Name:	
Class:	

JAMES RUSE AHS

GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2012

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard integral Sheet is provided.
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labeled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ~~●~~ C ○ D ○

Trial HSC Examination
Mathematics Extension 2, 2012

Multiple Choice Answer Sheet

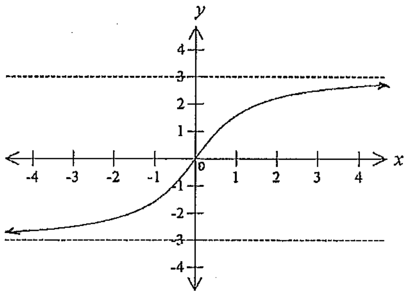
Student id number: ☐☐☐☐

Completely colour in the response oval representing the most correct answer.

1	A ○	B ○	C ○	D ○
2	A ○	B ○	C ○	D ○
3	A ○	B ○	C ○	D ○
4	A ○	B ○	C ○	D ○
5	A ○	B ○	C ○	D ○
6	A ○	B ○	C ○	D ○
7	A ○	B ○	C ○	D ○
8	A ○	B ○	C ○	D ○
9	A ○	B ○	C ○	D ○
10	A ○	B ○	C ○	D ○

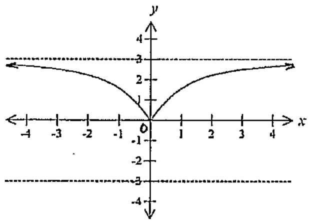
Mark: /10

1. The diagram shows the graph of the function $y = f(x)$.

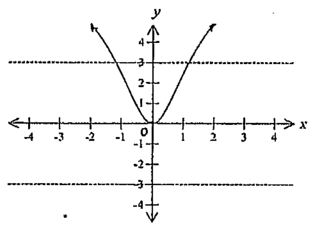


Which of the following is the graph of $y = \sqrt{f(x)}$?

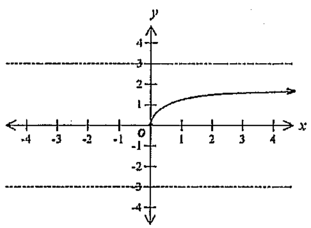
(A)



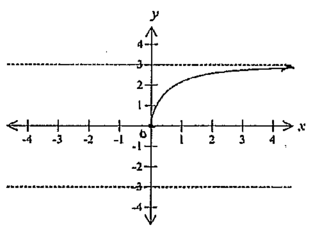
(B)



(C)



(D)



2. Let $z = 3 - i$. What is the value of $\overline{iz}$?

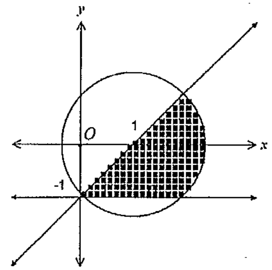
(A) $-1-3i$.

(B) $-1+3i$.

(C) $1-3i$.

(D) $1+3i$.

3. Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$.

(B) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$.

(C) $|z-1| \le 1$ and $0 \le Arg(z-i) \le \frac{\pi}{4}$.

(D) $|z-1| \le 1$ and $0 \le Arg(z+i) \le \frac{\pi}{4}$.

4. Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

5. What is the solution to the inequality: $\frac{x(5-x)}{x-4} \ge -3$?

(A) $2 \le x < 4$ or $x \ge 6$.

(B) $1 \le x < 4$ or $x \ge 5$.

(C) $4 < x \le 6$ or $x \le 2$.

(D) $4 > x \le 5$ or $x \le 1$.

6. The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$.

(B) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$.

(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$.

(D) $\tan\theta\tan\phi=\frac{a^2}{b^2}$.

7. What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=2$ and $q=4$.

(B) $p=2$ and $q=-4$.

(C) $p=-2$ and $q=4$.

(D) $p=-2$ and $q=-4$.

8. A particle of mass m is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle or mkv^2. Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x}=-\left(g+kv^2\right)$ where g ms^{-2} is the acceleration due to gravity. Assume $k=10$ and the acceleration due to gravity is 10 ms^{-2}.

Which of the following gives the correct expression for the time taken?

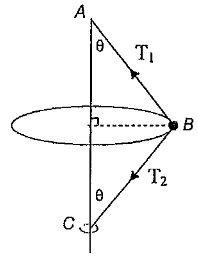
(A) $t=\frac{1}{10}\left(\tan^{-1}u-\tan^{-1}v\right)$.

(B) $t=\frac{1}{10}\left(\tan^{-1}v-\tan^{-1}u\right)$.

(C) $t=\frac{1}{10}\left(\tan^{-1}u+\tan^{-1}v\right)$.

(D) $t=\frac{1}{10}\left(\tan^{-1}v+\tan^{-1}u\right)$.

9. A body of mass m kg is attached by two light rods AB and BC. Both rods are l metres in length. Rod AB is hinged at point A and makes an angle θ with the vertical shaft. Rod BC is attached to a ring that can slide freely along the vertical shaft.



What are the tensions in the rods?

(A) $T_1=\frac{1}{2}\left(mg\sec\theta+ml\omega^2\right)$ and $T_2=\frac{1}{2}\left(ml\omega^2-mg\sec\theta\right)$.

(B) $T_1=\frac{1}{2}\left(mg\sin\theta+ml\omega^2\right)$ and $T_2=\frac{1}{2}\left(ml\omega^2-mg\sin\theta\right)$.

(C) $T_1=\frac{1}{2}\left(mg\sec\theta-ml\omega^2\right)$ and $T_2=\frac{1}{2}\left(ml\omega^2+mg\sec\theta\right)$.

(D) $T_1=\frac{1}{2}\left(mg\sin\theta-ml\omega^2\right)$ and $T_2=\frac{1}{2}\left(ml\omega^2+mg\sin\theta\right)$.

10. A skydiver falls from a plane which is flying horizontally at 2 000 m. Initially his motion is determined by the acceleration due to gravity of 10 m/s^2 and any resistance is negligible. After 5 seconds, he opens his parachute and his motion is determined by the equation: $\ddot{x}=10-\frac{5}{4}v$, where downwards direction is taken as positive.

Hence his terminal velocity will be 8 m/s.

Which statement best reflects the situation after the skydiver opens his parachute?

(A) He hits the ground with a vertical speed of 50 m/s.

(B) He hits the ground with a vertical speed of 8 m/s

(C) His vertical speed never exceeds 8 m/s.

(D) His vertical speed never drops below 8 m/s.

End of Section I

Section II

Total Marks is 90
Attempt Question 11 – 16.
Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question on a new sheet of paper with your **student id number** in the top right hand corner and the question number on the left hand side of your paper.

All necessary working must be shown in each and every question.

Question 11. [Start a New Page] **Marks**

(a) Given $z = 1 + 2i$ and $w = -2 + i$, find:

(i) $|z|$. **1**

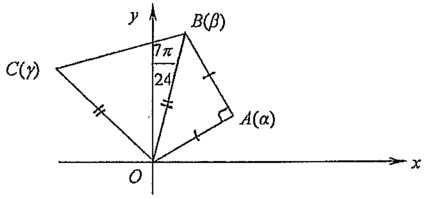
(ii) zw. **2**

(iii) $\dfrac{5}{iw}$. **2**

(b) Find the two complex numbers z that satisfy: $z\bar{z} = 37$ and $\dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$. **3**

(c) If $w = \left(-1 + i\sqrt{3}\right)^{2012}$, find $Arg\, w$. **2**

(d) Points A, B and C represent the complex numbers α, β and γ in the Argand diagram respectively.



