Name:
Class:

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2012

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged working.

Total Marks 70

Section I: **10 marks**

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: **60 Marks**

- Attempt Question 11 - 14
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \textit{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

SECTION I MULTIPLE CHOICE (10 marks)

Attempt Question 1 – 10 (1 mark each)
Allow approximately 15 minutes for this section

1 A bowl of soup at temperature T°C, when placed in a cooler environment, loses heat according to the law $\frac{dT}{dt} = k(T - T_0)$ where t is the time elapsed in minutes and T_0 is the temperature of the environment in degrees Celsius. A bowl of soup at 96°C is left to stand in a room at a temperature of 18°C. After 3 minutes the soup cools down to 75°C. What is the value of k correct to 4 decimal places?

(A) -0.0784
(B) -0.0856
(C) -0.1046
(D) -0.1236

2 Which of the following is an expression for $\int \cos^2 2x\, dx$?

(A) $x - \frac{1}{4}\sin 4x + c$
(B) $x + \frac{1}{4}\sin 4x + c$
(C) $\frac{x}{2} - \frac{1}{8}\sin 4x + c$
(D) $\frac{x}{2} + \frac{1}{8}\sin 4x + c$

3 The velocity of a particle moving in a straight line is given by $v = 2x + 3$ where x metres is the distance from fixed point O and v is the velocity in metres per second. What is the acceleration of the particle when it is 4 metres from O?

(A) $a = 11\text{ms}^{-2}$
(B) $a = 19\text{ms}^{-2}$
(C) $a = 22\text{ms}^{-2}$
(D) $a = 23.5\text{ms}^{-2}$

4 Which of the following is an expression for $\int x\sqrt{1 - x^2}dx$?
Use the substitution $u = 1 - x^2$.

(A) $\frac{-(1-x^2)^3}{3} + c$
(B) $\frac{(1-x^2)^3}{3} + c$
(C) $\frac{-(1-x^2)^{\frac{3}{2}}}{3} + c$
(D) $\frac{(1-x^2)^{\frac{3}{2}}}{3} + c$

5 What are the solutions to the equation $e^{6x} - 7e^{3x} + 6 = 0$?

(A) $x = 1$ and $x = 6$
(B) $x = 0$ and $x = \frac{\ln 6}{2}$
(C) $x = 0$ and $x = \frac{\ln 6}{3}$
(D) $x = 1$ and $x = \frac{\ln 6}{2}$

6 A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin in metres and v is its velocity in ms^{-1}. The motion is simple harmonic. What is the amplitude?

(A) 2π metres
(B) 3 metres
(C) 8 metres
(D) 9 metres

7 How many distinct permutations of the letters of the word 'ATTAINS' are possible in a straight line when the word begins and ends with the letter T?

(A) 60
(B) 120
(C) 360
(D) 1260

8 If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$
(B) $f^{-1}(x) = e^{y+2}$
(C) $f^{-1}(x) = \log_e x - 2$
(D) $f^{-1}(x) = \log_e x + 2$

9 What is the coefficient of x^5 in the expansion of $(1 - 3x + 2x^3)(1 - 2x)^6$?

(A) -792
(B) -720
(C) 120
(D) 312

10 A die is tossed 3 times. What is the probability of 0 or 1 six turning up?

(A) $\frac{2}{27}$
(B) $\frac{25}{27}$
(C) $\frac{91}{216}$
(D) $\frac{125}{216}$

SECTION II EXTENDED RESPONSE (60 marks)

Total Marks is 60
Attempt Question 11 – 14.
Allow approximately 1 hour & 45 minutes for this section.

Answer all questions, starting each new question on a new sheet of paper with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.

QUESTION 11 *(15 Marks)*

Marks

(a) Solve $\frac{4}{3x+1} < 5$. **2**

(b) If α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x - 5 = 0$, find the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$. **2**

(c) Use the substitution $x = u^2 + 1$ for $u > 0$ to evaluate the integral: **4**

$$\int_1^5 (x+1)\sqrt{x-1}\,dx$$

(d) A series is given by $1 + \frac{1-p}{p} + \left(\frac{1-p}{p}\right)^2 + \cdots$, where p is positive.

(i) Find the domain of p such that the series has a sum to infinity. **2**

(ii) Find this sum to infinity in terms of p. **1**

(e) Prove that the tangent to a parabola $x^2 = 4ay$ at a given point $P(2ap, ap^2)$ is equally inclined to the axis of the parabola and the focal chord through the point. **4**

QUESTION 12 *(15 Marks)* START A NEW PAGE

Marks

(a) Solve: $x^3 + 2x^2 - 5x - 6 = 0$ **2**

(b) When the polynomial $P(x)$ is divided by $x^2 - 1$, the remainder is $3x + 1$. What is the remainder when $P(x)$ is divided by $x + 1$? **2**

(c) In how many ways can 4 men and 4 women be arranged around a circular table if:

(i) All women sit together? **2**

(ii) All the men are in pairs separated by two pairs of women? **2**

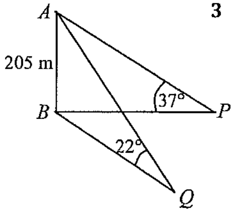
(d) Find the general solution to: $\cos 5\theta - \cos 2\theta = 0$ **2**

(e) A thin-walled cone-shaped cup is to hold 36π cm^3 of water when full. What dimensions will minimize the amount of material needed for the cup? [You may make use of the formula $A = \pi rs$, where s is the slant height of a cone] **5**

QUESTION 13 *(15 Marks)* START A NEW PAGE **Marks**

(a) A is 205 metres above the horizontal plane BPQ. AB is vertical. The angle of elevation of A from P is 37° and the angle of elevation of A from Q is 22°. P is due East of B and Q is South 47° East from B. **3**

Calculate the distance from P to Q, to the nearest metre.



