KAMBALA

EXTENSION 2 MATHEMATICS

HSC ASSESSMENT TASK #1 (Complex No's Only.)

NOVEMBER 2004

Time Allowed: 50 minutes

INSTRUCTIONS

- This task contains 3 questions of 12 marks each. Marks for each part question are shown.
- Answer all questions on the writing paper provided.
- Start each question on a new page.
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.

Question 1 **Start a new page** **Marks**

(a) Simplify, leaving your answer in the form $a+ib$:

(i) $(2+3i)(1-2i)$ **1**

(ii) $\dfrac{\sqrt{3}-i}{\sqrt{2}-i}$ **2**

(b) (i) Write $1+i\sqrt{3}$ in mod-arg form **2**

(ii) Hence find the two square roots of $1+i\sqrt{3}$ **3**

(d) If $\overset{A}{z_1}=4i$, $\overset{B}{z_2}=2\sqrt{3}-2i$ and $\overset{C}{z_3}=-2\sqrt{3}-2i$ represent A, B and C respectively, show that Δ ABC is equilateral **4**

Question 2 **Start a new page** **Marks**

(a) (i) Find $(1-i)^5$ in mod-arg form **2**

(ii) Using De Moivres Theorem or otherwise find the values of n for which $(1-i)^n$ is purely real. **2**

(b) For what values of a and b is $z=1$ a solution to the equation
$$2iz^2+(ia+1)z-(i+3b)=0\,?$$ **3**

(c) (i) Solve the equation $z^5 - i = 0$, giving the roots z_1, z_2, z_3, z_4, z_5 in mod-arg form with principal arguments. 3

(ii) Show that the perimeter of the pentagon $z_1 z_2 z_3 z_4 z_5$ formed by the roots in the Argand plane is given by

$$P = 5\sqrt{2\left(1 - \cos\frac{2\pi}{5}\right)}$$ 2

Question 3 **Start a new page** **Marks**

(a) Sketch the region on the Argand plane where both
$1 \le |z - i| \le 2$ and $\frac{\pi}{3} < \arg z < \frac{2\pi}{3}$ apply 3

(b) By looking at the values of i^n for $n = 1, 2, 3, 4, \ldots$ or otherwise, find the possible values of the series $S_n = 1 + i + i^2 + i^3 + i^4 + \ldots\ldots + i^n$ 3

(c) z is the complex number $z = x + iy$

(i) Find the equation of the curve for which $\text{Im}(z^2) = 4$ and sketch on a suitably labelled Argand plane 2

(ii) Show that the locus given by $\text{Re}(z^2) = 0$ is the pair of intersecting lines $y = \pm x$ 2

(iii) The region R in the Argand plane consists of all points which satisfy both $0 \le \text{Im}(z^2) \le 4$ and $\text{Re}(z^2) \le 0$.
Complete a sketch of the region R on your answer for part (i) 2

(0,0)

End of task

9:05

Su-Min Lim

$\frac{31}{36}$ great

$\frac{10\frac{1}{2}}{12}$

Q1

$\frac{1}{1}$ a) i) $2 - 4i + 3i - 6(i)^2$

$= 2 - i + 6$

$= 8 - i$ ✓

ii) $\dfrac{\sqrt{3} - i}{\sqrt{2} - i}$

$= \dfrac{(\sqrt{3} - i)(\sqrt{2} + i)}{2 + 1}$

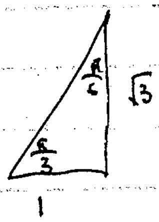
$= \dfrac{\sqrt{6} + i\sqrt{3} - i\sqrt{2} + 1}{3}$