Not to scale

ΔOAB is right isosceles at A, ΔCOB is isosceles with $OB = OC$ and $\angle OBC = \dfrac{7\pi}{24}$.

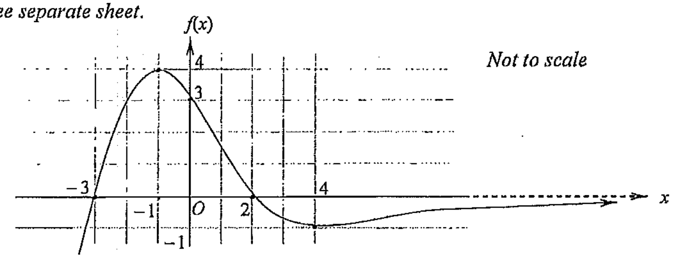
(i) Copy or trace the diagram onto your writing and find $\angle AOC$. **1**

(ii) Explain why $\gamma = \sqrt{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)\alpha$ **2**

(iii) Hence find the value of $2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$. **2**

Question 12. [Start a New Page] **Marks**

(a) Given the sketch of the function $f(x)$, sketch each of the following on separate diagrams:
See separate sheet.



Not to scale

(i) $y = -f(x)$. **1**

(ii) $y = f(-x)$. **2**

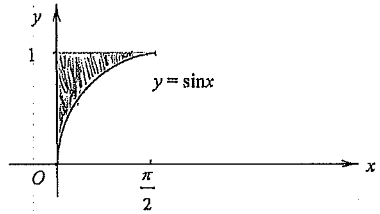
(iii) $y = f(x^2)$. **2**

(b) (i) Show that the equation of the tangent to the polynomial function $y = P(x)$ at $x = \alpha$ is $y = P'(\alpha)(x - \alpha) + P(\alpha)$. **1**

(ii) Explain why: when the polynomial $P(x)$ (of degree greater than 2) is divided by the quadratic $(x-\alpha)^2$, then $P(x) = (x-\alpha)^2 Q(x) + ax + b$, where $Q(x)$ is the quotient and a and b are real numbers. **1**

(iii) Hence show that when $P(x)$ is divided by $(x-\alpha)^2$, the remainder is equivalent to $P'(\alpha)(x-\alpha) + P(\alpha)$, the expression from the tangent in part (b) (i). **3**

(c) The area bounded by the curve $y = \sin x$, for $0 \le x \le \dfrac{\pi}{2}$, the lines $x = 0$ and $y = 1$ is rotated about the y-axis.



Not to scale

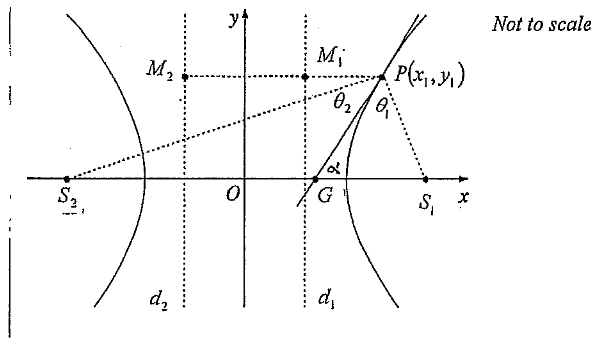
(i) By using the method of cylindrical shells, show that the volume V of the solid of revolution about the y-axis is given by: **2**

$$V = 2\pi\int_0^{\frac{\pi}{2}} x(1 - \sin x)\,dx.$$

(ii) Hence calculate the volume of the solid. **3**

Question 13. [Start a New Page] **Marks**

(a)



The point $P(x_1, y_1)$ lies on the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

The two foci of the hyperbola are S_1 and S_2 and the two directrices are d_1 and d_2, as shown.

(i) Show that the length $S_1P = \frac{\sqrt{34}}{5}x_1 - 5$. **2**

(ii) Show that the equation of the tangent at P is $\frac{x_1x}{25} - \frac{y_1y}{9} = 1$. **2**

(iii) The tangent at P intersects the transverse axis at point G. Find the coordinates of point G. **1**

(iv) Given $\angle S_1PG = \theta_1$, $\angle GPS_2 = \theta_2$ and $\angle S_1GP = \alpha$,

(1) By using the sine rule, show that: $\sin\alpha = \frac{x_1}{5}\sin\theta_1$. **2**

(2) Hence show that: $\sin\theta_1 = \sin\theta_2$. **2**

(3) Hence deduce that GP bisects $\angle S_1PS_2$. **2**

(b) Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x\, dx$, for $n = 1, 2, \ldots$

(i) Show that $I_1 = \frac{1}{2}\ln 2$. **1**

(ii) Show that: $I_{n-2} + I_n = \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$, for $n = 2, 3, 4, \ldots$ **2**

(iii) Find I_5. **1**

Question 14. [Start a New Page] **Marks**

(a) Given that: $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n < \frac{\pi}{2}$, where $0 \le \theta_i < \frac{\pi}{2}$ for $i = 1, 2, 3, \ldots, n$. **3**

Prove, by mathematical induction for $n = 1, 2, 3, \ldots$, that:

$$\tan(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) \ge \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \ldots + \tan\theta_n.$$

(b) (i) Find the constants a, b and c such that: **3**

$$\frac{300x}{1000 + x^3} = \frac{a}{10 + x} + \frac{bx + c}{100 - 10x + x^2}.$$

(ii) A particle of mass m kg is projected vertically upwards in a highly resistive medium at a velocity of 5 m/s.
The particle is subjected to the force of gravity and to a resistance due to the medium of magnitude $\frac{mv^3}{100}$ newtons. Given the acceleration due to gravity is 10 m/s^2,

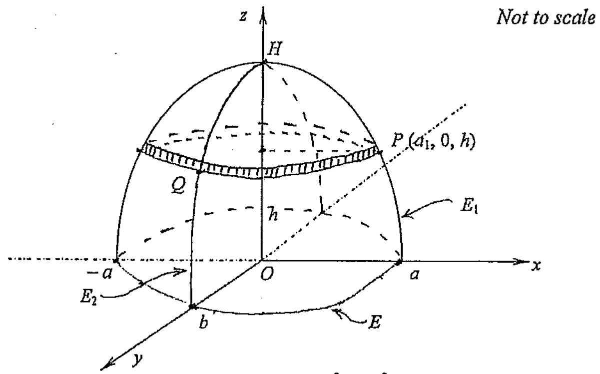
(1) State the equation of motion (if upwards is the positive direction) **1**

(2) Hence find the maximum height reached by the particle, (giving your answer correct to 1 decimal places). **3**

Question 14 (c) *continued over page*

Question 14. *continued* **Marks**

(c) A right solid has an elliptical base whose equation is E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
The height of the solid is H such that the sections in the x-z plane and the y-z plane are the semi-ellipses E_1 and E_2 respectively.
Every cross-section parallel to the base is elliptical in shape as shown in the diagram.



(i) Given that the equation of ellipse E_1 is $\frac{x^2}{a^2}+\frac{z^2}{H^2}=1$, **1**
State the equation for ellipse E_2.

(ii) By taking a slice parallel to the base at height of h of thickness Δh, **2**
Hence point P can be stated as $(a_1, 0, h)$ and Q as $(0, b_1, h)$ for the elliptical slice, as shown in the diagram.