NOT TO SCALE

(b) Use mathematical induction to show: **4**

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

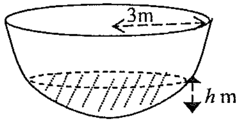
for $n \geq 2$ where n is an integer.

(c) A particle moves in SHM on a horizontal line and its acceleration is $\frac{d^2x}{dt^2} = 36 - 9x$, where x is the displacement after t seconds.

(i) Find the centre of its motion. **1**

(ii) If the particle is initially at rest at $x = 6$, find the amplitude. **1**

(d) A hemi-spherical bowl has a radius of 3m. Oil is poured in at a constant rate of $\frac{\pi}{3}$m^3/min.



NOT TO SCALE

(i) Show that, when the depth of the oil is h metres, the volume of oil is: **2**

$$V = \frac{\pi}{3}(9h^2 - h^3)\text{m}^3$$

(ii) How deep is the oil after 8 minutes? **2**

(iii) At what rate is h increasing at this time? **2**

QUESTION 14 *(15 Marks)* START A NEW PAGE **Marks**

(a) A particle is moving in a straight line and its position x, in metres, from the origin O at time t seconds is given by

$$x = 3\cos 2t + 4\sin 2t + 2.$$

(i) Express $3\cos 2t + 4\sin 2t$ in the form **2**

$$R\cos(2t - \alpha) \quad \text{where } 0 < \alpha < \frac{\pi}{2} \text{ and } R > 0.$$

(ii) Prove that the particle is undergoing simple harmonic motion. Find the amplitude of the motion. **2**

(iii) Find the maximum speed of the particle. When does the particle **first** reach this maximum speed? Provide your answer to 2 decimal places. **2**

(b) Given the binomial expansion of

$$(1+x)^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \text{ and}$$

$$(1+x)^{n+1} = b_0 + b_1x + b_2x^2 + \cdots + b_{n+1}x^{n+1}$$

(i) Find the relationship for co-efficient b_k in terms of a_r. **1**

(ii) Hence find the expression, in terms of n only, of: **3**

$$\frac{1}{a_0a_1\ldots a_n} \times (a_0 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n) \quad \text{for } n = 1, 2, 3 \ldots$$

(c) (i) Show that $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right)$ for $n \geq 1$. **2**

(ii) Hence or otherwise show that: **3**

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) + \frac{3\pi}{4}$$

END OF PAPER

M. EXTENSION 1 TRIAL 2012

Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

2 + 4 = ?   (A) 2   (B) 6   (C) 8   (D) 9

A ○   B ●   C ○   D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie   A ●   B ☒   C ○   D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ●   B ☒   C ○   D ○

Trial HSC Examination
Mathematics Extension 1, 2012

Multiple Choice Answer Sheet

Student id number: ☐☐☐☐

Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	●	D	○
2	A	○	B	○	C	○	D	●
3	A	○	B	○	C	●	D	○
4	A	○	B	○	C	●	D	○
5	A	○	B	○	C	●	D	○
6	A	○	B	●	C	○	D	○
7	A	●	B	○	C	○	D	○
8	A	○	B	○	C	●	D	○
9	A	●	B	○	C	○	D	○
10	A	○	B	●	C	○	D	○

Mark: /10

1/5

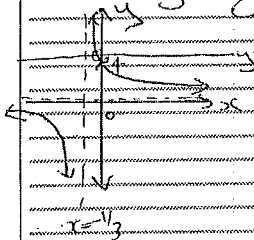
MATHEMATICS Extension 1 : Question 11

Suggested Solutions | **Marks** | **Marker's Comments**

(a) $\frac{4}{3x+1} < 5$

could have been done using a graph

or by algebra



pt of intersection:

$\frac{4}{3x+1} = 5$

$4 = 15x + 5$

$-1 = 15x$

$x = -\frac{1}{15}$

$\therefore x > -\frac{1}{15}$ or $x < -\frac{1}{3}$

Or using algebra:

$(3x+1)^2 \frac{4}{3x+1} < 5 (3x+1)^2$

METHOD II:

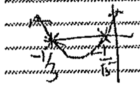
$4(3x+1) < 5(9x^2 + 6x + 1)$

$12x + 4 < 45x^2 + 30x + 5$

$-45x^2 - 18x - 1 < 0$

$45x^2 + 18x + 1 > 0$

$(15x+1)(3x+1) > 0$



$\therefore x < -\frac{1}{3}$ or $x > -\frac{1}{15}$

(b) $\alpha + \beta + \gamma = -b/a = -2$ ½

$\alpha\beta\gamma = -d/a = 5$ ½

$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$ ½

$= -2/5$ ½

(c) $x = u^2 + 1$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$ ½

Marker's Comments:

* ½ mk off if the signs were wrong

* If you only got $x > -\frac{1}{15}$, you scored one mk only

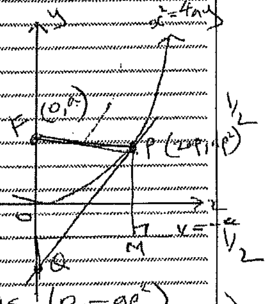
*A lot of students were confused with the signs!

