QUESTION 1 (START A NEW PAGE)

(a) Differentiate:

(i) $\dfrac{1}{1 + 4x^2}$

(ii) $e^{2x} \log_e 2x$

(b) Write down primitive functions of :

(i) $\sqrt{2x + 1}$

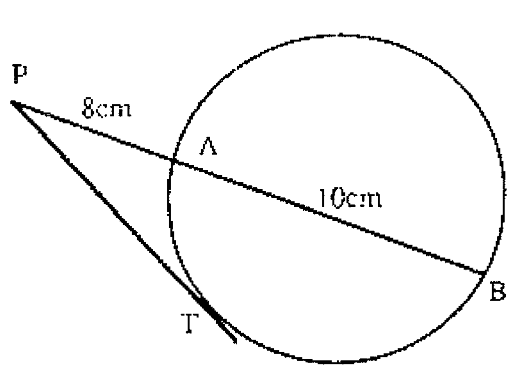
(ii) $\dfrac{1}{1 + 9x^2}$

(c) A and B are the points $(-4, 3)$ and $(2, -1)$ respectively. Find the coordinates of the point Q which divides AB externally in the ratio 4 : 5.

(d) Draw the graph of a function $y = f(x)$ for $1 \le x \le 2$ such that $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$ for $1 \le x \le 2$.

QUESTION 2 (Start A New Page)

(a) PT is a tangent to a circle ABT. PAB is a secant intersecting the circle in A and B. PA = 8cm and AB = 10cm. Find the length of PT giving reason(s) for your answers.



(b) Find the gradients of the 2 lines which make angles of 45° with the line whose equation is $2x + 3y - 6 = 0$.

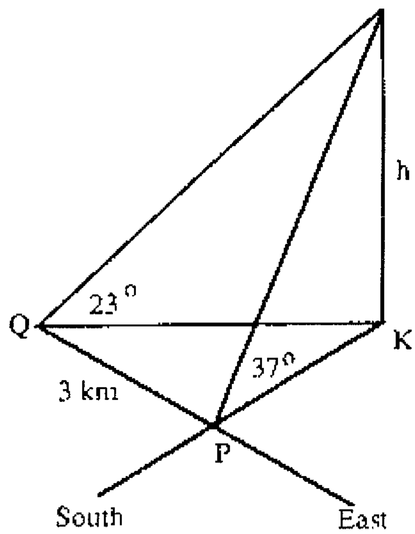
(c) A particle moves along a straight line with a displacement $x(t)$ metres from 0 given by $x(t) = t(2t - 3)(t - 4)$ where t is measured in seconds.

(i) Draw the displacement time graph $x(t)$ and the velocity-time graph $v(t)$. (**Note**: Co-ordinates of stationary points need not be shown.)

(ii) Describe the motion of the particle for $\frac{3}{2} \le t \le 4$.

QUESTION 3 (Start a New Page)

(a) The angle of elevation of a hill top from a place P due south of it is 37°. The angle of elevation of this same hill top from a place Q, due west of P, is 23°. The distance of Q from P is 3 kilometres. If the height of the hill is h kilometres:



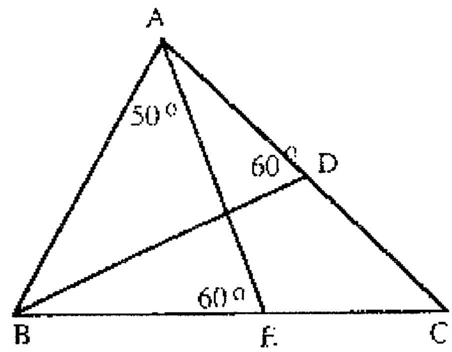
(i) Prove that $PK = h \cot 37^{\circ}$.

(ii) Find a similar expression of QK.

(iii) Calculate the height of the hill to the nearest 10 metres.