By assuming that the area of ellipse E is πab square units,

Show that the cross-sectional area A of this slice is given by $A=\pi ab\left(1-\frac{h^2}{H^2}\right)$.

(iii) Hence find the volume of the solid. **2**

Question 15. [Start a New Page] **Marks**

(a) Let α, β and γ be the distinct roots of the cubic equation $x^3+ax^2+bx-54=0$, where a and b are real numbers. Suppose that $\alpha^2+\beta^2=0$ and $\alpha^2+\gamma^2=0$,

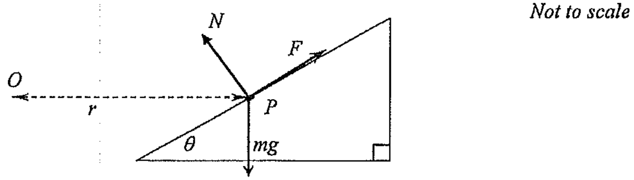
(i) Explain why $\beta+\gamma=0$. **2**

(ii) Hence explain why α is real. **2**

(iii) Hence, or otherwise explain why β and γ are complex and purely imaginary. **2**

(iv) Find a and b. **2**

(b) On a racetrack for small cars of mass m kg, a circular bend of radius r m is banked at an angle of θ to the horizontal.
The maximum frictional force is F Newtons (up or down the bank) and the acceleration due to gravity is 10 m/s^2 ie $g=10$. The normal reaction to the surface is N Newtons.
Let point P represent the small car on the banked track as shown in the diagram.



(i) From the diagram, the vertical resolution for motion downwards at P is: **1**
$$N\cos\theta+F\sin\theta=mg.$$
Find the horizontal resolution when the car is travelling at speed v m/s at P.

(ii) Hence, if $r=80$, $\theta=45^0$ and the maximum frictional force is at most $\frac{1}{9}$ of the normal reaction force N. Find the minimum speed that the car can safely negotiate the bend without slipping down the incline. **2**

(iii) For the upwards motion of the car, find the maximum speed that the car can safely negotiate the bend. **3**

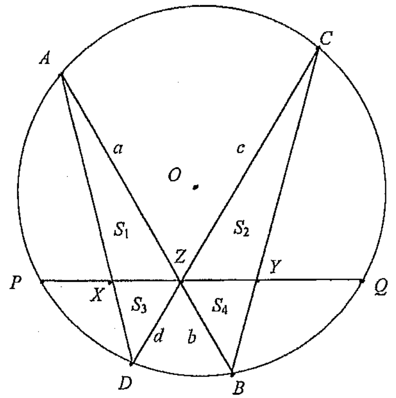
(iv) Hence or otherwise, determine the designed speed (no slipping) for this angle. **1**

Question 16. [Start a New Page] **Marks**

(a) Show that $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$, using the substitution $x = \frac{1-u}{1+u}$. **4**

(b) The roots of $x^4 + 3x - 1 = 0$ are α, β, γ and δ. Find $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. **2**

(c)



Not to scale

Given Z is any point on the chord PQ of the given circle. Chord AB and CD pass through Z. Let X and Y be the points of intersection of the chords AD and CB with PQ respectively, as shown in the diagram.

Given $ZA = a$; $ZB = b$; $ZC = c$ and $ZD = d$. **Let** $ZP = p$; $ZQ = q$; $ZX = x$ and $ZY = y$.

Given S_1 denotes the *area* of ΔZAX; S_2 for ΔZCY; S_3 for ΔZDX and S_4 for ΔZBY.

(i) By detaching page **12** and stapling it to your Question 16,
Show that: $\frac{S_1}{S_2} = \frac{a.AX}{c.CY}$ and $\frac{S_1}{S_4} = \frac{a.x}{b.y}$. **3**

(ii) Hence deduce that: $\frac{S_1S_3}{S_2S_4} = \frac{a.d \times AX.XD}{b.c \times CY.YB} = \frac{a.d.x^2}{b.c.y^2}$. **2**

(iii) Hence explain why: $\frac{x^2}{y^2} = \frac{(p-x)(x+q)}{(p+y)(q-y)}$. **1**

(iv) Hence show that: $\frac{1}{x} - \frac{1}{y} = \frac{1}{p} - \frac{1}{q}$. **2**

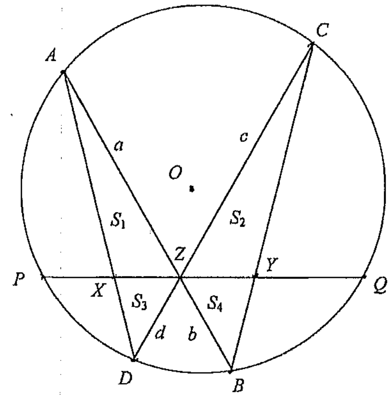
(v) If Z is the midpoint of PQ, what is the relationship between x and y. **1**

THE END

☺ 😐 ☹

Student id No.

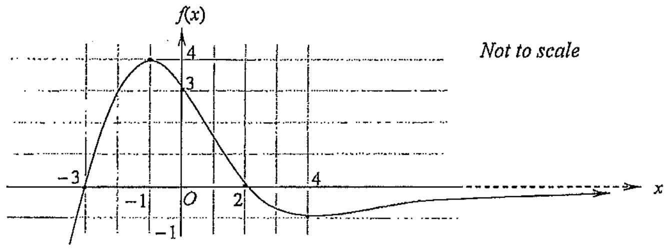
Question 16. (c) Attach this diagram to your **Q16** (c) solutions.



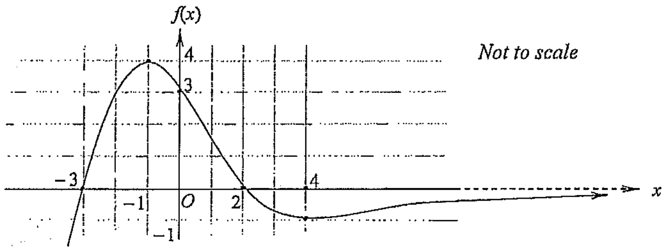
M Ext 2
Question 12 (a)

Student id No.				

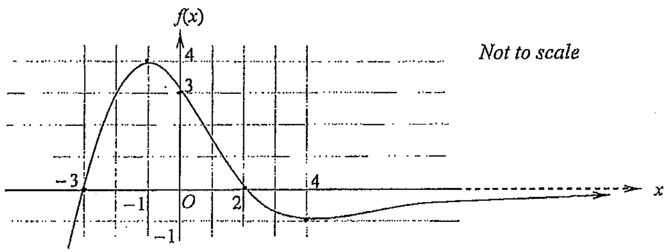
(i)



(ii)



(iii)



Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

2 + 4 = ? (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ~~●~~ C ○ D ○

Trial HSC Examination
Mathematics Extension 2, 2012

Multiple Choice Answer Sheet

SOLUTIONS

Student id number: ANSWERS.

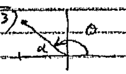
Completely colour in the response oval representing the most correct answer.