MATHEMATICS Extension 1 : Question...11...

Suggested Solutions	Marks	Marker's Comments
when $x=1$ $u=0$ $x=5$ $u=2$	$\}$ 1/2	
$\int_1^5 (x+1)\sqrt{x-1}\,dx = \int_0^2 (u^2+1+1)\sqrt{u^2+1-1}\,.2u\,du$		1/2 mk
$=\int_0^2 (u^2+2)u.2u\,du$		1/2
$=2\int_0^2 (u^4+2u^2)\,du$		1/2
$=2\left[\frac{u^5}{5}+\frac{2u^3}{3}\right]_0^2$		1/2
$=2\left(\frac{32}{5}+\frac{2\times8}{3}-0\right)$		1/2
$=\frac{64}{5}+\frac{32}{3}$		* Answers of "$\frac{64}{5}$" or "8" → 3mks max.
$=\frac{352}{15}$		
$=23\frac{7}{15}$	1/2	
(d) (i) For there to be a sum to infinity		
$\lvert r\rvert<1$		* lost 1mk if they wrote $0<r<1$ or if they wrote $r<1 \Rightarrow$ 1mk.
ie $-1<r<1$		
$\therefore -1<\frac{1-p}{p}<1$ $\}$	1mk	
$-1<\frac{1-p}{p}$ or $\frac{1-p}{p}<1$		* lost one mark if they squared the absolute value expression
$-\cancel{p}<1-\cancel{p}$ $1-p<p$ $\frac{1}{2}$mk.		
no soln, $\frac{1}{2}$mk $1<2p$		
$\frac{1}{2}<p$		or $\left\lvert\frac{1-p}{p}\right\rvert<1$
$\therefore p>\frac{1}{2}$		$\frac{\lvert 1-p\rvert}{\lvert p\rvert}<1$
$p>0$ (data) $\therefore p>\frac{1}{2}$		$p\neq0$ $\lvert 1-p\rvert<\lvert p\rvert$ etc

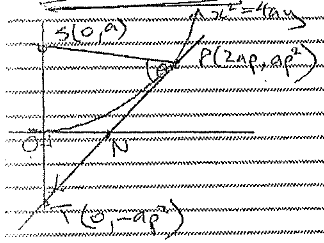
---

MATHEMATICS Extension 1 : Question...11...

Suggested Solutions	Marks	Marker's Comments
d (ii) $S_\infty=\frac{a}{1-r}$		0 mks if they got the formula wrong!!
$=\frac{1}{1-\frac{1-p}{p}}$	$\}$ 1/2	
$=\frac{p}{p-(1-p)}$		
$=\frac{p}{p-1+p}$		
$=\frac{p}{2p-1}$, $p>\frac{1}{2}$	$\}$ $\frac{1}{2}$	
(e) METHOD ONE 		
when $x=2ap$, $y=ap^2$		
$\frac{dy}{dx}=p$	1/2	
eqn of tangent is:		
$y-ap^2=p(x-2ap)$		
$y=px-ap^2$	1/2	
y intercept is Q, Q is $(0,-ap^2)$		
$\therefore d_{FQ}=ap^2+a$	$\}$ $\frac{1}{2}$	
$=a(p^2+1)$		
$d_{FP}=\sqrt{(2ap-0)^2+(ap^2-a)^2}$		$d_{FP}=d_{PM}$ focus directrix defn
$=\sqrt{4a^2p^2+a^2p^4-2a^2p^2+a^2}$	$\}$ $\frac{1}{2}$	$FP=PM$ $=ap^2-(-a)$ $=ap^2+a$
$=\sqrt{a^2p^4+2a^2p^2+a^2}$		
$=a\sqrt{p^4+2p^2+1}$		
$=a\sqrt{(p^2+1)^2}$	$\}$ $\frac{1}{2}$	
$=a(p^2+1)$		
$\therefore d_{FP}=d_{FQ}$ (both $a(p^2+1)$)	1/2	

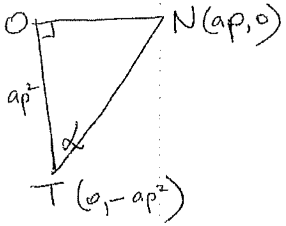
4/5

MATHEMATICS Extension 1 : Question 11....

