,2^3.3^6$ $(= 489888)$	1 1	
(b) $\boxed{3}$  1. $\angle DCE = \alpha$ (Angles in same segment standing on arc DE are equal) ✓ 2. $\angle DAB = \alpha$ (Exterior angle of cyclic Quad ABCD equals interior opposite angle) ✓ 3. As $\angle DFE = \angle DAB = \alpha$ ✓ $\therefore$ AGED is a cyclic Quad as (Exterior angle equals interior opposite angle [converse])		
(c) (i) Nº of ways $= 1\ 0\ 0\ 0 + 3\ 0\ 0\ 0 + 5\ 0\ 0\ 0$ $= {}^6C_1\times{}^5C_5 + {}^6C_3\times{}^5C_3 + {}^6C_5\times{}^5C_1$ $= 6 + 20\times10 + 6\times5$ $\boxed{2}$ $= 236$ $\boxed{3}$	1 1	Note: O+O = E E+E = E Need odd Nº of odd Nº: for sum to be odd
(ii) $P(E) = \frac{236}{{}^{11}C_6}$ ✓ $= \frac{236}{462} = \frac{118}{231}$ ①	1	
(d) Let $P = 2000$ int rate $= 0.08$, n is... $R = 1.08$ $M = 200$ ③ (i) After 1st prize: $B_1 = P\times R - 200$ After 2nd prize awarded: $B_2 = B_1R - 200 = (PR - 200)R - 200$ $= PR^2 - 200(1+R)$ After 3rd: $B_3 = B_2R - 200 = (PR^2 - 200(1+R))R - 200$ $= PR^3 - 200(1+R+R^2)$ $\therefore$ After n^{th} $B_n = PR^n - 200(1+R+R^2+\ldots+R^{n-1})$ $= PR^n - 200\left(\frac{R^n-1}{R-1}\right)$ $= 2000R^n - \frac{200(R^n-1)}{0.08}$ $= 2000R^n - 2500(R^n-1)$ $= -500R^n + 2500 = 500[5 - 1.08^n]$ reqd. $\boxed{4}$	$\frac{1}{2}$ $\frac{1}{2}$ 1 1	(ii) ① Set $B_n = 0$ $\Rightarrow 1.08^n = 5$ $n = \frac{\log 5}{\log 1.08} = 20.912$ $\therefore$ Nº of years is 20

MATHEMATICS Extension 1 : Question 5

Suggested Solutions	Marks	Marker's Comments
Q5(a) (i) $(9+5x)^{29} = {}^{29}C_0 9^{29} + {}^{29}C_1 9^{28}(5x) + {}^{29}C_2 9^{27}(5x)^2 + \dots$ $\therefore \underline{p_k = {}^{29}C_k 9^{29-k} . 5^k}$ ✓ $k = 0, 1, 2, \dots, n$	①	
(ii) ② $\frac{p_{k+1}}{p_k} = \frac{{}^{29}C_{k+1} \cdot 9^{29-(k+1)} \cdot 5^{k+1}}{{}^{29}C_k \cdot 9^{29-k} \cdot 5^k}$ $= \frac{\cancel{29!}}{(k+1)!(28-k)!} \times \frac{k!(29-k)!}{\cancel{29!}} \times 9^{-1} \times 5^1$ $= \frac{(29-k)}{(k+1)} \times \frac{1}{9} \times 5 = \frac{5(29-k)}{9(k+1)}$ ✓	1 1	1 For showing how to get the result
(iii) Find the least positive integer k such that $\frac{p_{k+1}}{p_k} = \frac{5(29-k)}{9(k+1)} \le 1$ ② so $145 - 5k \le 9k + 9$ as $k>0$ $136 \le 14k$ $\therefore k \ge \frac{136}{14} = 9.714\ldots$ $\therefore \underline{k = 10}$ ✓ $\therefore$ Largest coefft. is $p_{10} = {}^{29}C_{10} \cdot 9^{19} \cdot 5^{10}$ $= {}^{29}C_{10} 3^{38} . 5^{10}$	1 1	$\underline{\underline{\text{If}}}$ do $\frac{p_{k+1}}{p_k} \geq 1$ $k = 9$ but still $p_{9+1} = p_{10}$
(b) (i) $\frac{d}{dt}(We^{kt}) = \frac{dW}{dt}.e^{kt} + W.ke^{kt}$ ② $= -k(W+15)e^{kt} + kWe^{kt}$ $= -kWe^{kt} - 15ke^{kt} + kWe^{kt}$ $\therefore \frac{d}{dt}(We^{kt}) = -15ke^{kt}$ q.e.d.	1 1	
(ii) As $\frac{d}{dt}(We^{kt}) = -15ke^{kt}$ ② $\therefore We^{kt} = -15e^{kt} + C$ when $t = 0$ $W = 20$ so $20 = -15 + C$ i.e. $C = 35$ $\therefore We^{kt} = -15e^{kt} + 35$ so $W = -15 + 35e^{-kt}$ q.e.d.	1 1	
(iii) (α) As $t = 5$ $W = 6$ ② $\therefore 6 = -15 + 35e^{-5k}$ so $e^{-5k} = \frac{21}{35} = \frac{3}{5} = 0\cdot6$ $-5k = \ln 0\cdot6$; $k = -\frac{\ln 0\cdot6}{5}$ ✓	1	
$\therefore$ Rate $= -\left(-\frac{\ln 0\cdot6}{5}\right)(6+15) = 21\frac{\ln 0\cdot6}{5}$ using (i) $= -2\cdot145\ldots$ $\underline{\text{Rate} = -2\cdot1\ ^\circ C/\min}$ ✓	1	
(β) $-15 + 35e^{kt} = -10$ ① $e^{-kt} = \frac{5}{35} = \frac{1}{7} \Rightarrow t = \frac{\ln(\frac{1}{7})}{-k}$ $= 19\cdot0467\ldots$		