1	A ○	B ○	C ●	D ○
2	A ○	B ○	C ●	D ○
3	A ○	B ●	C ○	D ○
4	A ○	B ○	C ●	D ○
5	A ○	B ○	C ●	D ○
6	A ○	B ●	C ○	D ○
7	A ○	B ○	C ●	D ○
8	A ●	B ○	C ○	D ○
9	A ●	B ○	C ○	D ○
10	A ○	B ○	C ○	D ●

Mark: /10

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments
Q11 (a)(i) $z = 1+2i$, $w = -2+i$ $\lvert z\rvert = \sqrt{1^2+2^2}$ $\therefore \lvert z\rvert = \sqrt{5}$		$\boxed{1}$
(ii) $zw = (1+2i)(-2+i)$ $= -2 + i - 4i + 2i^2 = -2 + i - 4i - 2$ $\therefore zw = -4 - 3i$		$\boxed{2}$
(iii) $\dfrac{5}{iw} = \dfrac{5}{i(-2+i)} = \dfrac{5}{-i-2i} \times \dfrac{-1+2i}{-1+2i}$ $= \dfrac{5(-1+2i)}{(-1)^2+4} = \dfrac{5(-1+2i)}{5}$ $\therefore \dfrac{5}{iw} = -1+2i$		$\dfrac{5}{iw} \times \dfrac{\overline{iw}}{\overline{iw}} = \dfrac{5\,\overline{iw}}{\lvert w\rvert^2}$ $= \dfrac{-5i(-2-i)}{5} =$ $\boxed{2}$
(b) $z\bar{z} = 37$ $\quad \dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$ Now $\dfrac{z}{\bar{z}} \times \dfrac{z}{z} = \dfrac{z^2}{z\bar{z}} = \dfrac{z^2}{37}$ $\therefore \dfrac{z^2}{37} = \dfrac{35}{37} + \dfrac{12i}{37} \Rightarrow z^2 = 35 + 12i$ Let $z = x + iy$ $\quad x, y \in \mathbb{R}$ $z^2 = x^2 - y^2 + 2xyi$ $z\bar{z} = x^2 + y^2$ $\therefore x^2 - y^2 = 35$ — (1) $2xy = 12$ $xy = 6$ — (2) $x^2 + y^2 = 37$ — (3) (1) and (3) $\quad 2x^2 = 72$ $x^2 = 36$ $x = \pm 6$ sub in (2) $\quad y = \dfrac{6}{x} = 1$ or -1 $\therefore z = 6+i$ or $-6-i$ $\qquad \pm(6+i)$		$z = x+iy$ $\bar{z} = x - iy$ $z\bar{z} = x^2+y^2$ $z^2 = x^2-y^2+2xyi$ $\boxed{3}$
(c) $w = (-1+i\sqrt{3})^{2012}$ $\quad (-1, \sqrt{3})$  $\tan\alpha = \dfrac{\sqrt{3}}{1} \Rightarrow \alpha = \dfrac{\pi}{3}$ $\operatorname{Arg} w = 2012 \operatorname{Arg}(-1+i\sqrt{3})$ $\quad \therefore \theta = \dfrac{2\pi}{3}$ $= 2012 \times \dfrac{2\pi}{3} = \dfrac{4024\pi}{3}$ $(\div 2\pi)$ $= -\dfrac{2\pi}{3}$		$\dfrac{2012}{3}$ $\boxed{2}$

\\TITAN\StaffHome$\woh08\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

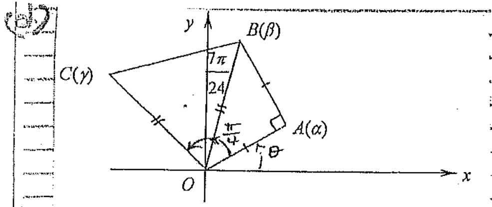
/

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments

(d)



(i) $\angle AOB = \frac{\pi}{4}$ right isosceles $\triangle OAB$

$\angle OCB = \frac{7\pi}{24}$ (equal angles opposite equal sides $OB = OC$)

$2 \times \frac{7\pi}{24} + \angle COB = \pi$ (Angle sum of $\triangle COB$ is π)

$\angle COB = \pi - \frac{7\pi}{12} = \frac{5\pi}{12}$

$\therefore \angle AOC = \frac{\pi}{4} + \frac{5\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$ — 1

(ii) $\vec{OC} = \gamma$, $\operatorname{Arg}\gamma = \theta + \angle AOC = \theta + \frac{2\pi}{3}$, $\operatorname{Arg}\alpha = \theta$

$OC = OB = |\beta| = \sqrt{2}.OA$ (right isosceles $\triangle$, ... Pyth. Thm)

but $OC = |\gamma| = |\beta| = \sqrt{2}\,OA$

$\therefore \gamma = \beta \operatorname{cis}\frac{2\pi}{3} = \sqrt{2}.\alpha.\operatorname{cis}\frac{2\pi}{3} = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\alpha.$ — 2

Enlarge by and Rotate by

(iii) $LHS = 2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2} = 2\alpha^2 + \left[\sqrt{2}\operatorname{cis}\frac{2\pi}{3}\alpha\right]^2 + \alpha.\sqrt{2}\operatorname{cis}\frac{2\pi}{3}.\alpha.\sqrt{2}$

$= 2\alpha^2 + 2\alpha^2\operatorname{cis}\frac{4\pi}{3} + 2\alpha^2\operatorname{cis}\frac{2\pi}{3}$ (de Moivres thm)

$= 2\alpha^2\left[1 + \operatorname{cis}\frac{4\pi}{3} + \operatorname{cis}\frac{2\pi}{3}\right]$

$= 2\alpha^2\left[1 + -\frac{1}{2} - \frac{\sqrt{3}}{2}i + -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

$= 0$ — 2

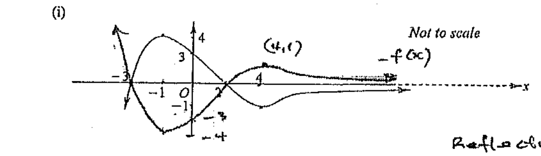
JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments

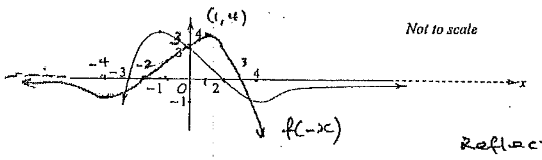
(a) Question 12 (a)

(i)



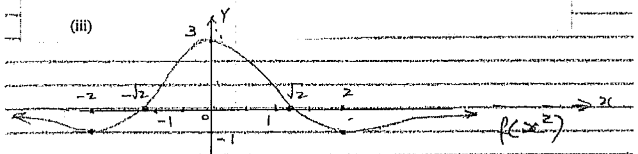
Reflection in x-axis — 1

(ii)



Reflection in y-axis. — 2

(iii)



New x-int: $x^2 = -3$; -1; 2

$\therefore x = \cancel{x}$; $\cancel{x}$; $\pm\sqrt{2}$

$x = 0$ $x^2 = 0$ $f(x^2) = f(0) = 3$ $(0, 3)$

$x = \pm 2$ $x^2 = 4$ $f(x^2) = f(4) = -1.$ $(\pm 2, -1)$ — 2

(b) (i) $y = P(x)$ $\therefore \frac{dy}{dx} = P'(x)$

Gradient of Tangent at $(\alpha, P(\alpha))$ $m_T = P'(\alpha)$

Eqn of Tangent: $y - P(\alpha) = P'(\alpha)(x - \alpha)$

ie $y = P'(\alpha)(x - \alpha) + P(\alpha)$ QED — 1

(ii) $P(x) = (x-\alpha)^2 Q(x) + R(x)$.

Since the divisor $(x-\alpha)^2$ is of degree 2

$\therefore$ deg of remainder $R(x)$ is at most 1.