Suggested Solutions	Marks	Marker's Comments
$\therefore \angle FRP = \angle FPQ$ (equal angles are opposite equal sides in ΔFRP)		½
$\therefore$ the tangent is equally inclined to the focal chord and the axis		} ½
METHOD 2:  		½ mk for a decent diagram with data marked on it!!
gradient of tangent $= p$ gradient of focal chord $= \frac{ap^2-a}{2ap-0}$ $= \frac{a(p^2-1)}{2ap}$ $= \frac{p^2-1}{2p}$	}	½ mk for both correct gradients
In ΔSPT, $\tan\theta = \left\lvert \frac{m_1-m_2}{1+m_1m_2} \right\rvert$ $= \left\lvert \frac{p-\left(\frac{p^2-1}{2p}\right)}{1+\frac{p(p^2-1)}{2p}} \right\rvert$	½	
$= \left\lvert \frac{2p^2-(p^2-1)}{2p+p(p^2-1)} \right\rvert$ $= \left\lvert \frac{p^2+1}{p^3+2p-p} \right\rvert$	½	
$= \left\lvert \frac{p^2+1}{p^3+p} \right\rvert$ $= \left\lvert \frac{p^2+1}{p(p^2+1)} \right\rvert = \frac{1}{p}$	½	

5/5

the gradient of the y-axis is undefined



In ΔONT,

$\tan\alpha = \frac{ap}{ap^2}$

$\therefore \tan\alpha = \frac{1}{p}$

} 1mk

$\therefore \tan\alpha = \tan\theta = \frac{1}{p}$

conclusion

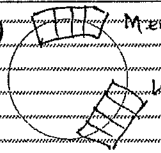
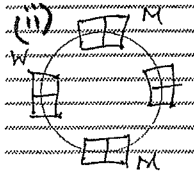
} ½mk

* Fudging resulted in a maximum of 2mks.

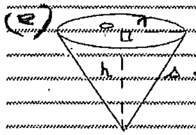
* students were making statements without any justifications, no evidence ∴ they lost marks!!!

JRAHS MATH EXT 1 TRIAL, 2012

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	**Marks**	**Marker's Comments**
(a) Let $P(x) = x^3 + 2x^2 - 5x - 6$ possible integer/rational zeros $\pm1, \pm2, \pm3, \pm6$ TEST : $x-1$: $P(1) = 1+2-5-6 = -8 \neq 0$ $x+1$: $P(-1) = -1+2+5-6 = 0$ $\therefore$ $x+1$ is a factor // $x=-1$ is a root (Factor Thm) $x+1 \overline{)\,x^3+2x^2-5x-6}$ quotient x^2+x-6 x^3+x^2 $0 \quad x^2-5x$ x^2+x $0 \quad -6x-6$ $-6x-6$ $-1)\ 1 \quad 2 \quad -5 \quad -6$ $-1 \quad -1 \quad 6$ $1 \quad 1 \quad -6 \quad 0$ $\therefore (x+1)(x^2+x-6) = (x+1)(x+3)(x-2) = 0$ $\therefore$ $\underline{x = -3, -1 \text{ or } 2}$	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\boxed{2}$ $\frac{1}{2}$ For either −3 or 2 $-\frac{1}{2}$ if +1 or +3
(b) $P(x) = A(x)Q(x) + R(x)$ $= (x^2-1)Q(x) + 3x+1$ $x+1$ $P(-1) = 0 \times Q(-1) + 3(-1) + 1$ $P(-1) = -2$ $\therefore$ the remainder is -2 when divide by $x+1$ (Rem. Thm)	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\boxed{2}$
(c) (i)  Men 4M together and 4W together Women $n(E) = \overset{M}{4!} \times \overset{W}{4!} = (4!)^2$ $= \underline{576 \text{ ways}}$	1 (1)	$\frac{2!}{2!} \times 4! \times 4!$ $\boxed{2}$ $\frac{5!}{5} \times 4!$ $(4W, M_1, M_2, M_3, M_4)$
(ii)  M_1 other M W Method I : $(2 \times 3!) \times 4!$ $n(E) = (2 \times 3!) \times 4! = \underline{288}$ Method II $n(E) = \left(\frac{{}^4C_2}{2!} \times 2!\right) \times 4! = \underline{288}$ 1st 2M other METHOD III $n(E) = 3 \times 2! \times 2! \times \overset{W}{4!} = \underline{288}$ who pair with M Method IV $n(E) = 2 \times 2^4 \times 3 \times 3 = \underline{288}$ opposites in 2, each group, W M		1 For '2×3!' or equiv. $\frac{1}{2}$ 4! ✓ $\frac{1}{2}$ 288 $\boxed{2}$ Very UNCLEAR ARGUMENTS by students 144 common 576 but …

