Student ID: ________________

KAMBALA

Mathematics Extension 2

HSC Assessment Task 1

February 2008

Time Allowed: 50 minutes working time

Outcomes Assessed

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E3 uses the relationship between algebraic and geometric representations of complex numbers

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

E9 communicates abstract ideas and relationships using appropriate notation and logical argument

INSTRUCTIONS

- This task contains 3 questions of 10 marks each. Marks for each part question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks will be allocated to questions involving higher order thinking.

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 *(Start a new page.)* 10 Marks

(a) If $z = 2 + 3i$ and $w = 4 - i$, find:

(i) $z + w$ **1**

(ii) iw **1**

(b) If $(1 - i)z = 3 - 4i$, find:

(i) z **2**

(ii) $z\bar{z}$ **2**

(c) Given that $\vec{p}$ is the vector that represents the complex number $-2-3i$ and $\vec{q}$ is the vector that represents the complex number $1-i$:

(i) Show these vectors on an Argand diagram. **2**

(ii) Show the position vector $\vec{p}+\vec{q}$. **1**

(iii) What complex number does $\vec{p}+\vec{q}$ represent? **1**

Question 2 *(Start a new page.)* 10 Marks

(a) Sketch $|z+i| \le 2$ and describe the locus geometrically. **2**

(b) (i) Write $\sqrt{3} - i$ in mod-arg form. **2**

(ii) Hence find the two square roots of $\sqrt{3} - i$. **3**

(c) Solve $z^2 + (1+i)z + 2i = 0$ expressing the roots in the form $a + ib$ where a and b are real. **3**
Let $z = x + iy$ where x and y are real.

Question 3 *(Start a new page.)* 10 Marks

(a) (i) Sketch the graph of $y = \dfrac{2x}{x^2+1}$, indicating the co-ordinates of its stationary points. **3**

(ii) What geometrical feature of the graph of the curve $y = \dfrac{2x}{x^2+1}$ shows that the function is odd? **1**

(b) Consider the function $f(x) = 6x - x^2$:

On separate diagrams, sketch the graphs of the following:

(i) $y = f(x)$, clearly showing the co-ordinates of any turning points. **1**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y = |f(x)|$ **1**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $|y| = f(x)$ **1**

End of Assessment Task

Q1 (a) (i) $2+3i+4-i = \underline{6+2i}$

(ii) $i(4-i) = 4i+1 = \underline{1+4i}$

(b) (i) $z = \dfrac{3-4i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{(3+4)+i(3-4)}{1+1}$

$= \dfrac{5-i}{2}$

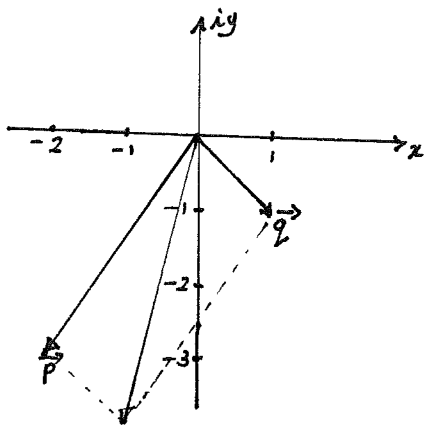
(ii) $z\bar{z} = \frac{1}{2}(5-i) \cdot \frac{1}{2}(5+i)$

$= \frac{1}{4}(25+1)$

$= \frac{26}{4}$

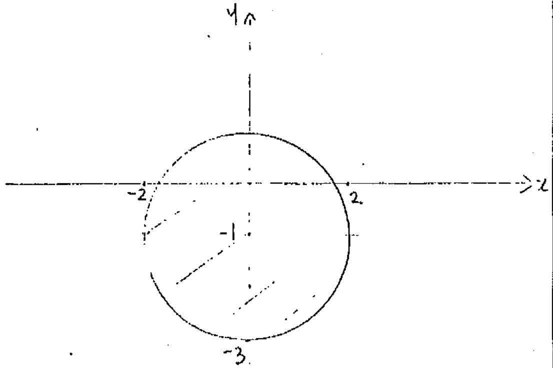
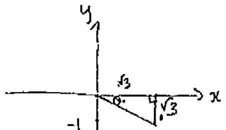
$= \underline{6\frac{1}{2}}$

(c) (i)