$\therefore R(x)$ is linear

ie $R(x) = ax + b$ — 1

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments

(b)(iii) $P(x) = (x-\alpha)^2 Q(x) + ax + b$

∴ $P(\alpha) = 0 + a\alpha + b$

$P(\alpha) = a\alpha + b$ — (1)

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x) + a$

∴ $P'(\alpha) = 0 + 0 + a$

∴ $a = P'(\alpha)$ — (2)

so $R(x) = ax + b$

$= P'(\alpha)x + P(\alpha) - a\alpha$

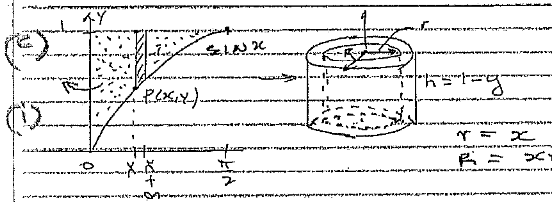
$= P'(\alpha)x - P'(\alpha)\alpha + P(\alpha)$

$= P'(\alpha)(x-\alpha) + P(\alpha)$

∴ Remainder is the same as the tangent at $x=\alpha$ when divide by $(x-\alpha)^2$.

$\boxed{3}$

(c)(i)



Marker's comments: "$r = x - \frac{1}{2}\delta x$" $R = x + \frac{1}{2}\delta x$; $A \doteqdot \pi \times 2x \times \delta x$; "Rectangular Prism"; $A \doteqdot 2\pi x \times (1-y)$; $\delta V \doteqdot 2\pi x(1-y)\delta x$

$A \doteqdot \pi[R^2 - r^2] = \pi[(R+r)(R-r)]$

$A \doteqdot \pi[(2x+\delta x)(\delta x)] = \pi[2x\delta x + (\delta x)^2]$

$A \doteqdot 2\pi x\,\delta x$, neglect $(\delta x)^2 \doteqdot 0$

∴ Volume of shell $\delta V \doteqdot 2\pi x\,\delta x(1-y) = 2\pi x(1-y)\delta x$

Volume of solid $V \doteqdot \sum_1^n 2\pi x(1-\sin x)\delta x$

$= \lim_{\substack{\delta x \to 0 \\ n \to \infty}} \sum_0^{\pi/2} 2\pi x(1-\sin x)\delta x$

$= 2\pi\int_0^{\pi/2} x(1-\sin x)\,dx$.

$\boxed{2}$

(ii) $V = 2\pi\int_0^{\pi/2} x(1-\sin x)\,dx$ $\quad u = x \quad du = dx \quad dV = (1-\sin x)\,dx \quad V = x + \cos x$ METHOD I

$= 2\pi\left\{[x(x+\cos x)]_0^{\pi/2} - \int_0^{\pi/2}(x+\cos x)\,dx\right\}$

$= 2\pi\left\{\frac{\pi}{2}\cdot\frac{\pi}{2} - 0 - \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2}\right\}$

$= 2\pi\left\{\frac{\pi^2}{4} - \left(\frac{\pi^2}{8}+1\right) - 0\right\}$

$= 2\pi\left\{\frac{\pi^2}{4} - \frac{\pi^2}{8} - 1\right\} = 2\pi\left\{\frac{\pi^2}{8} - 1\right\}$

Volume is $\left(\frac{\pi^3}{4} - 2\pi\right) u^3$

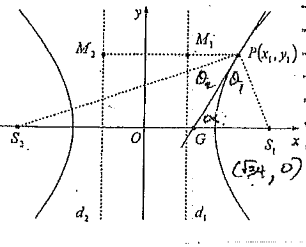
Marker's comment: $2\pi\left[\frac{x^2}{2} + x\cos x - \sin x\right]_0^{\pi/2}$

$\boxed{3}$

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments

(a)(i)



$\frac{x^2}{25} - \frac{y^2}{9} = 1$ $\quad a=5,\ b=3$

$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{25} = \frac{34}{25}$

$e = \frac{\sqrt{34}}{5}$; $e > 1$

$S_1(ae, 0) = (\sqrt{34}, 0)$

$d_1: x = \frac{a}{e} = \frac{25}{\sqrt{34}}$

∴ $S_1P = ePM_1$ (focus-directrix defn.)

$= \frac{\sqrt{34}}{5}\left(x_1 - \frac{25}{\sqrt{34}}\right)$

$S_1P = \frac{\sqrt{34}}{5}x_1 - 5$ qed.

$\boxed{2}$

Marker's comment: Distance formula $S_1P = \sqrt{(\sqrt{34} - x_1)^2 + (0 - y_1)^2}$

(ii) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

$\frac{d}{dx}\left[\frac{x^2}{25} - \frac{y^2}{9}\right] = \frac{d}{dx}1$

$\frac{2x}{25} - \frac{2y\,y'}{9} = 0$

$\frac{dy}{dx} = \frac{9x}{25y}$

Gradient of tangent at P: $m_T = \frac{9x_1}{25y_1}$

Eqn. of Tangent $y - y_1 = \frac{9x_1}{25y_1}(x - x_1)$

$\times \frac{y_1}{9}$ $\quad \frac{y_1 y}{9} - \frac{y_1^2}{9} = \frac{x_1 x}{25} - \frac{x_1^2}{25}$

so $\frac{x_1 x}{25} - \frac{y_1 y}{9} = \frac{x_1^2}{25} - \frac{y_1^2}{9}$

ie $\frac{x_1 x}{25} - \frac{y_1 y}{9} = 1$ qed as $\frac{x_1^2}{25} - \frac{y_1^2}{9} = 1$

$\boxed{2}$

Marker's comment: as $P(x_1, y_1)$ is on Hyp.

(iii) For G $y = 0$ ∴ $\frac{x_1 x}{25} = 1$

ie $x = \frac{25}{x_1}$

ie $G = \left(\frac{25}{x_1}, 0\right)$

$\boxed{1}$

(iv) For $\triangle S_1PG$: $\frac{S_1P}{\sin\alpha} = \frac{S_1G}{\sin\theta_1}$

(1) $\sin\alpha = \frac{S_1P\sin\theta_1}{S_1G} = \frac{\left(\frac{\sqrt{34}}{5}x_1 - 5\right)\sin\theta_1}{\sqrt{34} - \frac{25}{x_1}}$

$= \frac{(\sqrt{34}x_1 - 25)x_1\sin\theta_1}{(\sqrt{34}x_1 - 25)5}$; $\quad x_1 \neq \frac{25}{\sqrt{34}}$.

∴ $\sin\alpha = \frac{x_1}{5}\sin\theta_1$ qed.