(b) ABC is a triangle. D lies on AC and $\angle ADB = 60^{\circ}$. E lies on BC and $\angle AEB = 60^{\circ}$ $\angle BAE = 50^{\circ}$

Copy this diagram onto your page and find the size of $\angle CDE$ giving full reasons for your answer.



(c) Find the number of ways of arranging 2 men, 2 boys and 2 girls in a circle if:

(i) there is no restriction.

(ii) the two boys sit next to one another.

(d) In solving a problem it is necessary to find a value of r for which $\pi r^2 + 2\pi rh$ is a minimum knowing that $\pi r^2 h = 5$. Write down a problem which could be solved using this information.

Question 4 (Start a new page)

(a) (i) On the same set of axes draw neat sketches of $y = x^2$ and $y = 4x - x^2$ showing the co-ordinates of the points of intersection.

(ii) Find the volume of the solid generated when the region bounded by these two curves is rotated one revolution about the x-axis.

(b) A body is projected with speed $24.5\ \text{mm}^{-1}$ from the top of a cliff 58.8m high at an angle of α to the horizontal where $\alpha = \tan^{-1}(\frac{4}{3})$. Take the bottom of the cliff as the origin and take the acceleration due to gravity g as 9.8ms^{-2}

(i) Show that $x = 14.7t$ and $y = 58.8 + 19.6t - 4.9t^2$.

(ii) Find the range of the horizontal plane through the foot of the cliff.

(iii) Find the speed of the body when it reaches this point.

Question 5. (Start a New Page)

(a) Sketch the graph of $y = 2\cos^{-1}x \quad \frac{\pi}{4}$ stating its natural domain and range.

(b) Find the general solution of the trigonometric equation $\cos 3\theta = \cos\theta$.

(c) The rise and fall of the tide at a certain harbour may be taken to be simple harmonic, the interval between successive high tides being 12 hours 30 minutes. The harbour entrance has a depth of 11 metres at high tide and 7 metres at low tide. If low tide occurs at 9.05 a.m. on a certain day find the earliest time thereafter that a ship drawing 10 metres can pass through the entrance.

Question 6 (Start a New Page)

(a) Prove by induction that $9^{n+2} - 4^n$ is divisible by 5 for integers $n \geq 1$.

(b) Newton's law of cooling states that the rate at which a body loses heat to its surroundings is proportional to the difference between the temperature T of the body and the temperature S of its surrounding medium. This can be expressed by the differential equation

$$\frac{dT}{dt} = k(T - S)$$

where t is the time in minutes and k is a constant.

(i) Show that $T = S + Be^{-kt}$, where B is a constant, is a solution.

(ii) If the temperature of a beaker of water falls from 90°C to 60°C in 5 minutes at a room temperature of 20°C, find

(α) the time taken for the temperature of the water to cool to 50°C (Give your answer correct to 1 decimal place).

(β) the temperature of the water 15 minutes after reaching 60°C. (Give your answer correct to 1 decimal place.)

QUESTION 7 (Start a New page.)

(a) The chord PQ joining the points $P(2p, p^2)$ and $Q(2q, q^2)$ on $x^2 = 4y$ always passes through the point A(2, 0) when produced.

(i) Show $(p + q) = pq$

(ii) Find the co-ordinates of M, the midpoint of PQ.

(iii) Find the equation of the parabola on which M always lies as P varies. On the same set of axes sketch this parabola and the parabola $x^2 = 4y$ showing co-ordinates of vertices and points of intersection.

(iv) Write down the equation of the locus of M indicating any restriction which exists for the domain.

(b) The structural steel work of a new office building is finished. Across the street 60m from the foot of a freight elevator shaft in the building a spectator is standing, watching the freight elevation ascend at a constant rate of 15m/s. How fast is the angle of elevation of the spectators line of sight to the elevator increasing 6 seconds after his line of sight passed the horizontal? [Give your answer to 2 significant figures in rad./sec.]

THIS IS THE END OF THE PAPER

Daniel Ong.