$\frac{2}{2}$ $= \dfrac{\sqrt{6} + 1}{3} + i\dfrac{(\sqrt{3} - \sqrt{2})}{3}$ ✓✓

b) i) $r = \sqrt{1 + 3} = 2$

$\theta = \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$



$\frac{2}{2}$ $1 + i\sqrt{3} = 2 \text{ cis } \frac{\pi}{3}$ ✓✓

ii) let $z^2 = 2 \text{ cis } \frac{\pi}{3}$

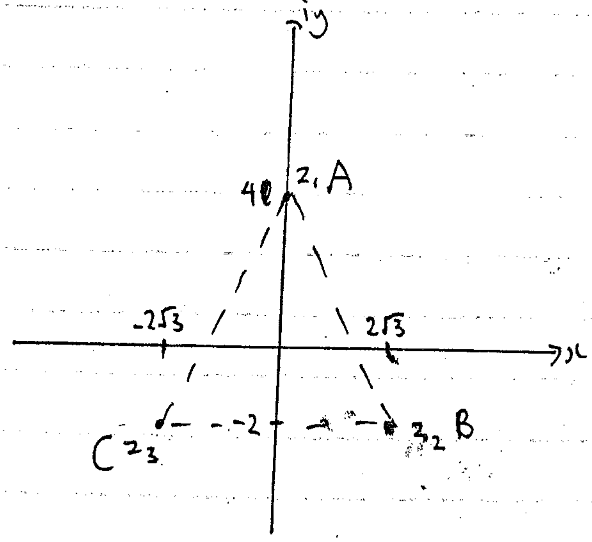
$z^{\frac{1}{2}} = 2^{\frac{1}{2}} \text{ cis } \frac{1}{2}\left(\frac{\pi}{3} + 2k\pi\right) \ldots$

$\frac{1\frac{1}{2}}{3}$ $z_1 = \sqrt{2} \text{ cis } \frac{\pi}{6}$ ✓

$z_2 = -\sqrt{2} \text{ cis } \frac{\pi}{6}$ ✗

XX

d)



$AB = |z_1 - z_2|$
$= |4i - 2\sqrt{3} + 2i|$
$= |-2\sqrt{3} + 6i|$
$= \sqrt{12 + 36}$
$= \sqrt{48} = 4\sqrt{3}$

$AC = |z_1 - z_3|$
$= |4i + 2\sqrt{3} + 2i|$
$= |6i + 2\sqrt{3}|$
$= \sqrt{36 + 12}$
$= \sqrt{48} = 4\sqrt{3}$

4/4

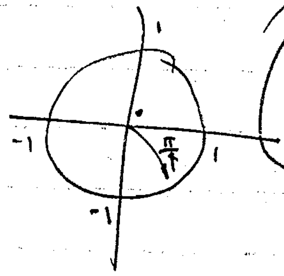
$BC = |z_2 - z_3|$
$= |2\sqrt{3} - 2i + 2\sqrt{3} + 2i|$
$= |4\sqrt{3}|$
$= \sqrt{(4\sqrt{3})^2}$
$= 4\sqrt{3}$

$AB = AC = BC$

$\therefore \triangle ABC$ is equilateral as all sides are =

9:15 -20

Q 2



$1\frac{1}{2}$ / 12

a) i) $1-i = \sqrt{1+1}\ \text{cis}\ (\tan^{-1} -1)$

$= \sqrt{2}\ \text{cis}\ (-\frac{\pi}{4})$ ✓

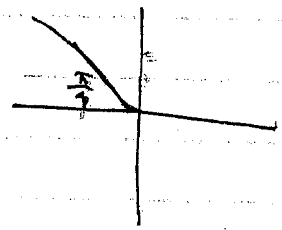
2/2

$(1-i)^5 = (\sqrt{2})^5 \text{cis}\ (5 \times -\frac{\pi}{4})$

$= 4\sqrt{2}\ \text{cis}\ (-\frac{5\pi}{4})$ $= ?\ 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ is more 'polite'.

ii) ~~$(1+i)^5 = 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$~~

~~$= 4\sqrt{2}\ (\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$~~



~~For $(1+i)^5$ to be real, $\sin\frac{3\pi}{4}$~~

$1\frac{1}{2}$ / 2

$(1+i)^n = (\sqrt{2})^n \text{cis}\ (-\frac{n\pi}{4}) = (\sqrt{2})^n (\cos(-\frac{n\pi}{4}) + i\sin(-\frac{n\pi}{4}))$

For $(1+i)^n$ to be real, $\sin(-\frac{n\pi}{4})$ must be 0

$n = \pm 4, \pm 8, \pm 12, \pm 16 \ldots \Rightarrow$ multiple of 4 or 0

careful...

b) ~~$2i + ia + 1 - i - 3b = 0$~~

~~$i(2 + a - 1) - 3b = 0$~~

~~$i(1+a) - 3b = 0$~~

$2i + ia - i - 3b + 1 = 0$

$i(2 + a - 1) - 3b + 1 = 0$

$i(1+a) - 3b + 1 = 0$

Equating Re:/Im...