$\boxed{2}$

Marker's comment: $S_1G = \sqrt{34} - \frac{25}{x_1}$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments

(iv) (2) In $\triangle S_2PG$, $\angle PGS_2 = \pi - \alpha$

$$\frac{S_2P}{\sin(\pi-\alpha)} = \frac{S_2G}{\sin\theta_2}$$

$$\frac{S_2P}{\sin\alpha} = \frac{S_2G}{\sin\theta_2}$$

$$\sin\alpha = \frac{S_2P\sin\theta_2}{S_2G} \qquad S_2P = ePM_2 = \frac{\sqrt{34}}{5}x_1 + 5 = \frac{\sqrt{34}x_1+25}{5}$$

$$S_2G = \frac{25}{x_1} + \sqrt{34} = \frac{\sqrt{34}x_1+25}{x_1}$$

$$= \frac{x_1}{5}\sin\theta_2$$

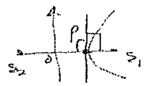
so $\sin\alpha = \frac{x_1}{5}\sin\theta_1 = \frac{x_1}{5}\sin\theta_2$

$\therefore \sin\theta_1 = \sin\theta_2$ $\boxed{2}$

(3) $\sin\theta_1 = \sin\theta_2$

$\therefore \theta_1 = \theta_2$ or $\theta_1 = \pi - \theta_2$

Now $\theta_1 = \pi - \theta_2$ only when $\theta_1 = \theta_2 = \frac{\pi}{2}$ ie. P is at Vertex A



$\therefore \theta_1 \equiv \theta_2$

$\therefore$ GP bisects $\angle S_1PS_2$ $\boxed{2}$

(b) $I_n = \int_{\pi/6}^{\pi/4} \cot^n x\,dx \quad n = 1, 2, 3, \ldots$

(i) $I_1 = \int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x}\,dx = \ln(\sin x)\Big]_{\pi/6}^{\pi/4}$ $\quad \sin x > 0 \ [\frac{\pi}{6}, \frac{\pi}{4}]$

$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2} = \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2} = \ln 2^{\frac{1}{2}}$

$I_1 = \frac{1}{2}\ln 2$ good $\boxed{1}$

(ii) $I_{n-2} + I_n = \int \cot^{n-2}x + \cot^n x\,dx$

$= \int \cot^{n-2}x\,(1 + \cot^2 x)\,dx$

$= \int \cot^{n-2}x \cdot \operatorname{cosec}^2 x\,dx$

$= -\frac{1}{n-1}\cot^{n-1}x\Big]_{\pi/6}^{\pi/4}$

$= -\frac{1}{n-1}\left[1 - (\sqrt{3})^{n-1}\right] = \frac{1}{n-1}\left[3^{\frac{1}{2}(n-1)} - 1\right]$ $\boxed{2}$

(iii) I_5: $n=5$ $\quad I_3 + I_5 = \frac{1}{4}[3^2 - 1] = 2$

$n=3$ $\quad I_1 + I_3 = \frac{1}{2}[3 - 1] = 1$

$\therefore I_5 = 2 - I_3 = 2 - (1 - I_1) = 1 + I_1$

$= 1 + \frac{1}{2}\ln 2.$ $\boxed{1}$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments

(a) Let $S_n = \theta_1 + \theta_2 + \ldots + \theta_n < \frac{\pi}{2}$

Let $P(n)$: $\tan(S_n) \geq \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_n$

For $P(1)$:

LHS $= \tan\theta_1$ $\qquad$ RHS $= \tan\theta_1$

$\therefore P(1)$ is true

Assume $P(n)$ is true up to some integer $k \geq 1$

ie $S_k = \theta_1 + \theta_2 + \ldots + \theta_k < \frac{\pi}{2}$ and

$\tan(S_k) \geq \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_k$ —— (1)

RTP $P(k+1)$ is true

ie $\tan(S_{k+1}) \geq \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_k + \tan\theta_{k+1}$

Proof $P(k+1)$

Now $\tan S_{k+1} = \tan(S_k + \theta_{k+1})$

$= \frac{\tan S_k + \tan\theta_{k+1}}{1 - \tan S_k \cdot \tan\theta_{k+1}}$ $\quad$ * using assumption (1)

$\geq \frac{\tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_k + \tan\theta_{k+1}}{1 - \tan S_k \cdot \tan\theta_{k+1}}$

Now $\theta_1 + \theta_2 + \ldots + \theta_k + \theta_{k+1} < \frac{\pi}{2}$

$\therefore S_k < \frac{\pi}{2} - \theta_{k+1}$ $\qquad 0 < \theta_i < \pi/2$

$\therefore \tan S_k < \tan(\frac{\pi}{2} - \theta_{k+1}) = \frac{1}{\tan\theta_{k+1}}$

$0 < \tan S_k \tan\theta_{k+1} < 1.$

$\therefore 0 < 1 - \tan S_k \tan\theta_{k+1} < 1$

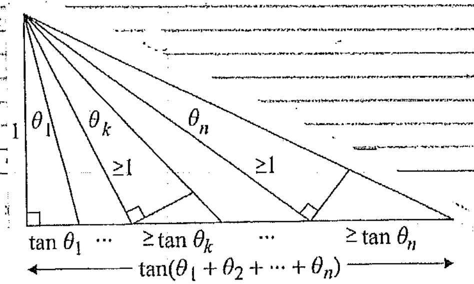
$\therefore \frac{1}{1 - \tan S_k \tan\theta_{k+1}} > 1$

Hence $\tan S_{k+1} \geq \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_{k+1}$

$\therefore P(k+1)$ is true

$\therefore$ By the PMI $P(n)$ is true for $n = 1, 2, 3, \ldots$ $\boxed{3}$

NOTE:



MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments

(b) (i) $\dfrac{300x}{1000+x^3} = \dfrac{a}{10+x} + \dfrac{bx+c}{100-10x+x^2}$

$\therefore\ 300x \equiv a(100-10x+x^2) + (bx+c)(10+x)$

$x=-10$: $-3000 = 300a$
$\therefore\ a = -10$

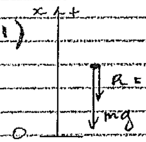
$x=0$: $0 = 100a + 10c$
$10c = -100a = 1000$
$c = 100$

$x=10$: $3000 = 100a + (10b+c)\times 20$
$= -1000 + 20(10b+100)$
$4000 = 200b + 2000$
$\therefore\ b = 10$

$\Rightarrow \begin{cases} a = -10 \\ b = 10 \\ c = 100 \end{cases}$

$\boxed{3}$

(ii) (1)



Data $t=0,\ x=0,\ v=5,\ g=10,\ \ddot{x}=?$

Equation of motion

$m\ddot{x} = -mg - \dfrac{mv^3}{100}$

$\ddot{x} = -10 - \dfrac{v^3}{100} = -\dfrac{1}{100}(1000+v^3)$

$\boxed{1}$

(2) $\ddot{x} = v\dfrac{dv}{dx} = -\dfrac{1}{100}(1000+v^3)$

Max height when $x = H$ $v \equiv 0$

$\displaystyle\int_5^0 \frac{100v\,dv}{1000+v^3} = -\int_0^H dx$

so $\displaystyle\frac{1}{3}\int_5^0 \frac{300v\,dv}{1000+v^3} = \frac{1}{3}\int_5^0 \left(\frac{-10}{10+v} + \frac{10v+100}{100-10v+v^2}\right) dv \equiv -[x]_0^H$

ie $\displaystyle\frac{10}{3}\int_5^0 \left(\frac{-1}{10+v} + \frac{v+10}{100-10v+v^2}\right) dv = -H$

$\displaystyle\frac{10}{3}\int_5^0 \left(\frac{-1}{10+v} + \frac{\frac{1}{2}(2v-10)}{v^2-10v+100} + \frac{15}{75+(v-5)^2}\right) dv = -H$ $\qquad \dfrac{15}{5\sqrt{3}} = \sqrt{3}$

$\dfrac{10}{3}\left[-\ln(10+v) + \dfrac{1}{2}\ln(v^2-10v+100) + \dfrac{15}{5\sqrt{3}}\tan^{-1}\dfrac{(v-5)}{5\sqrt{3}}\right]_5^0 = -H$