James Ruse Trial Bunk 1992.

Q1(a)(i) $\frac{d}{dx}\ \frac{1}{(1+4x^2)}$

$= -1 \times 8x\ (1+4x^2)^{-2} = \frac{-8x}{1+4x^2}$ ✓

(ii) $\frac{d}{dx}\ (e^{2x}\ \ln_e 2x)$

$u = e^{2x}$ $\quad v = \ln 2x$

$u' = 2e^{2x}$ $\quad v' = \frac{1}{x}$ ✓

$\frac{d}{dx} = vu' + uv'$

$= 2e^{2x}\ln 2x + \frac{e^{2x}}{x}$

$= e^{2x}[2\ln 2x + \frac{1}{x}]$ ✓

(b)(i) $\int (2x+1)^{1/2}.dx$

$= \frac{(2x+1)^{3/2}}{3/2 \times 2} + c.$ ✓

$= \frac{(2x+1)\sqrt{2x+1}}{3} + c.$

(ii). $\int \frac{1}{1+9x^2}$

$= \frac{1}{3}\tan^{-1}(\frac{x}{3}) + c.$ ✓

(c). $(-4, 3)\ (2, 1)$ $\quad 4:5$ Externally

$x = \frac{mx_2 - nx_1}{m-n}$ $\quad y = \frac{my_2 - ny_1}{m-n}$

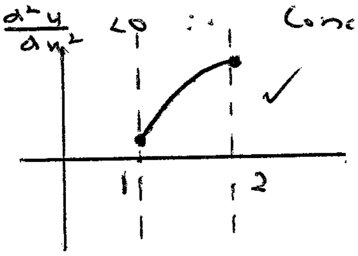
$= \frac{8+20}{-1}$ ✓ $\quad = \frac{-4-15}{-1}$ ✓

$= -28$ ✓ $\quad = +19$

$Q = (-28, 19)$

(d). $\frac{dy}{dx} > 0.\ \therefore$ Increasing

$\frac{d^2y}{dx^2} < 0\ \therefore$ Concave Down.



✓

Question

(a). $PT^2 = PA \times PB$ (secant tan tangent rule).

$PT^2 = 10 \times 18$

$PT = \sqrt{180}$ ✓

$= 6\sqrt{5}.$

(b). $2x + 3y - 6 = 0$

$3y = -2x + 6$

$y = -2/3 x + 2$ ✓

$m_1 = -2/3$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$1 = \left|\frac{-2/3 - m_2}{1 - 2/3 m_2}\right|$

$1 - 2/3 m_2 = -2/3 - m_2$

$3 - 2m_2 = -2 - 3m_2$

$5 = -m_2$

$m_2 = -5$ ✓

OR

$-1 + 2/3 m_2 = -2/3 - m_2$

$-3 + 2m_2 = -2 - 3m_2$

$-1 = -5m_2$

$m_2 = 1/5$ ✓

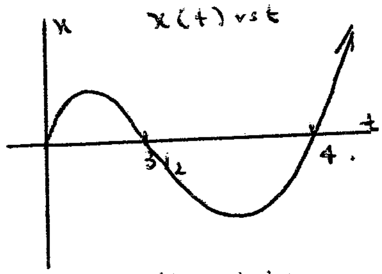
Q 2.

