Student ID: ________________

KAMBALA

Mathematics Extension 2

HSC Assessment Task 1

February 2011

Time Allowed: 50 minutes working time

Outcomes Assessed

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E3 uses the relationship between algebraic and geometric representations of complex numbers
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions
E9 communicates abstract ideas and relationships using appropriate notation and logical argument

INSTRUCTIONS

- This task contains 3 questions. Marks for each part question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Board-approved calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks will be allocated to questions involving higher order thinking.

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (*Start a new page.*) 11 Marks

(a) Given that $z_1 = 3 - i$ and $z_2 = 2 + 5i$, express in the form $a + ib$, where a and b are real:

(i) $\left(\overline{z_1}\right)^2$ 2

(ii) $\dfrac{z_1}{z_2}$ 2

(iii) $\left|\dfrac{\overline{z_1}}{z_2}\right|$ 2

(b) Let $z = x + iy$, where x and y are real and $\arg z = \dfrac{3\pi}{5}$.

(i) Sketch the locus of z. 1

(ii) Find $\arg(-z)$. 1

(c) (i) Express $-1 - i$ in modulus argument form. 1

(ii) Hence evaluate $(-1-i)^{12}$. 2

Question 2 (*Start a new page.*) 11 Marks

(a) (i) By solving $z^5 = 1$ determine the complex 5th roots of unity. 2

(ii) Hence factorise $z^5 - 1$ over the complex field. 1

(iii) Hence also factorise $z^5 - 1$ over the real field. 2

(b) Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$ given that it has a root of multiplicity 2. 3

(c) Give a geometric and algebraic description of the locus of all complex numbers z such that:

$$2|z| = z + \bar{z} + 4$$ 3

Question 3 (*Start a new page.*) 12 Marks

(a) The roots of $x^3 + ax^2 + bx + c = 0$ are in arithmetic sequence. By denoting the roots as $(\alpha - d), \alpha$ and $(\alpha + d)$ prove that:

(i) one of the roots is $-\dfrac{a}{3}$. 1

(ii) $2a^3 - 9ab + 27c = 0$. 2

(iii) Hence or otherwise solve the equation $8x^3 - 36x^2 + 22x + 21 = 0$ given that its roots are in arithmetic sequence. 3

(b) Consider $f(x) = (x+2)(x-1)(x-3)$.

Sketch the following graphs, on separate one-third page diagrams, and without using calculus.

Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x)$ 1

(ii) $y = |f(x)|$ 1

(iii) $y^2 = f(x)$ 2

(iv) $y = \dfrac{1}{f(x)}$ 2

End of Assessment Task

Kambala Ext 2 HSC Task #1 Feb 2011

Question 1

(a) $z_1 = 3 - i, \quad z_2 = 2 + 5i$

(i) $\bar{z}_1 = 3 + i$ | 1

$(\bar{z}_1)^2 = 9 + 6i - 1$

$= 8 + 6i$ | 1

(ii) $\dfrac{z_1}{z_2} = \dfrac{3-i}{2+5i} \times \dfrac{2-5i}{2-5i}$ | 1

$= \dfrac{6 - 15i - 2i - 5}{4 + 25}$

$= \dfrac{1 - 17i}{29}$ | 1

(iii) $\left|\dfrac{\bar{z}_1}{z_2}\right|$

$= \left|\dfrac{3+i}{2+5i} \times \dfrac{2-5i}{2-5i}\right|$

$= \left|\dfrac{6 - 15i + 2i + 5}{4 + 25}\right|$

$= \left|\dfrac{11 - 13i}{29}\right|$ ①

$= \dfrac{1}{29}\sqrt{11^2 + 13^2}$

$= \dfrac{\sqrt{290}}{29}$ ①

Method II

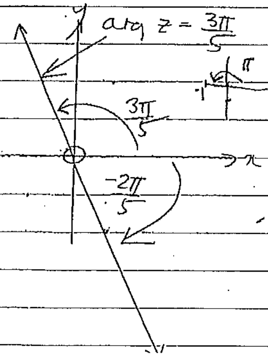
$\left|\dfrac{\bar{z}_1}{z_2}\right| = \dfrac{|\bar{z}_1|}{|z_2|}$

$= \dfrac{\sqrt{3^2 + 1^2}}{\sqrt{2^2 + 5^2}}$ | 1

$= \dfrac{\sqrt{10}}{\sqrt{29}} \times \dfrac{\sqrt{29}}{\sqrt{29}}$

$= \dfrac{\sqrt{290}}{29}$ | 1

(b)



$\arg(-z)$

$= \arg z + \arg(-1)$

ie rotation by π

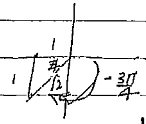
Rotn 180°

$\therefore \arg(-z) = \dfrac{-2\pi}{5}$ | 1

1 | for locus with (0,0) excluded.

Q1 ctd.