$-H = \dfrac{10}{3}\left[\left(\cancel{-\ln 10 + \tfrac{1}{2}\ln 100} + \sqrt{3}\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)\right) - \left(-\ln 15 + \tfrac{1}{2}(\ln 75 + 0)\right)\right]$ $\quad \boxed{3}$

$-H = \dfrac{10}{3}\left[-\sqrt{3}\cdot\dfrac{\pi}{6} + \ln 15 - \dfrac{1}{2}\ln 75\right]$

$H = \dfrac{10}{3}\left[\dfrac{\pi\sqrt{3}}{6} + \dfrac{1}{2}\ln\dfrac{5}{3}\right] \doteq 1.19197\ldots$

$\therefore$ Max height is $\underline{1.2\text{ m}}$ (1dp)

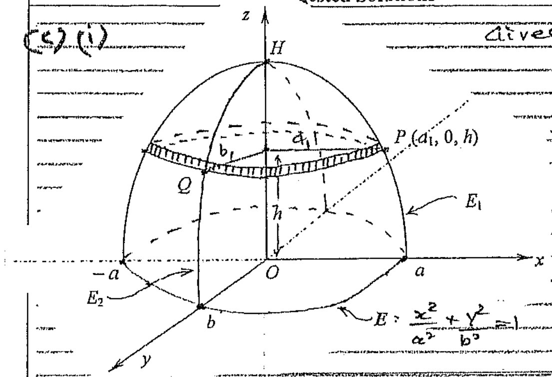
$\boxed{8.}$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments

(c) (i)



Given

$E_1:\ \dfrac{x^2}{a^2} + \dfrac{z^2}{H^2} = 1$

so

$E_2:\ \dfrac{y^2}{b^2} + \dfrac{z^2}{H^2} = 1$

$\boxed{1}$

(ii) Since P lies on E_1 at $z=h$

$\dfrac{a_1^2}{a^2} + \dfrac{h^2}{H^2} = 1$ ie $a_1^2 = a^2\left(1-\dfrac{h^2}{H^2}\right)$

And Q lies on E_2 at $z=h$

$\dfrac{b_1^2}{b^2} + \dfrac{h^2}{H^2} = 1$ ie $b_1^2 = b^2\left(1-\dfrac{h^2}{H^2}\right)$

$\therefore$ Area of slice at $z=h$: $A \doteq \pi a_1 b_1$

ie $A \doteq \pi a\sqrt{1-\dfrac{h^2}{H^2}} \times b\sqrt{1-\dfrac{h^2}{H^2}}$

$A = \pi ab\left(1-\dfrac{h^2}{H^2}\right)$ qed

$\boxed{2}$

(iii) Volume of slice $\delta V \doteq \pi ab\left(1-\dfrac{h^2}{H^2}\right)\times\delta h$ $\qquad$ Let $\delta h \equiv \Delta h$

Volume of solid $V \doteq \displaystyle\sum_1^n \pi ab\left(1-\frac{h^2}{H^2}\right)\delta h$

$= \displaystyle\lim_{\delta h\to 0}\sum_0^H \pi ab\left(1-\frac{h^2}{H^2}\right)\delta h$

$V = \pi ab\displaystyle\int_0^H 1-\frac{h^2}{H^2}\,dh$

$= \pi ab\left[h - \dfrac{h^3}{3H^2}\right]_0^H$

$V = \pi ab\left[H - \dfrac{H^3}{3H^2}\right] = \dfrac{2}{3}\pi abH$

$\boxed{2}$

$\therefore$ volume is $\dfrac{2}{3}abH\pi$ u³

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 15

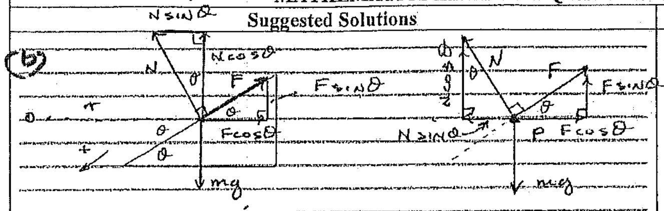
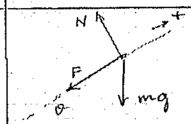
Suggested Solutions	Marks	Marker's Comments
(a) $x^3+ax^2+bx-54=0$		
(i) As $\alpha^2+\beta^2=0$ — (1) $\alpha^2+\gamma^2=0$ — (2) $\therefore\ \beta^2=\gamma^2 \Rightarrow \beta^2-\gamma^2=0$ $\beta=\pm\gamma$ but α,β and γ are distinct $\therefore\ \beta\neq\gamma$ $\therefore\ \beta=-\gamma \Rightarrow \beta+\gamma=0.$	2	
(ii) Now $s_1=\Sigma\alpha=\alpha+\beta+\gamma=-a$ ie $\alpha+0=-a$ $\alpha=-a$ As a is real (data) $\therefore\ \alpha$ is real	2	$a\in\mathbb{R}$ $\alpha=-a$ $\Rightarrow \alpha\in\mathbb{R}$
(iii) Now $\alpha^2+\beta^2=0$ $\beta^2=-\alpha^2=i^2\alpha^2$ $\beta=\pm i\alpha$ As $\alpha\in\mathbb{R}$ $\therefore\ \beta$ is purely imaginary As $\gamma=-\beta=\mp i\alpha$. So again γ is purely imaginary	2	
(iv) Now $s_3=\Sigma\alpha\beta\gamma=\alpha\beta\gamma=-(-54)=54$ but $\gamma=-\beta$ $\therefore\ -\alpha\beta^2=54$ but $\beta^2=-\alpha^2$ $\therefore\ \alpha^3=54$ $\alpha=\sqrt[3]{54}\in\mathbb{R}$ $\therefore\ a=-\alpha=-\sqrt[3]{54}=-3\sqrt[3]{2}=-(54)^{\frac{1}{3}}$ $s_2=\Sigma\alpha\beta=\alpha\beta+\beta\gamma+\gamma\alpha=b$ $\alpha(\beta+\gamma)+\beta\gamma=b$ $\Rightarrow\ \beta(-\beta)=b$ $-\beta^2=b$ $+\alpha^2=b$ $\therefore\ b=(-\sqrt[3]{54})^2=\sqrt[3]{54^2}=54^{\frac{2}{3}}$ So $a=-\sqrt[3]{54}$ and $b=\sqrt[3]{54^2}$	2	