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	**Marks**	**Marker's Comments**
(d) $\therefore \cos 5\theta = \cos 2\theta$ METHOD I $5\theta = 2m\pi \pm 2\theta$ ✓ $\theta = \begin{cases} \frac{2m\pi}{7} \\ \frac{2m\pi}{3} \end{cases}, m \in \mathbb{Z}$ ✓ METHOD II $2\theta = 2n\pi \pm 5\theta$ $\theta = \begin{cases} \frac{2n\pi}{7} \\ -\frac{2n\pi}{3} \end{cases}, n \in \mathbb{Z}$		Expansions ☹ $\boxed{2}$ $\frac{1}{2}$ $\frac{1}{2}$ each
(e)  $V = \frac{1}{3}\pi r^2 h = 36\pi$ $r^2 h = 108$ $h = \frac{108}{r^2}$ $s^2 = r^2 + h^2$ (Pyth. thm) $s^2 = r^2 + \frac{108^2}{r^4} = \frac{r^6 + 108^2}{r^4}$ $s = \frac{\sqrt{r^6 + 108^2}}{r^2}$ Now $A(r) = \pi r s$ $= \pi r \frac{\sqrt{r^6 + 108^2}}{r^2}$ $A(r) = \frac{\pi}{r}\sqrt{r^6 + 108^2}$ $\frac{dA}{dr} = \pi\left[-\frac{1}{r^2}\sqrt{r^6 + 108^2} + \frac{1}{r}\frac{6r^5}{2\sqrt{r^6+108^2}}\right]$ $= \pi\left[\frac{3r^4}{\sqrt{r^6+108^2}} - \frac{\sqrt{r^6+108^2}}{r^2}\right]$ For possible max/min value of A to occur $\frac{dA}{dr} = 0$ $\therefore \frac{3r^4}{\sqrt{r^6+108^2}} = \frac{\sqrt{r^6+108^2}}{r^2}$ $3r^6 = r^6 + 108^2$ $2r^6 = 108^2 = 11664$ $r^6 = 5832 = 54 \times 108 = 2 \times 3^6$ $r = \sqrt[6]{5832} = 3\sqrt{2} = 4.24\ldots$ as $r > 0$	$\frac{1}{2}$ $\frac{1}{2}$ $(\frac{1}{2})$ 1 1 $\frac{1}{2}$	$108^2 = 11664$ $s^2 = \frac{108}{h} + h^2 = \frac{108 + h^3}{h}$ $s = \sqrt{h^2 + \frac{108}{h}} = \frac{\sqrt{h^3+108}}{\sqrt{h}}$ $A = \pi\sqrt{\frac{108}{h}}\sqrt{h^2 + \frac{108}{h}}$ $A(h) = \pi\sqrt{108h + \frac{108^2}{h^2}}$ $A'(h) = \frac{\pi}{2\sqrt{\ldots}}\left(108 - \frac{2 \times 108^2}{h^3}\right)$ $= \frac{\pi}{\sqrt{\ldots}}\left(54 - \frac{108^2}{h^3}\right)$ $54 = \frac{108^2}{h^3}$ $h^3 = \frac{108^2}{54} = 216$ $h = 6, (h > 0)$

$r = 4.24 \quad A' = -0.06$

MATHEMATICS Extension 1 : Question 12

Suggested Solutions | **Marks** | **Marker's Comments**

CONT:

(e) $r = 3\sqrt{2} = \sqrt{18} = 4.24\ldots$

$\therefore h = \frac{108}{r^2} = \frac{108}{18} = 6$; $[A = 18\pi\sqrt{3}]$

$\boxed{5}$

TEST NATURE at $r = 3\sqrt{2} = 4.24\ldots$

r	4	4.1	4.2	4.24...	4.25	4.3	4.5	5
$\frac{dA}{dr}$	−5.43 ⊖ \	−3.16 ⊖ \	−0.93 ⊖ \	0 —	0.16.. ⊕ /	1.24 ⊕ /	5.43 ⊕ /	14.9c ⊕ /

$\frac{1}{2}$ each

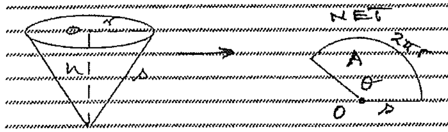
$\therefore$ a Relative min T.P at $r = 3\sqrt{2}$

And since there exist only 1TP for $r > 0$

$\therefore$ absolute min Area occurs when $r = 3\sqrt{2}$, $h = 6$ $\quad \frac{1}{2}$

dimensions $3\sqrt{2}$ cm for radius and 6 cm for height

NOTE: Background Y10



$\frac{\theta}{2\pi} = \frac{2\pi r}{2\pi s} = \frac{A}{\pi s}$

$\therefore \theta = \frac{2\pi r}{s}$

and $A = \frac{1}{2}s^2\theta = \frac{1}{2}s^2 \times \frac{2\pi r}{s}$

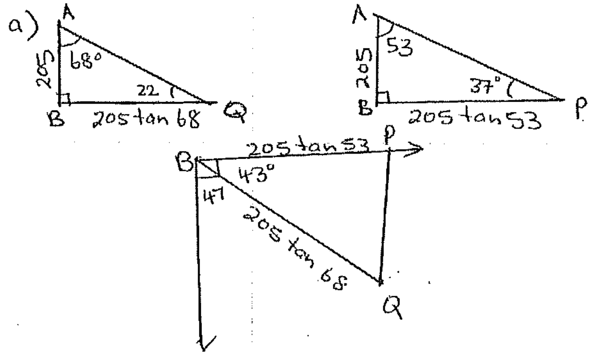
$A = \pi r s$ [Y10 recall formula]

MATHEMATICS Extension 1 : Question 13

Pg 1

Suggested Solutions | **Marks** | **Marker's Comments**

a)



$PQ^2 = (205\tan 53)^2 + (205\tan 68)^2 - 2(205^2 \tan 53 \tan 68)\cos 43°$

$= 205^2(\tan^2 53 + \tan^2 68 - 2\tan 53 \tan 68 \cos 43$

$PQ = \sqrt{205^2(\tan^2 53 + \tan^2 68 - 2\tan 53 \tan 68 \cos 43}$

$= 359.935\,0248\ldots$

$= 360$m (nearest metre).