MATHEMATICS Extension 1 : Question 6.

Suggested Solutions	Marks	Marker's Comments
Q6(a) (i) $t = 0 \begin{cases} x = 0 \\ y = 0 \end{cases}$  $\dot{y} = U\sin\alpha$ ②  $\ddot{y} = -g$ $\dot{y} = \int -g\,dt$ $\dot{y} = -gt + c$ but $t = 0$ $\dot{y} = U\sin\alpha$ $\therefore c = U\sin\alpha$ ✓ $\dot{y} = U\sin\alpha - gt$ $y = \int(U\sin\alpha - gt)\,dt$ $y = Ut\sin\alpha - \frac{1}{2}gt^2 + D$ $t = 0$ $y = 0$ so $D = 0$ ✓ $\Rightarrow y = Ut\sin\alpha - \frac{1}{2}gt^2$	1 1	
(ii) For the range : $y = 0$ so $t(U\sin\alpha - \frac{1}{2}gt) = 0$ ② $\therefore t = 0$ or $t = \frac{2U\sin\alpha}{g}$ $\therefore R = x = \frac{U.2U\sin\alpha.\cos\alpha}{g}$ ✓ $= \frac{U^2\sin2\alpha}{g}$	1 1	
(iii) (α) Max height is 3·5m when $t = \frac{1}{2} \times \frac{2U\sin\alpha}{g} = \frac{U\sin\alpha}{g}$ so $3\cdot5 = U.\frac{U\sin\alpha}{g}.\sin\alpha - \frac{1}{2}g \times \frac{U^2\sin^2\alpha}{g^2}$ $= \frac{U^2\sin^2\alpha}{g} - \frac{U^2\sin^2\alpha}{2g}$ ② $3\cdot5 = \frac{U^2\sin^2\alpha}{2g}$ so $U^2 = \frac{3\cdot5\times2g}{\sin^2\alpha} = 7g\,\mathrm{cosec}^2\alpha$	1 1	or $\dot{y} = U\sin\alpha - gt = 0$
(β) Max R will then be $R = \frac{U^2\sin2\alpha}{g}$ ① $= \frac{7g}{\sin^2\alpha} . \frac{2\sin\alpha\cos\alpha}{g}$ $= 14\frac{\cos\alpha}{\sin\alpha}$ $\therefore$ max $R = 14\cot\alpha$ q.e.d.	1	1 For subst (iii)(α) into (ii) and showing how $\frac{14\cos\alpha}{\sin\alpha}$

MATHEMATICS Extension 1 : Question 6

Suggested Solutions | **Marks** | **Marker's Comments**

Q6(b) (i) $f_2(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!}$

$= \frac{2 + 2x + x(x+1)}{2} = \frac{2+2x+x^2+x}{2}$

$= \frac{x^2+3x+2}{2}$ ✓ 1

1 For getting to $\frac{x^2+3x+2}{2}$

$= \frac{1}{2!}(x+1)(x+2)$ ②

and the zeros are $\underline{-1 \text{ and } -2}$ 1

(ii) Let P(n) be the proposition that :

$f_n(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)x\ldots(x+n-1)}{n!} = \frac{1}{n!}(x+1)(x+2)\ldots(x+n)$

• Now P(1) was given
P(2) was shown true in part (i)

* Assume P(n) is true for some integer $k \geq 1$ 1

i.e.

$f_k(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)\ldots(x+k-1)}{k!} = \frac{1}{k!}(x+1)(x+2)\ldots(x+k)$ —(*)

RTP : P(k+1) is true

i.e. $f_{k+1}(x) = \frac{1}{(k+1)!}(x+1)(x+2)\ldots(x+k)$

PROOF : For P(k+1)

$f_{k+1}(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)\ldots(x+k-1)}{k!} + \frac{x(x+1)(x+2)\ldots(x+k)}{(k+1)!}$

$= \frac{1}{k!}(x+1)(x+2)\ldots(x+k) + \frac{x(x+1)\ldots(x+k-1)(x+k)}{(k+1)!}$ using assumption (*) 1

1 For using/ substituting assumption

$= \frac{(x+1)(x+2)\ldots(x+k)}{k!}\left\{1 + \frac{x}{k+1}\right\}$

$= \frac{1}{k!}(x+1)(x+2)\ldots(x+k)\left\{\frac{k+1+x}{k+1}\right\}$ 1

$= \frac{1}{(k+1)!}(x+1)(x+2)\ldots(x+k+1)$ ③

$\therefore$ P(k+1) is true

* $\therefore$ by the PMI P(n) is true for n=1, 2, 3, ...