\\TITAN\StaffHome$\woh08\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

$54^2=2916$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 15

Suggested Solutions	Marks	Marker's Comments
(b) 		
(i) Given vertical resolution at P is $N\cos\theta+F\sin\theta=mg$ HORIZONTAL is $N\sin\theta-F\cos\theta=\frac{mv^2}{r}$	1	(Newton's II)
(ii) $r=80,\ \angle\theta=45^\circ,\ g=10$ $\quad F\le\frac{N}{9}$ So min. speed when $F=\frac{N}{9}$ or $N=9F$ $\therefore\ \frac{9F}{\sqrt2}+\frac{F}{\sqrt2}=m\times10$ $F=m\sqrt2$ and $\frac{9F}{\sqrt2}-\frac{F}{\sqrt2}=\frac{mv^2}{80}$ $\therefore\ mv^2=\frac{800F}{\sqrt2}=640m$ $\therefore\ v^2=640$ $v=8\sqrt{10}$ min speed is $8\sqrt{10}$ m/s [91.1 kmph]	2	$\frac{8F}{\sqrt2}=\frac{mv^2}{80}$ $mv^2=\frac{640F}{\sqrt2}$
(iii) For upwards motion VERTICAL RESOLUTION: $N\cos\theta-F\sin\theta=mg$ ie $\frac{N}{\sqrt2}-\frac{F}{\sqrt2}=10m$ ie $\frac{8F}{\sqrt2}=10m$ ie $F=\frac{10m\sqrt2}{8}$ HORIZONTAL: $\frac{N}{\sqrt2}+\frac{F}{\sqrt2}=\frac{mv^2}{80}$ $mv^2=\frac{800F}{\sqrt2}=\frac{800}{\sqrt2}\times\frac{10\times m\sqrt2}{8}=1000m$ $v^2=10\,000$ $v=\sqrt{1000}=10\sqrt{10}$ max speed is $10\sqrt{10}$ m/s [113.8 kmph]	3	 $\frac{10F}{\sqrt2}=\frac{mv^2}{80}$
(iv) when no slipping $F\equiv0$ so from (ii) or (iii) $\frac{N}{\sqrt2}=10m$ and $\frac{N}{\sqrt2}=\frac{mv^2}{80}$ $\Rightarrow\ v^2=800$ ie $v=20\sqrt2$ [101.8 kmph] $\therefore$ the optimum speed for no slippage at 45°, is $20\sqrt2$ m/s	1	• can show $V^2=V_{min}\times V_{max}$ and/or use $\tan\theta=\frac{V^2}{rg}$

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments

(a) $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

x	u
1	0
0	1

$x = \frac{1-u}{1+u}$

$\frac{dx}{du} = \frac{-1(1+u) - (1-u)\times 1}{(1+u)^2}$

$= \frac{-1-u-1+u}{(1+u)^2} = \frac{-2}{(1+u)^2}$

$dx = \frac{-2}{(1+u)^2}\,du$

$1+x = 1 + \frac{1-u}{1+u} = \frac{2}{1+u}$

$1+x^2 = 1 + \frac{(1-u)^2}{(1+u)^2} = \frac{1+2u+u^2+1-2u+u^2}{(1+u)^2}$

$1+x^2 = \frac{2(1+u^2)}{(1+u)^2}$

$\therefore I = \int_1^0 \frac{\ln\frac{2}{1+u} \times \frac{-2}{(1+u)^2}}{\frac{2(1+u^2)}{(1+u)^2}}\,du$

$= \int_1^0 \frac{-2\ln\frac{2}{1+u}}{2(1+u^2)}\,du$

$I = +\int_0^1 \frac{\ln 2}{1+u^2} - \frac{\ln(1+u)}{1+u^2}\,du$

$= \int_0^1 \frac{\ln 2\,dx}{1+x^2} - I$ u is a dummy variable

$2I = \ln 2.\left[\tan^{-1}x\right]_0^1 = \ln 2.\left(\frac{\pi}{4}\right)$

$\therefore I = \ln 2.\frac{\pi}{8}$ q.e.d.

$\boxed{4}$

(b) $x^4 + 3x - 1 = 0$ METHOD I

so $\alpha^4 + 3\alpha - 1 = 0$

so $\sum^4 \alpha_i^4 + 3\sum^4 \alpha_i - \sum 1 = 0$

$\sum \alpha^4 = -3(\alpha+\beta+\gamma+\delta) + 4\times 1 = -3\times 0 + 4$

$\alpha^4+\beta^4+\gamma^4+\delta^4 = 4$

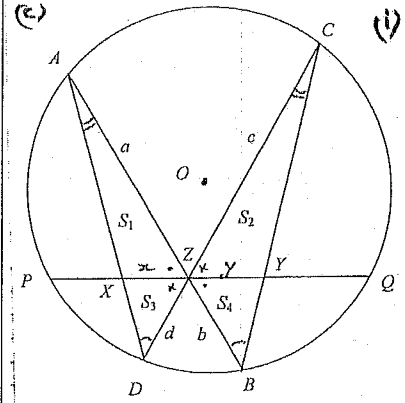
Method II construct Polyn whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ ie $81x = (1-x)^4$ ie $x^4 - 4x^3 + 6x^2 - 85x + 1 = 0$ hence ...

$\boxed{2}$

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments

(c)



could use ||| Δs

(i) $S_1 = \frac{1}{2} a.AX \sin A$

$S_2 = \frac{1}{2} c.CY.\sin C$

As $\angle C = \angle A$ (angles subtended at circumference are equal standing on arc DB)

$\therefore \sin C = \sin A$

$\therefore \frac{S_1}{S_2} = \frac{a.AX}{c.CY}$ QED

$S_1 = \frac{1}{2} a.x.\sin X\hat{Z}A$

$S_4 = \frac{1}{2} b y \sin B\hat{Z}Y$

but $\angle BZY = \angle XZA$ (vertically opposite angles equal)

$\therefore \frac{S_1}{S_4} = \frac{a.x}{b.y}$ QED

$\boxed{3}$

(ii) similarly $\angle D = \angle B$ (angles subtended at circumference are equal standing on arc AC)

so $\frac{S_3}{S_4} = \frac{\frac{1}{2}DX.d.\sin D}{\frac{1}{2}BY.b.\sin B} = \frac{d.XD}{b.YB}$

so $\frac{S_1}{S_2} \times \frac{S_3}{S_4} = \frac{a.AX}{c.CY} \times \frac{d.XD}{b.YB} = \frac{a.d \times AX.XD}{b.c \times CY.YB}$

Now $\frac{S_3}{S_2} = \frac{\frac{1}{2}.d.x.\sin D\hat{Z}X}{\frac{1}{2}.c.y.\sin C\hat{Z}Y} = \frac{d.x}{c.y}$

$\therefore \frac{S_1}{S_4} \times \frac{S_3}{S_2} = \frac{a.x}{b.y} \times \frac{d.x}{c.y} = \frac{a.d.x^2}{b.c.y^2}$ QED

$\boxed{2}$

(iii) $\therefore \frac{a.d \times AX.XD}{b.c \times CY.YB} = \frac{a.d.x^2}{b.c.y^2}$

so $\frac{x^2}{y^2} = \frac{AX.XD}{CY.YB} = \frac{PX.XQ}{PY.YQ} = \frac{(p-x)(x+q)}{(p+y)(q-y)}$

(the products of the intercepts of 2 intersecting chords are equal)

$\boxed{1}$

MATHEMATICS Extension 2: Question 16		
Suggested Solutions	Marks	Marker's Comments

(c) (iv) $\dfrac{x^2}{y^2} = \dfrac{(p-x)(x+q)}{(p+y)(q-y)}$

$= \dfrac{px + pq - x^2 - qx}{pq - py + qy - y^2}$

$x^2pq - x^2yp + x^2yq - \cancel{x^2y^2} = xy^2p + y^2pq - \cancel{x^2y^2} - xy^2q$

$pq(x^2 - y^2) = x^2yp - x^2yq + xy^2p - xy^2q$

$pq(x-y)(x+y) = x^2y(p-q) + xy^2(p-q)$

$pq(x-y)(x+y) = xy(p-q)(x+y)$

so $pq(x-y) = xy(p-q)$

ie $\dfrac{x-y}{xy} = \dfrac{p-q}{pq}$

$\therefore \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{q} - \dfrac{1}{p} \Rightarrow \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{p} - \dfrac{1}{q}$

$\boxed{2}$

(v) If Z is the mid point

$\therefore p = q$

$\therefore \dfrac{1}{p} - \dfrac{1}{q} = 0$

$\Rightarrow x = y$.

$\boxed{1}$