Alternatively

BP could also be.

$BQ = \frac{205}{\tan 22}$

$BP = \frac{205}{\tan 37}$

✓ 1 mark for getting lengths of BP and BQ

✓ 1 mark calculating angle PBQ as 43°

✓ 1 mark for correct formula and working

b)

$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \cdots\cdots \left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$

Step 1 Prove true for $n = 2$: $P(2)$

LHS

$1 - \frac{1}{2^2} = 1 - \frac{1}{4}$

$= \frac{3}{4}$

RHS

$\frac{2+1}{2 \times 2}$

$= \frac{3}{4} = $ LHS

$\therefore$ statement is true for $n=2$.

✓ 1 mark step 1

1 mark for ½ steps 2 ~~+3~~

2 step 3

Prove LHS = RHS

½ mark for conclusion statement must mention true for all integers $n \geq 2$ or PMI.

MATHEMATICS Extension 1 : Question 13

Pg 2

Suggested Solutions | **Marks** | **Marker's Comments**

Step 2 : Assume true for $n=k$.

i.e $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \text{------} \left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$.

$\frac{1}{2}$.

Step 3 prove true for $n=k+1$

i.e $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \text{------} \left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+1}$

Some students had this step missing

LHS $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right) \text{-----} \left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$ (by assumption)

$-\frac{1}{2}$ mark if assumption is not mentioned

$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$

$= \frac{k^2+2k+1-1}{2k(k+1)}$

$= \frac{k^2+2k}{2k(k+1)}$

$= \frac{k^2+2k}{2k(k+1)}$

$= \frac{k+2}{2(k+1)} = RHS$

Step 4 statement is true for $n=k+1$ if assumed true for $n=k$. Since the statement has been proven true for $n=2$, it is true for all integers $n \geq 2$.

or ∴ by the PMI $P(n)$ is ~~true for~~ $n=2,3,4 \ldots$

MATHEMATICS Extension 1 : Question 13

Pg. 3

Suggested Solutions | **Marks** | **Marker's Comments**

c (i) $\frac{d^2x}{dt^2} = 36-9x$

Compare $\ddot{x} = -n^2(x-x_0)$ ↑centre.

$\ddot{x} = -9(x-4)$

∴ centre is $x=4$

✓ 1 mark

ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 36-9x$

$\frac{1}{2}v^2 = \int(36-9x).dx$

$\frac{v^2}{2} = 36x - \frac{9x^2}{2} + c$.

For 1 mark be careful.

when $x=6$, $v=0$

$0 = 216 - 162 + c$

∴ $c = -54$.

If students got $6-4=2$ only they must justify and mention particle is at end point

Equation is $\frac{v^2}{2} = 36x - \frac{9x^2}{2} - 54$

$v^2 = 72x - 9x^2 - 108$

$= 9(8x - x^2 - 12)$

$= -9(x^2 - 8x + 12)$

$v^2 = -9((x-4)^2 - 4)$

$= -9(4 - (x-4)^2)$

Compare $v^2 = n^2(a^2 - (x-x_0)^2)$

$a^2 = 4$

$a = 2$. $(a>0)$.

$v^2 = n^2[a^2 - (x-x_0)^2]$

$v^2 = 9[a^2 - (x-4)^2]$

$x=6$ $v=0$

Leads to $a = 2$

OR.

write it in the form

$x = b + a\cos nt + c$ ↖centre

so $x = 4 + a\cos 3t$ (from part (i) $n^2=9$ so $n=3$)

1 mark

when $t=0$, $x=6$

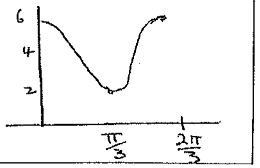
$6 = 4 + a\cos 3t$

$a\cos 3t = 2$.