(ii) $\vec{p}+\vec{q}$

(iii) $\vec{p}+\vec{q} = -1-4i$

Qn	Solutions	Marks	Comments+Criteria
2. (a)		1	
	Locus is interior and boundary of a circle, centre $(0,-1)$ and radius 2 units.	1	
(b) (i)	$\sqrt{3} - i = r \operatorname{cis} \theta$ $r = \sqrt{(\sqrt{3})^2 + (-1)^2}$ $= \sqrt{3+1}$ $= \sqrt{4}$ $\therefore r = 2$  $\tan\theta = -\frac{1}{\sqrt{3}}$ $\theta = -\frac{\pi}{6}$ $\therefore \sqrt{3} - i = 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$ $= 2\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]$	2	
(ii)	let $\sqrt{\sqrt{3}-i} = r \operatorname{cis} \theta$ $\sqrt{2 \operatorname{cis}\left(-\frac{\pi}{6}\right)} = r \operatorname{cis} \theta$ $2 \operatorname{cis}\left(-\frac{\pi}{6}\right) = (r \operatorname{cis} \theta)^2$ $2 \operatorname{cis}\left(-\frac{\pi}{6}\right) = r^2 \operatorname{cis} 2\theta$ (De Moivre's Thm) $r^2 = 2$ $\quad 2\theta = -\frac{\pi}{6} + 2k\pi$ $r = \sqrt{2}$ $(r>0)$ $\quad \theta = -\frac{\pi}{12} + k\pi$ when $k=0$, $\theta = -\frac{\pi}{12}$ $k=1$, $\theta = \frac{11\pi}{12}$ $\therefore$ The two square roots are $\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{12}\right)$ and $\sqrt{2} \operatorname{cis}\left(\frac{11\pi}{12}\right)$	3	

Qn	Solutions	Marks	Comments+Criteria
2(c)	$z^2 + (1+i)z + 2i = 0$ $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $= \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4 \times 1 \times 2i}}{2}$	1	
	$= \frac{-(1+i) \pm \sqrt{1+2i-1-8i}}{2}$ $= \frac{-(1+i) \pm \sqrt{-6i}}{2}$	1	
	let $\sqrt{-6i} = a + ib$ $-6i = a^2 - b^2 + i \cdot 2ab$ $a^2 - b^2 = 0$ $2ab = -6$ $b = \frac{-3}{a}$ $a^2 - \frac{9}{a^2} = 0$ $a^4 - 9 = 0$ $(a^2-3)(a^2+3) = 0$ $a = \pm\sqrt{3}$ $\quad a^2 = -3$ no soln, a is real $\therefore b = \frac{-3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \mp\sqrt{3}$ $\therefore \sqrt{-6i} = \sqrt{3} - \sqrt{3}i$ or $-\sqrt{3} + \sqrt{3}i$ $\therefore z = \frac{-(1+i) + \sqrt{3} - \sqrt{3}i}{2}$ or $\frac{-(1+i) + (-\sqrt{3} + \sqrt{3}i)}{2}$ $= \frac{(-1+\sqrt{3}) + i(-1-\sqrt{3})}{2}$ or $\frac{(-1-\sqrt{3}) + i(-1+\sqrt{3})}{2}$	1	Note that for $\pm\sqrt{-6i}$ 4 cases simplify to 2: $+\sqrt{3} - \sqrt{3}i$ or $-(\sqrt{3} - \sqrt{3}i)$ $= -\sqrt{3} + \sqrt{3}i$ $+(-\sqrt{3} + \sqrt{3}i)$ or $-(-\sqrt{3} + \sqrt{3}i)$ $= \sqrt{3} - \sqrt{3}i$ $\therefore$ Used just $+\sqrt{3} - \sqrt{3}i$ and $+(-\sqrt{3} + \sqrt{3}i)$

Q3(a)

$y = 6x - x^2$

$y = x(6 - x) \Rightarrow (0,0), (6,0)$

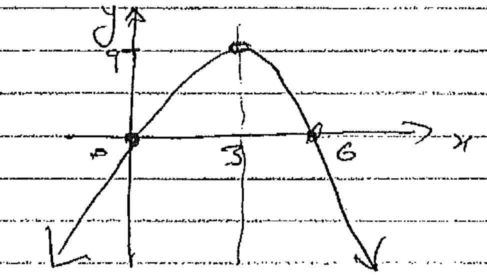
$y' = 6 - 2x$

$y' = 0 \Rightarrow x = 3, \; y = 9$

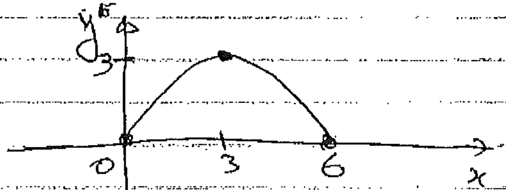
$\Rightarrow (3, 9)$

$y'' = -2 < 0$ Max at $(3, 9)$

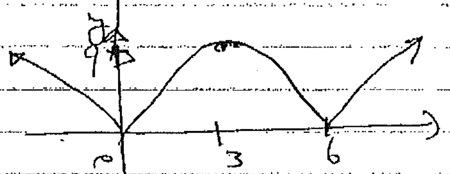
(i) $y = f(x)$



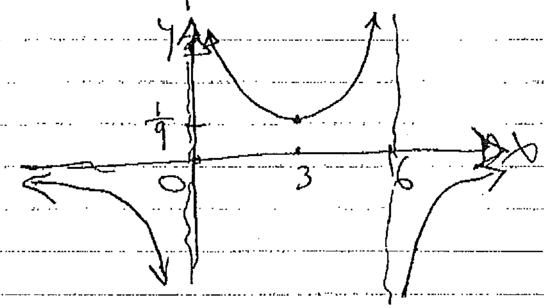
(ii) $y = \sqrt{f(x)}$



(iii) $y = |f(x)|$



(iv) $y = \dfrac{1}{f(x)}$



Q3(b) (v) $|y| = f(x)$

$\therefore y = \pm f(x)$