3/3

$1 + a = 0$

$a = -1$

$-3b + 1 = 0$

$1 = 3b$

$b = \frac{1}{3}$ ✓✓

~~c) i) $z^5 - i = 0$~~

~~$z^5 = i$~~

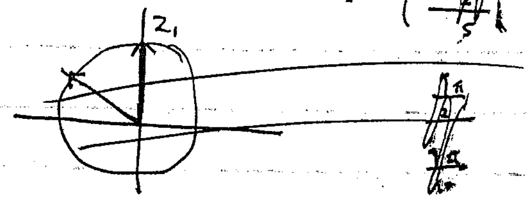
~~$z_1 = i = cis\frac{\pi}{2}$~~

~~$z_2 = cis\frac{7\pi}{10}$~~

~~$z_3 = cis\frac{9\pi}{10}$~~

~~z_4~~

$\left(\frac{\frac{\pi}{2}+4\pi}{5}\right)$



traditional to write

$r\, cis\, 5\theta = cis\left(\frac{\pi}{2}+2k\pi\right)$ $k = 0, \pm 1, \pm 2 \ldots$

$r = 1$

$\therefore 5\theta = \left(\frac{\pi}{2}+2k\pi\right)$

$\theta_1 = \frac{\pi}{10}$ etc.

c) i) $z^5 - i = 0$

$z^5 = i$

$z_1 = i = cis\frac{\pi}{2}$

$z_2 = cis\left(\frac{\frac{\pi}{2}+2\pi}{5}\right)$

$= cis\left(\frac{9\pi}{10}\right)$ ✓

3/3

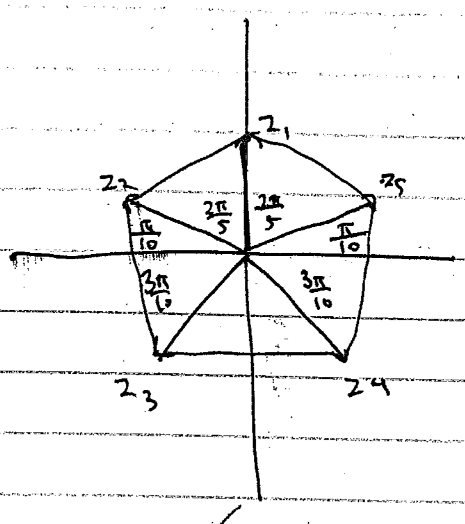
$z_3 = cis\frac{13\pi}{10} = cis\left(-\frac{7\pi}{10}\right)$ ✓