MATHEMATICS Extension 1 : Question 7

Suggested Solutions | **Marks** | **Marker's Comments**

(a) $V = \pi\int_{r-w}^{r}(r^2 - x^2)\,dx$ $\boxed{2}$

$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{r-w}^{r}$

$= \pi\left[\left(r^3 - \frac{1}{3}r^3\right) - \left(r^2(r-w) - \frac{1}{3}(r-w)^3\right)\right]$ 1

$= \pi\left[\frac{2}{3}r^3 - \frac{(r-w)}{3}\left(3r^2 - (r-w)^2\right)\right]$

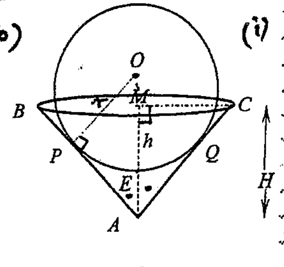
$= \frac{\pi}{3}\left[2r^3 - (r-w)\left(3r^2 - r^2 + 2rw - w^2\right)\right]$

$= \frac{\pi}{3}\left[2r^3 - (r-w)\left(2r^2 + 2rw - w^2\right)\right]$

$= \frac{\pi}{3}\left[2r^3 - \left(2r^3 + \cancel{2r^2w} - rw^2 - \cancel{2r^2w} - 2rw^2 + w^3\right)\right]$ 1

$= \frac{\pi}{3}\left[3rw^2 - w^3\right] = \frac{\pi}{3}(3r - w)w^2$

(b)



(i) As $\Delta OPA \,|||\, \Delta CMA$ (equiangular)

$\frac{r}{R} = \frac{OA}{AC}$ ✓ (corresponding sides in similar Δs are in the same radio) 1

$\frac{r}{R} = \frac{H + (r-h)}{L}$

$rL = HR + rR - hR$

$r(L-R) = (H-h)R$ ✓ 1

② $\therefore r = \frac{(H-h)R}{L-R}$

(ii) Using (a) where $h = w$, $r = \frac{(H-h)R}{L-R}$

$\therefore V = \frac{\pi}{3}\left(3.\frac{(H-h)R}{L-R} - h\right)h^2$ 1

1 For subst and simplifying to

$= \frac{\pi}{3(L-R)}\left[3HR - 3hR - hL + hR\right]h^2$ ①

$= \frac{\pi}{3(L-R)}\left[3RHh^2 - (L+2R)h^3\right]$

(iii) $\frac{dV}{dh} = \frac{\pi}{3(L-R)}\left[6RHh - 3(L+2R)h^2\right]$ $

$= \frac{\pi}{L-R}\left[2RHh - (L+2R)h^2\right]$

For possible max/min values of V to occur $\frac{dV}{dr} = 0$ 1

$\therefore h(2RH - (L+2R)h) = 0$

④ $\therefore h = 0$ or $h = \frac{2HR}{L+2R}$ 1

but $h \neq 0$

TEST : $\frac{d^2V}{dh^2} = \frac{\pi}{L-R}\left[2RH - 2(L+2R)h\right]$ 1 + 1

at $h = \frac{2RH}{L+2R}$ $\frac{d^2V}{dh^2} = \frac{\pi}{L-R}\left[2RH - 4RH\right] = \frac{-2RH}{L-R} < 0$

$\therefore$ a relative max t.p. at $h = \frac{2RH}{L+2R}$. $r = \frac{RHL}{(L-R)(L+2R)}$.

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
(c) (i) ① $(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \binom{2n}{3}x^3 + \dots + \binom{2n}{2n}x^{2n}$		1
(ii) By differentiating both sides wrt x $-2n(1-x)^{2n-1} = -\binom{2n}{1} + 2\binom{2n}{2}x - 3\binom{2n}{3}x^2 + \dots + 2n\binom{2n}{2n}x^{2n-1}$ ② put $x = 1$ $0 = -\binom{2n}{1} + 2\binom{2n}{2} - 3\binom{2n}{3} + \dots - (2n-1)\binom{2n}{2n-1} + 2n\binom{2n}{2n}$ so $\binom{2n}{1} + 3\binom{2n}{3} + \dots + (2n-1)\binom{2n}{2n-1} =$ $= 2\binom{2n}{2} + 4\binom{2n}{4} + \dots + 2n\binom{2n}{2n}$ q.e.d.	1 1	✓ ✓ 1 For Differe... 1 For subst $x=1$ and