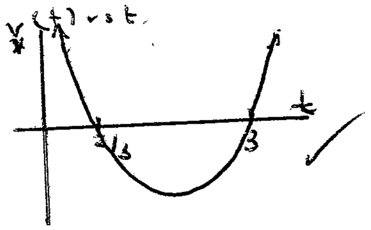
(c). (i) $x(t) = t(2t-3)(t-4)$

$= t[2t^2 - 8t - 3t + 12]$

$= 2t^3 - 11t^2 + 12t$ ✓

$v(t) = \underline{6t^2 - 22t + 12} = \underline{(t-3)(3t-2)}$





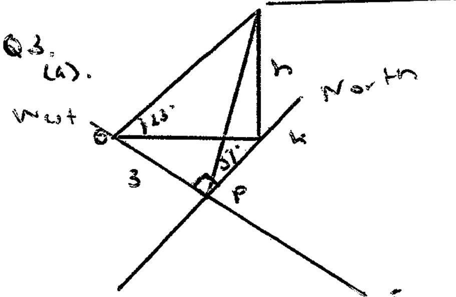
✓

(ii). From $3/2 < t \leq 4$

Particle moves to the left. and changes ~~dif~~ direction after 3 s.

Q3.

(a).



(i). $\tan 37° = \frac{h}{PK}$

$\cot 37° = \frac{PK}{h}$ ✓

$\underline{PK = h \cot 37°}$

(ii) $\tan 23° = h/QK$

$\underline{QK = h \cot 23°}$ ✓

(iii). $QK = \sqrt{PK^2 + 9}$

$QK^2 = PK^2 + 9$

$h^2 \cot^2 23° = h^2 \cot^2 37 + 9$

$h^2[\cot^2 23° - \cot^2 37°] = 9$

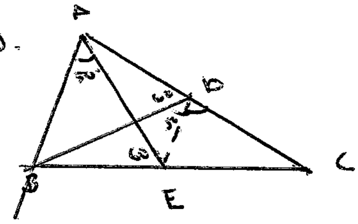
$\frac{[\cot^2 23 - \cot^2 37]}{9} = h^2$

$0.272 = h^2$

$0.5216 = h$

$\underline{h \doteqdot 520 \text{ m}}$

(b).



(b). ABDE is a cyclic quad

since $A\hat{D}B = 60° = A\hat{E}B$

(Angles equal subtended on the same arc

$\therefore B\hat{D}E = 50°$

$\therefore E\hat{D}C = \pi - 60° - 50°$

$= \underline{70°}$ ✓

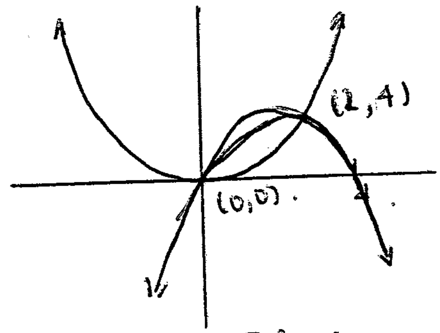
(c). 2M, 2B, 2G

(i) $\underline{5!}$ ✓

(ii) $\underline{4! \times 2}$ ✓

(d). Q. Find the radius which minimises the surface area of a cylinder with ~~a~~ an open end and a fixed Vol. of $5u^3$.

Q4(a).



(i) $x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$x(2x-4) = 0$

$x = 0 \quad x = 2$

$\therefore$ P.O.I $(0,0)$ & $(2,4)$ ✓ ✓

$V = \pi \int_0^2 (4x-x^2)^2 - (x^2)^2 \, dx$ ... try again

(ii). $V = \pi \int_0^2 4x - x^2 - x^2 \, dx$

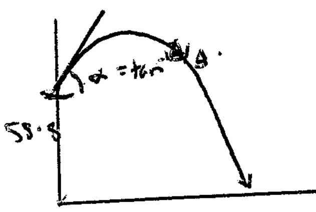
$= \pi \int_0^2 4x - 2x^2 \, dx$

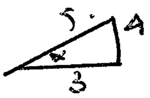
$= \pi \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2$

$= \pi \left[8 - \frac{16}{3} \right]$

$= \frac{8\pi}{3}$

(b) $V = 24.5$





$\sin\alpha = 4/5$

$\cos\alpha = 3/5$

(i). $\ddot{x} = 0$

$\dot{x} = C_1$

At $t = 0$ $\dot{x} = V\cos\alpha$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c$ ✓

At $t = 0$, $x = 0$

$= Vt\cos\alpha$

$= \frac{24.5 \times t \times 3}{5}$

$= 14.7t$ ✓

$\ddot{y} = -g$

$\dot{y} = -gt + c$

At $t = 0$ $\dot{y} = V\sin\alpha$

$\dot{y} = -gt + V\sin\alpha$

$y = \frac{-gt^2}{2} + Vt\sin\alpha + c$ ✓

At $t = 0, y = 58.8$.