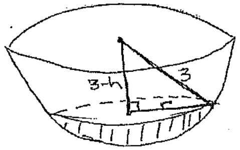
$a.(1) = 2$

$a = 2$

Because it is initially at rest and at end of motion $c=0$



MATHEMATICS: Question 13

Suggested Solutions	Marks Awarded	Marker's Comments
d) i)  Using pythagoras $3^2 = r^2+(3-h)^2$ $r^2 = 9-(3-h)^2$ $= 9-(9-6h+h^2)$ $= 6h-h^2$	2 marks	$x^2 = 9-y^2$ $\therefore V = \pi\int_0^h (9-y)^2$ is incorrect. limits should be 3 and 3-h.
Volume $= \pi\int_0^h (6h-h^2)\,dh$ ✓ $= \pi\left[3h^2-\frac{h^3}{3}\right]_0^h$ $= \pi\left(3h^2-\frac{h^3}{3}\right)$ ✓ $V = \frac{\pi}{3}\left(9h^2-h^3\right)$ as ✓ required ∴ Volume is $\frac{\pi}{3}(9h^2-h^3)$ m^3	/	If recognised circle is rotated with correct limits $\pi\int_{3-h}^{3}$ ―― dx is 1 mark. and 1 mark for correct subsequent working without making the question simpler.
Alternatively Rotated $x^2+y^2=9$ around x axis between $x=3$ and $x=3-h$. $\therefore y^2 = 9-x^2$ Volume $= \pi\int_{3-h}^{3}(y^2)\,dx$ $= \pi\int_{3-h}^{3}(9-x^2)\,dx$ ✓ $= \left[9x-\frac{x^3}{3}\right]_{3-h}^{3}$ $= \pi\left[(27-9)-\{9(3-h)-\frac{1}{3}(3-h)^3\}\right]$ $= \pi\left(3h^2-\frac{h^3}{3}\right)$ $= \frac{\pi}{3}\left(9h^2-h^3\right)$ m^3 ✓		$\pi[18-9\{\frac{(3-h)}{3}(27-(3-h)^2\}]$ $= \pi($ $=$ $=$

MATHEMATICS: Question 13

Suggested Solutions	Marks Awarded	Marker's Comments
d(ii) After 8 minutes, $V = \frac{8\pi}{3}$. $\therefore \frac{8\pi}{3} = \frac{\pi}{3}(9h^2-h^3)$ $8 = 9h^2-h^3$ $h^3-9h^2+8=0$ $(h-1)(h^2-8h-8)=0$ ⟶	1 mark.	h^2-8h-8 $h-1\,\big)\,h^3-9h^2+0h+8$ h^3-h^2 $-8h^2+6h$ $-8h^2+8h$ $-8h+8$ $-8h+8$ 0
$h-1=0$ or $h^2-8h-8=0$ $\therefore h=1$ $\quad \frac{8\pm\sqrt{64-(-32)}}{2}$ $= \frac{8\pm\sqrt{96}}{2}$ $= 4\pm2\sqrt{6}$ } as $6 \le h \le 3$ } ⟶	$\frac{1}{2}$ mark	Some ended up with h^2-8 instead of h^2-8h-8
$h=1$ is only th solution ⟶	½ mark	Some tested $h=1$ on h^3-9h^2+8 only Did not mention other solutions. not working
d(iii) $\frac{dV}{dt}=\frac{\pi}{3}$ (given) $\frac{dV}{dh}=\frac{\pi}{3}(18h-3h^2)$ $=\frac{\pi}{3}(6h-h^2)$ $\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt}$. $=\frac{1}{\pi(6h-h^2)}\times\frac{\pi}{3}$. ✓ 1mark. $\frac{dh}{dt}=\frac{1}{3(6h-h^2)}$ when $h=1$ $\frac{dh}{dt}=\frac{1}{3(6-1}=\frac{1}{15}$ ✓ 1 mark. ∴ when oil is 1m deep, h is increasing at a rate of 1/15 metres/min.		$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$ $\frac{\pi}{3}=\frac{\pi}{3}(6h-h^2)\frac{dh}{dt}$ $\Rightarrow \frac{dh}{dt}=\frac{1}{3(6h-h^2)}$ Some forgot to invert $\frac{dV}{dh}=\pi(6h-h^2)$ $\therefore \frac{dh}{dV}=\frac{1}{\pi(6h-h^2)}$ Some calculated the first few steps in part (ii).