$z_4 = cis\frac{17\pi}{10} = cis\left(-\frac{3\pi}{10}\right)$ ✓

$z_5 = cis\frac{21\pi}{10} = cis\frac{\pi}{10}$ ✓

ii) Perimeter $= 5|z_1 - z_2|$

Let $|z_1 - z_2| = x$



2/2

$x^2 = 1^2 + 1^2 - 2\cos\frac{2\pi}{5}$

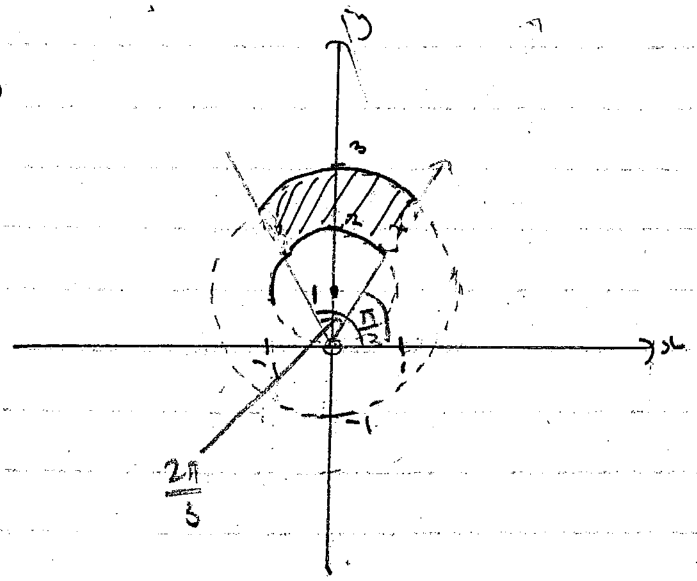
$= 2 - 2\cos\frac{2\pi}{5}$

$= 2\left(1 - \cos\frac{2\pi}{5}\right)$ ✓

$x = \sqrt{2\left(1-\cos\frac{2\pi}{5}\right)}$ so $5x = P = 5\sqrt{2\left(1-\cos\frac{2\pi}{5}\right)}$

Q3

a)



$\frac{3}{3}$ ✓✓✓ $\frac{9}{10}$

b) $i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

and so on ....

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(i^n - 1)}{i - 1}$$

$$= \frac{i^n - 1}{i - 1}$$

$i^n = i, -1, -i$ or 1 CAE

$\frac{i-1}{i-1} \doteq 0$?? Su Min !!

$\frac{1}{3}$

So $S_n = \frac{i-1}{i-1} = 0$, $\frac{-1-1}{i-1} = \frac{-2}{i-1}$?? simplify with real denoms, $\frac{i--i}{i-1} = \frac{2i}{i-1}$ = ?

when? × $r = i$?, or $\frac{1-i}{i-1} = -\frac{(i-1)}{i-1} = -1$ ×

and generally ...

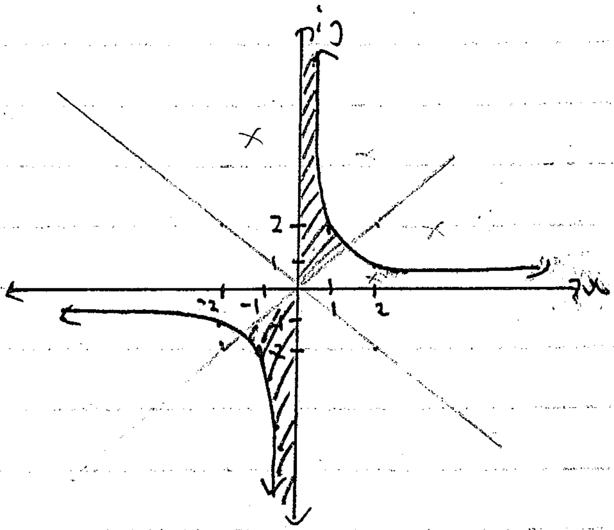
CAE & $\frac{1-1}{i-1} = 0$??

c) i) $\operatorname{Im}(z^2) = 4$

$\operatorname{Im}(x^2 + 2ixy - y^2) = 4$

$2xy = 4$

$xy = 2$



* for part (iii), Region R does not extend up indefinitely

2/2

ii) $\operatorname{Re}(x^2 + 2iyx - y^2) = 0$

$x^2 - y^2 = 0$

$x^2 = y^2$

$y = \pm x$

2/2

iii) $0 \le \operatorname{Im}(z^2) \le 4$

$0 \le 2xy \le 4$

$0 \le xy \le 2$

1/2

Pts tested: $(-1, 2)$

? $0 \le -2 \le 2$ ✗

$(2,3)$? $0 \le 6 \le 2$

$(3,2)$? $0 \le 6 \le 2$ ✗

$(1, 0.1)$? $0 \le 0.1 \le 2$ ✓

? $1 - (0.1)^2 \le 0$ ✗

iii) Region needed: $0 \leq 2xy \leq 4$

$0 \leq xy \leq 2$

$x^2 - y^2 \leq 0$

pt: $(0.1, 1)$

? $0 \leq 0.1 \leq 2$ ✓

$(0.1)^2 - 1^2 \leq 0$ ✓

$-0.1, 2$

? $0 \leq$