$y = \frac{-gt^2}{2} + Vt\sin\alpha + 58.8$

$= -4.9t^2 + 19.6t + 58.8$ ✓

(ii). $x = 14.7t$

$t = \frac{x}{14.7t}$ sub into y

$y = -4.9\left(\frac{x^2}{216.09}\right) + \frac{4x}{3} + 58.8$ ✓

$y = \frac{-10x^2}{441} + \frac{4x}{3} + 58.8$

Range occurs when $y = 0$.

$0 = -\frac{10x^2}{441} + \frac{4x}{3} + 58.8$

$= -10x^2 + 588x + 25930.8$ ✓

$x = -29.4$ or 88.2

$\therefore$ Range is 88.2m

(iii). ~~88.2 = 14.7t~~

~~t = 6~~

$\therefore$ ~~Time of Flight = 6.~~

~~Angle when hits~~

~~$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$~~

$= -gt + V\sin\alpha$

(b)(ii) $88.2 = 14.7t$

$t = 6$

Time of Flight $= 6$ sec.

$\Rightarrow \quad Speed = \sqrt{|\dot{x}^2 + \dot{y}^2|}$

Angle when it hits

$= \sqrt{(14.7)^2 + (19.6 - 9.8(6))^2}$

$= 41.9 m/s$

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

$= \left| \frac{-gt + Vt\sin\alpha}{V\cos\alpha} \right|$

$= \left| \frac{-9.8 \times 6 + 24.5 \times 6 \times \frac{4}{5}}{24.5 \times \frac{3}{5}} \right|$

$= \left| \frac{-58.8 + 117.6}{14.7} \right|$

$\theta = \tan^{-1}(4)$

$\cos\theta \ \frac{1}{\sqrt{17}}$

At $x = 88.2$

$88.2 = Vt\cos\theta$

$88.2 = V \times 6 \times \frac{1}{\sqrt{17}}$

$\frac{88.2\sqrt{17}}{6} = V$.

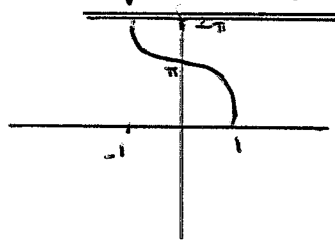
$V = 60.60 mm^{-1}$

Q5.(a). $y = 2\cos^{-1}x$ ✓

$D: -1 \le x \le 1$

$R: 0 \le y/2 \le \pi$ ✓

$0 \le y \le 2\pi$ ✓



(b). $\cos 3\theta = \cos\theta$

$\cos(2\theta + \theta) - \cos\theta = 0$.

$\cos 2\theta \cos\theta - \sin 2\theta \sin\theta - \cos\theta = 0$.

$\cos 2\theta \cos\theta - 2\cos\theta\sin\theta\sin\theta - \cos\theta = 0$.

$(\cos^2\theta - \sin^2\theta)\cos\theta - 2\cos\theta\sin^2\theta - \cos\theta = 0$.

$= 2\cos^3\theta - \cos\theta - 2\cos(1 - \cos^2\theta) - 2\cos\theta = 0$.

$= 2\cos^3\theta - 2\cos(1-\cos^2\theta) - 4\cos\theta = 0$. ✓

$= 2\cos^3\theta + 2\cos^3\theta - 4\cos\theta = 0$.

$= 4\cos^3\theta - 4\cos\theta = 0$.

$\cos^3\theta - \cos\theta = 0$.