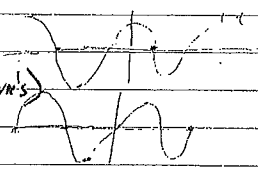
(c) $-1 - i = \sqrt{2}\,\text{cis}\left(\dfrac{-3\pi}{4}\right)$ | 1



$(-1-i)^{12}$

$= \left(\sqrt{2}\,\text{cis}\left(\dfrac{-3\pi}{4}\right)\right)^{12}$

$= 64\,\text{cis}\left(\dfrac{-36\pi}{4}\right)$ (De Moivre's)



$= 64\,\text{cis}(-9\pi)$ | 1

$= 64\left(\cos(-9\pi) + i\sin(-9\pi)\right)$

$= 64(-1 + i\sin 0)$

$= -64$ | 1

Question 2

(a) (i) $z^5 = 1$

Let $z = r\operatorname{cis}\theta$

$(r\operatorname{cis}\theta)^5 = 1\operatorname{cis}0$

$r^5\operatorname{cis}5\theta = 1\operatorname{cis}0$ (De Moivre's Thm)

$r = 1$

$5\theta = 0 + 2k\pi$

$\theta = \frac{2k\pi}{5}$ | 1

when

$k=0 \quad \theta = 0 \quad z_1 = 1\operatorname{cis}0 = 1$

$k=1 \quad \theta = \frac{2\pi}{5} \quad z_2 = 1\operatorname{cis}\frac{2\pi}{5} = \operatorname{cis}\frac{2\pi}{5}$

$k=-1 \quad \theta = -\frac{2\pi}{5} \quad z_3 = \operatorname{cis}\left(-\frac{2\pi}{5}\right)$

$k=2 \quad \theta = \frac{4\pi}{5} \quad z_4 = \operatorname{cis}\frac{4\pi}{5}$

$k=-2 \quad \theta = -\frac{4\pi}{5} \quad z_5 = \operatorname{cis}\left(-\frac{4\pi}{5}\right)$

$\therefore$ The complex 5th roots of unity are
$1, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\left(-\frac{2\pi}{5}\right), \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\left(-\frac{4\pi}{5}\right)$ | 1

(ii) $z^5 - 1 = (z-1)\left(z - \operatorname{cis}\frac{2\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z - \operatorname{cis}\frac{4\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)$ | 1

(iii) $\left(z - \operatorname{cis}\frac{2\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$

$= z^2 - \left(\operatorname{cis}\frac{2\pi}{5} + \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)z + \operatorname{cis}\frac{2\pi}{5}\operatorname{cis}\frac{-2\pi}{5}$

$= z^2 - 2\cos\frac{2\pi}{5}z + \overset{1}{\cancel{\cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5}}}$ | 1

Similarly

$\left(z - \operatorname{cis}\frac{4\pi}{5}\right)\left(z - \operatorname{cis}\left(\frac{-4\pi}{5}\right)\right) =$
$z^2 - 2\cos\frac{4\pi}{5}z + 1$

$\therefore z^5 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$ | 1

Q2ctd.

(b) $\overset{P(x)}{x^3 - 4x^2 - 3x + 18} = 0$
has a root of multiplicity 2
$\therefore$ Must be a root of multiplicity 1 of $P'(x)$

$P'(x) = 3x^2 - 8x - 3$
$= 3x^2 - 9x + x - 3$
$= 3x(x-3) + 1(x-3)$
$= (3x+1)(x-3)$

$P(3) = 3^3 - 4\times3^2 - 3\times3 + 18 = 0 \quad P\left(-\frac{1}{3}\right) = 16.5 \neq 0$

$\therefore (x-3)$ is a double root of $P(x)$

$\therefore x^2 - 6x + 9$ is a factor. | 1

$x^3 - 4x^2 - 3x + 18 = (x-a)(x^2 - 6x + 9)$

By inspection $a = -2$

$\therefore x^3 - 4x^2 - 3x + 18 = (x+2)(x-3)^2$ | 1

$\therefore$ Solns are $-2, 3$ and 3 | 1 | Must specify solutions not just give factorisation

(c) let $z = x + iy$

$2|z| = z + \bar{z} + 4$

$2\sqrt{x^2+y^2} = x + iy + x - iy + 4$ | 1

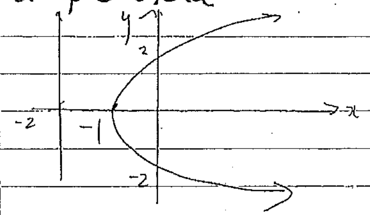
$2\sqrt{x^2+y^2} = 2x + 4$

$\sqrt{x^2+y^2} = x + 2$

$x^2 + y^2 = x^2 + 4x + 4$

$y^2 = 4x + 4$ is the required locus | 1

a parabola with vertex $(-1, 0)$



Question 3

(a) $x^3 + ax^2 + bx + c = 0$

Roots are
$(\alpha - d)$, α and $(\alpha + d)$

(i) sum of roots $= \frac{-b}{a}$

$\alpha - \cancel{d} + \alpha + \alpha + \cancel{d} = -a$ 1

$3\alpha = -a$

$\alpha = \frac{-a}{3}$

(ii)