MATHEMATICS Extension 1 : Question 14 V2

Suggested Solutions	Marks	Marker's Comments
a) $x = 3\cos 2t + 4\sin 2t + 2$ (i) $3\cos 2t + 4\sin 2t = R\cos(2t - \alpha)$ $R > 0$ $0 < \alpha < \pi/2$ $3\cos 2t + 4\sin 2t = R\cos 2t\cos\alpha + R\sin 2t\sin\alpha$ $\therefore R\cos\alpha = 3$ $R\sin\alpha = 4$ $\therefore R^2 = 3^2 + 4^2$ $R = 5$ $R > 0$. $\cos\alpha = 3/5$ $\sin\alpha = 4/5$ $\therefore 0 < \alpha < \pi/2$ $\therefore \alpha = \tan^{-1} 4/3$ or $\cos^{-1} 3/5$ or $\sin^{-1} 4/5$ $\therefore 3\cos 2t + 4\sin 2t = 5\cos(2t - \tan^{-1} 4/3)$	②	(½) for R=5 (½) Both sin + cos equations (½) α value. (½) answer. last answer only accepted.
(ii) $x = 5\cos(2t - \tan^{-1} 4/3) + 2$. $\dot{x} = -10\sin(2t - \tan^{-1} 4/5)$ $\ddot{x} = -20\cos(2t - \tan^{-1} 4/3)$ $\ddot{x} = -20\frac{(x-2)}{5}$ $= -4(x-2)$ $= -(2)^2(x-2)$ $\therefore \ddot{x}$ is in the form $\ddot{x} = -n^2(x-b)$ where $n = 2$ $b = 2$ $\therefore$ motion is SHM. amplitude is 5 metres	②	(½) $\dot{x}$ (½) $\ddot{x}$ (½) reason. (½) amplitude
(iii) $\dot{x} = -10\sin(2t - \tan^{-1} 4/3)$ $\therefore$ max speed is 10 m/s. $\therefore \lvert\sin(2t - \tan^{-1} 4/3)\rvert = 1$ $2t - \tan^{-1}(4/5) = (2n+1)\pi/2$ $n \in \mathbb{Z}$ $\therefore$ first time $2t = \frac{\pi}{2} + \tan^{-1}(4/3)$ $t = \frac{1}{2}[\pi/2 + \tan^{-1}(4/3)]$ $= 1.249045772$ $= 1.25$ (2dP)	②	Several methods. eg $\lvert\dot{x}\rvert = 10$ $\ddot{x} = 0$. (½) trig equation ① solution & 1dP. (½) max speed.
b) (i) $b_k = a_{k-1} + a_k$ from Pascal's triangle (alternatively $b_k = \left[\frac{n+1}{n+1-k}\right] a_k$	①	① correct relationship
(ii) $\frac{1}{a_0 a_1 \cdots a_n} \times (a_0 + a_1)(a_1 + a_2)\cdots(a_{n-1} + a_n)$ $= \frac{1}{a_0} \times \frac{1}{a_1 a_2 \cdots a_n} \times (b_1 b_2 \cdots b_n)$ $= 1 \times \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}$ $a_0 = 1$ now $\frac{b_k}{a_k} = \frac{{}^{n+1}C_k}{{}^{n}C_k} = \frac{(n+1)!}{(n-k+1)!\,k!} \times \frac{(n-k)!\,k!}{n!}$ $= \frac{n+1}{n-k+1}$	③	(½) product of b's (½) $a_0 = 1$ ① coefficients expansions ① answer with working

---

MATHEMATICS Extension 1 : Question 14 V2

Suggested Solutions	Marks	Marker's Comments
(iii) continued $\frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} = \frac{(n+1)(n+1)\cdots(n+1)}{(n)(n-1)\cdots(1)}$ $= \frac{(n+1)^n}{n!}$		
c) (i) To show $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}(2/n^2)$ $n \geq 1$ consider tan(LHS) Let $\tan^{-1}(n+1) = \alpha$ $\therefore \tan\alpha = n+1$ $\pi/4 < \alpha < \pi/2$ $\tan^{-1}(n-1) = \beta$ $\tan\beta = n-1$ $0 < \beta < \pi/2$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ $= \frac{(n+1)-(n-1)}{1+(n+1)(n-1)}$ $= \frac{2}{1+n^2-1} = \frac{2}{n^2}$ $\therefore \alpha - \beta = k\pi + \tan^{-1}(2/n^2)$ $k \in \mathbb{Z}$ But $-\pi/4 < (\alpha - \beta) < \pi/2$ $\therefore k = 0$ $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\left(\frac{2}{n^2}\right)$		(½) tan expansion (½) simplifying (½) solution (½) restrictions
(ii) To show $\sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) + \frac{3\pi}{4}$ LHS $= \sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1}(r-1)\right]$ $= \tan^{-1}2 - \tan^{-1}0 + \tan^{-1}3 - \tan^{-1}1 \cdots$ $\cdots \tan^{-1}n + \tan^{-1}(n-2) + \tan^{-1}(n+1) + \tan^{-1}(n-1)$ $= \tan^{-1}(n) + \tan^{-1}(n+1) - \tan^{-1}1$ consider $\tan\left(\tan^{-1}n + \tan^{-1}(n+1)\right)$ $= \frac{\tan(\tan^{-1}n) + \tan(\tan^{-1}(n+1))}{1 - \tan(\tan^{-1}n) \times \tan(\tan^{-1}(n+1))}$ $= \frac{n+n+1}{1-n(n+1)}$ $= \frac{2n+1}{1-n-n^2}$ $\therefore \tan^{-1}n + \tan^{-1}(n+1) = k\pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$ $k \in \mathbb{Z}$ But $\frac{\pi}{4} \leq \tan^{-1}n < \pi/2$ $n \geq 1$ and $\pi/4 < \tan^{-1}(n+1) < \pi/2$ $\therefore \frac{\pi}{2} < \tan^{-1}n + \tan^{-1}(n+1) < \pi$ $k = 1$ $\therefore \tan^{-1}n + \tan^{-1}(n+1) = \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$ $\therefore \sum_{r=1}^{n} \tan^{-1}\left(\frac{2}{r^2}\right) = \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \tan^{-1}(1)$ $= \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) - \pi/4$ $= \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) + \frac{3\pi}{4}$		(½) substitution (½) expansion + cancelling (½) tan expression and simplifying (½) general solution (½) correct restriction (½) answer with working