$\cos\theta[\cos^2\theta - 1] = 0$.

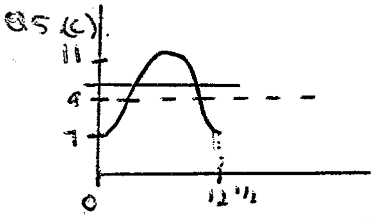
$\cos\theta = 0 \qquad \cos^2\theta = 1$

$\theta = \pi/2, \frac{3\pi}{2} \qquad \cos\theta = \pm 1$

$\theta = 0, \pi, 2\pi$ ✓

$\therefore \theta = \frac{n\pi}{2}$ for n integers.

Q5 (c).



Period: $12\frac{1}{2} = \frac{2\pi}{n}$

$25n = 4\pi$

$n = \frac{4\pi}{25}$.

$\therefore\ 9 \pm 2\cos(nt)$

$x = 9 - 2\cos\left(\frac{4\pi t}{25}\right)$

$10 = 9 - 2\cos\left(\frac{4\pi t}{25}\right)$

$-\frac{1}{2} = \cos\left(\frac{4\pi t}{25}\right)$

$\frac{2\pi}{3} = \frac{4\pi t}{25}$.

$\frac{50}{12} = t$.

$t = 4\frac{1}{6}$ hrs.

$\therefore$ Time. = 13:15 or 1:15pm.

Question 6.

(a) $9^{n+2} - 4^n \div 5$.

Let n = 1

$9^3 - 4 = 725$, which is divisable by 5

$\therefore$ True for n = 1

Assume true for n = k

$9^{k+2} - 4^k = 5M$ (where M is an integer).

$9^{k+2} = 5M + 4^k$

$9^{k+2} - 5M = 4^k$

Prove for n = k+1

$9^{k+3} - 4^{k+1}$

$= 9^{k+2}.9 - 4^k.4$

$= (5M + 4^k).9 - 4^{k+1}$

$= 45M + 9 \times 4^k - 4^{k+1}$

$= 45M + 4^k[9 - 4]$

$= 45 + 4^k.5$

$= 5[9 + 4^k] = 5N$ (where N is integer).

Which is divisible by 5.

~~$= 9^{k+3} - 4^k.4$~~

~~$= 9^{k+3} - (9^{k+2} - 5M)$~~

~~$= 9^{k+3} - 9^{k+2} + 5M$~~

~~$= 9^{k+2}[9 - 1] + 5M$~~

~~$= 8.9^{k+2} + 5M$.~~

$\therefore$ If true for n = k and n = k+1, then by the Principles of Induction it is true for all positive integers.

(b). (i) $T = 5 + Be^{-kt} \implies T + 5 = Be^{-kt}$

$\frac{dT}{dt} = -Bke^{-kt}$

$= k(T - 5)$

(ii) (α). $90 = 5 + Be^{-k0}$

$= 20 + Be^0$

$90 = 20 + B$

$B = 70$.

At t = 5, T = 60.

$60 = 20 + 70e^{-5k}$

$\frac{40}{70} = e^{-5k}$

$-\frac{\ln\left(\frac{40}{70}\right)}{5} = k$

$k \doteqdot .1119$

(β). t = 20.

$T = 20 + 70e^{20 \times -0.1119}$

$T = 27.$

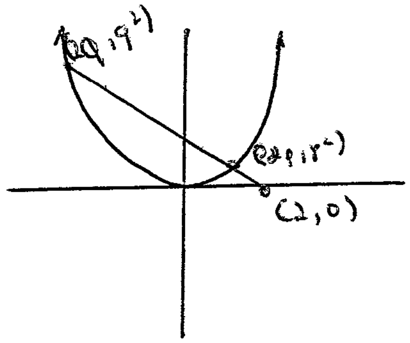
(γ) $50 = 20 + 70e^{-0.1119t}$

$\frac{30}{70} = e^{-0.1119t}$

$\ln\left(\frac{30}{70}\right) = -0.1119t$

$t = -\frac{\ln\left(\frac{30}{70}\right)}{0.1119}$

$= 7.6$ mins



Chord

$m = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{\cancel{(p-q)}(p+q)}{2\cancel{(p-q)}}$