One of the roots is $-\frac{a}{3}$

$\therefore P\left(\frac{-a}{3}\right) = 0$ 1

$\left(\frac{-a}{3}\right)^3 + a\left(\frac{-a}{3}\right)^2 + b\left(\frac{-a}{3}\right) + c = 0$

$-a^3 + 3a^3 - 9ab + 27c = 0$

$2a^3 - 9ab + 27c = 0$ 1

(iii) $8x^3 - 36x^2 + 22x + 21 = 0$.
let its roots be $(\alpha - d)$, α and $(\alpha + d)$

$x^3 - \frac{36}{8}x^2 + \frac{22}{8}x + \frac{21}{8} = 0$.

$\therefore$ One root is $\alpha = \frac{\frac{36}{8}}{3} = \frac{36}{24} = 1.5$ 1

Prod of roots

$(\alpha^2 - d^2) \cdot \alpha = \frac{-d}{a}$ (new one)

$((1.5)^2 - d^2) \cdot 1.5 = -\frac{21}{8}$ 1

$\frac{9}{4} - d^2 = -\frac{\cancel{21}^{7}}{\cancel{8}_{4}} \times \frac{\cancel{2}}{\cancel{3}}$

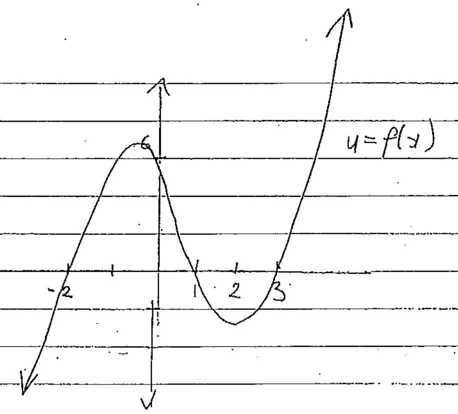
$-d^2 = -\frac{7}{4} - \frac{9}{4}$

$-d^2 = -4$

$d^2 = 4$

$d = \pm 2$ $\quad \therefore x = -0.5, 1.5, 3.5$ 1

Q3 ctd.



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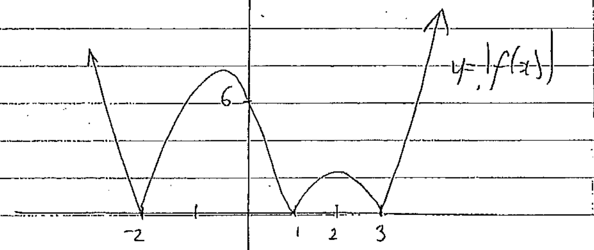
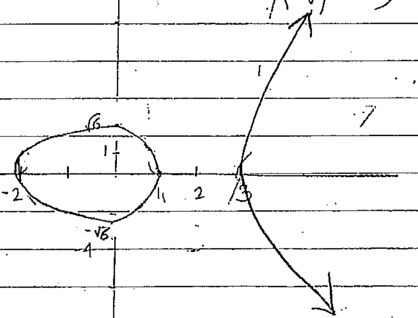


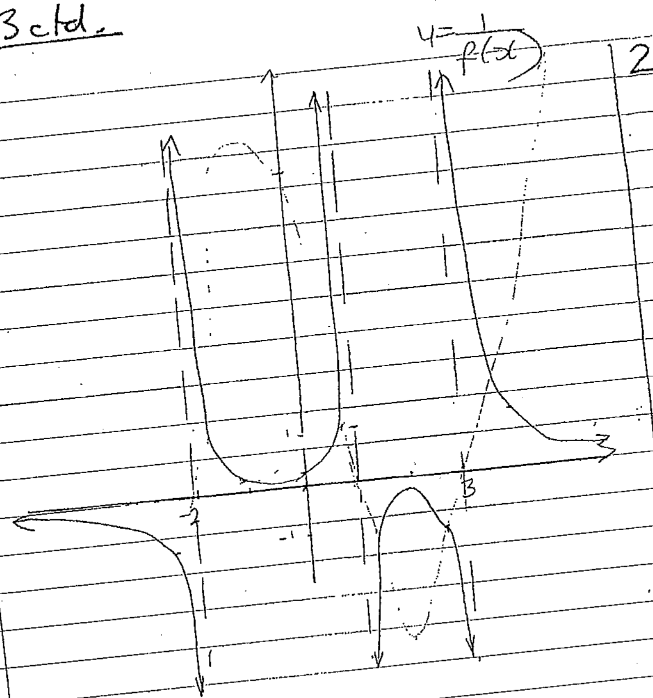
1

$y^2 = f(x)$

$y = \pm\sqrt{f(x)}$ 2



Q3 ctd.



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