$= \dfrac{p+q}{2}$ ✓

Eqn: $y - y_1 = m(x - x_1)$

$y - ap^2 = \dfrac{p+q}{2}(x - 2p)$

$y - p^2 = \dfrac{(p+q)x}{2} - p(p+q)$

$y - p^2 = \dfrac{(p+q)x}{2} - p^2 - pq$ ✓

$y = \dfrac{(p+q)x}{2} - pq$

But passes through $(2, 0)$

$0 = \dfrac{2(p+q)}{2} - pq$

$\underline{p+q = pq}$ ✓

(ii) Midpoint

$x = \dfrac{2p+2q}{2} \qquad y = \dfrac{p^2+q^2}{2}$

$= (p+q) \qquad = \dfrac{(p^2+q^2)}{2}$ ✓

$\therefore \underline{M\left[(p+q), \left(\dfrac{p^2+q^2}{2}\right)\right]}$

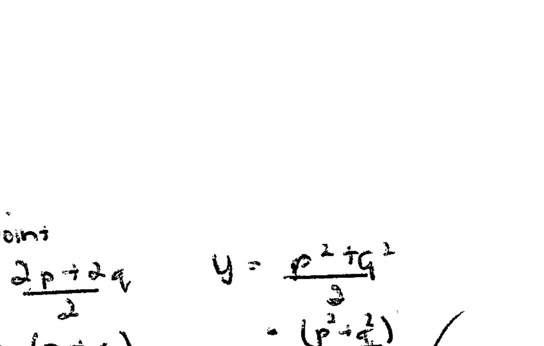
(iii). $x = p+q \qquad y = \dfrac{p^2+q^2}{2}$

$x = pq \qquad = \dfrac{(p+q)^2 - 2pq}{2}$ ✓

$y = \dfrac{(pq)^2 - 2pq}{2}$ ✓

Sub $x = pq$

$\underline{y = \frac{1}{2}(x^2 - 2x)}$ ✓



$x^2 = 4y$

$\underline{\text{vertice}: (0,0)}$

$y = \frac{1}{2}(x^2 - 2x)$

$y' = \frac{2x}{2} - 1 = 0.$

$\therefore$ vertice $(1, -\frac{1}{2})$

$\frac{x^2}{4} = \frac{x^2 - 2x}{2} \quad \Rightarrow x^2 = 2(x^2 - 2x)$ ✓

$x^2 = 4x^2 - 4x \qquad x^2 = 2x^2 - 4x$

$x^2 - 4x = 0.$

$\cancel{3x^2 - 4x = 0}$

$\cancel{x(3x - 4) = 0}$

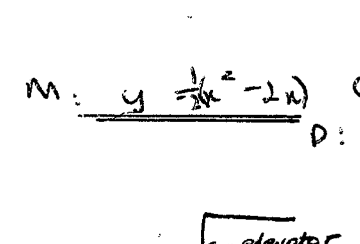
$\underline{P.O.I\ (0,0) \quad \cancel{(\tfrac{4}{3}, \tfrac{16}{9})}\ (4, 4)}$

(iv). $M: \underline{y = \frac{1}{2}(x^2 - 2x)}$ (proven above).

$D: x \geq 4 \cap x \leq 0.$ ✓

since $x \neq 4\ 0.$ ✓

(b) ??



$\dfrac{dy}{dt} = 15$ m/s

$\tan\theta = \dfrac{y}{60}$

$\therefore y = 60\tan\theta$

$\dfrac{dy}{d\theta} = 60\sec^2\theta$

Find $\dfrac{d\theta}{dt}$ ... continue.