Student Number: ____________

KAMBALA

AUGUST 2007

YEAR 12
HSC ASSESSMENT#4
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{x^2+6}{x^2+4}\,dx$ **2**

(b) (i) Find real numbers A, B and C such that **3**

$$\frac{9}{(2x-1)(x+1)^2} \equiv \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

(ii) Hence evaluate $\int_1^2 \frac{9}{(2x-1)(x+1)^2}\,dx$ **2**

(c) Use integration by parts to find the exact value of $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$. **4**

(d) Show that $\int \frac{x^n}{e^x}\,dx = n\int \frac{x^{n-1}}{e^x}\,dx - \frac{x^n}{e^x}$. **4**

Hence find $\int \frac{x^3}{e^x}\,dx$.

Marks

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $z = 1+\sqrt{3}\,i$ in modulus-argument form. **2**

(ii) Show that $z^7 - 64z = 0$ **3**

(b) (i) Solve the equation $z^4 = 1$. **1**

(ii) Hence find all solutions of the equation $z^4 = (z-1)^4$. **3**

(c) In an Argand diagram the points P, Q and R represent the complex numbers z, w and $z-w$ respectively. O is the origin and $z-w = iz$.

(i) Describe the geometric properties of ΔPOR, giving full reasons for your answer. **2**

(ii) Find the size of $\angle QPR$. **1**

(d) Find the Cartesian equation of the locus represented by $|z|^2 = z + \overline{z}$. **3**

Sketch this locus on an Argand diagram and describe it geometrically.

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x) = x\sqrt{4-x^2}$:

(i) State the natural domain and show that $f(x)$ is an odd function. **2**

(ii) Show that on the curve $y = f(x)$, stationary points occur at $x = \pm\sqrt{2}$. **3**
Find the co-ordinates of the stationary points and determine their nature.

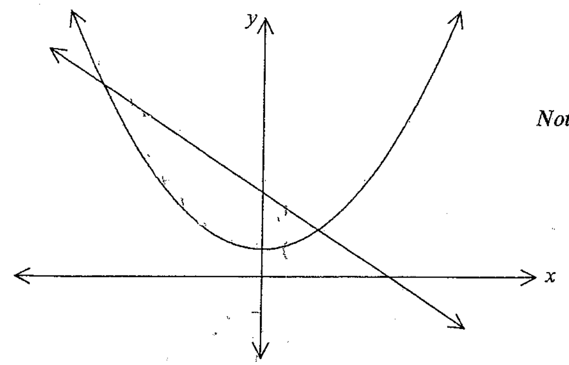
(iii) Draw a neat sketch of the curve $y = f(x)$, indicating the above features, and given that there is a point of inflexion at the origin. **2**

(iv) On separate diagrams, sketch the curves

1. $y^2 = x^2(4-x^2)$ **2**

2. $y = \dfrac{1}{f(x)}$ **2**

(b) The area bounded by the curve $y = x^2+1$ and the line $y = 3-x$ is rotated about the x-axis to form a solid.



Not to scale

(i) By considering slices perpendicular to the x-axis show that the area of one slice is given by $A = \pi(8-6x-x^2-x^4)$. **2**

(ii) Hence find the volume of the solid formed. **2**

Marks

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

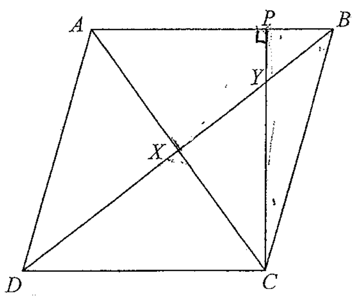
(a) A monic polynomial $P(x)$ of degree 4 with real coefficients has zeros $1+i$ and $-1+i$. **3**

Find $P(x)$, expressing your answer in expanded form.

(b) (i) Show that $\cos(A-B)-\cos(A+B)=2\sin A\sin B$ **2**

(ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 5x \sin x \, dx$. **2**

(c)



Not to scale

$ABCD$ is a rhombus whose diagonals intersect at X. The perpendicular CP from C to AB cuts BD at Y.

Prove that B, P, X, C are concyclic points. **3**

(d) Consider the hyperbola $\frac{y^2}{9}-\frac{x^2}{4}=1$. The area bounded by the lines $x=\pm 2$ and the hyperbola is rotated about the y-axis to form a solid.

(i) Using the method of cylindrical shells show that the volume of the solid so formed is given by $V=6\pi\int_0^2 x\sqrt{4+x^2}\,dx$. **3**

(ii) Hence find the volume of the solid. **2**

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

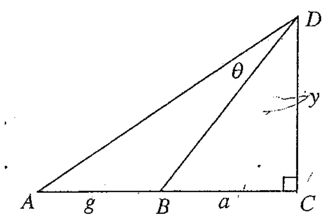
(a) If $(x-1)^2$ is a factor of $Q(x)=Ax^n+Bx^{n-2}+6$, show that $A=3n-6$ and $B=-3n$. **3**

(b) The tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at $P(4\sqrt{2}, 3)$ meets the asymptotes of the hyperbola at A and B.

(i) Show that P is the midpoint of AB. 4

(ii) Find the length of AB in exact form. 2

(c) In the diagram, A, B and C are collinear and DC is perpendicular to AC. AB, BC and DC are g, a and y units long respectively and $\angle ADB=\theta$.



Not to scale

(i) Show that $\tan\theta=\frac{gy}{a(g+a)+y^2}$. **3**

(ii) Find $\frac{d\theta}{dy}$ and hence show that the maximum value of θ is when $y=\sqrt{a(g+a)}$. **3**

Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) If α, β, γ are the roots of the equation $x^3+2x^2-2x+3=0$ form an equation whose roots are $\alpha^2, \beta^2, \gamma^2$. **2**

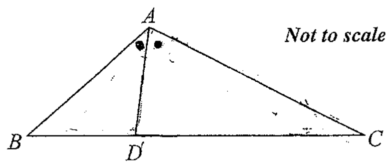
(b) A particle moves in a straight line and its position x at any time t is given by

$$x=\sqrt{3}\cos 3t-\sin 3t$$

(i) Show that the motion is simple harmonic. **2**

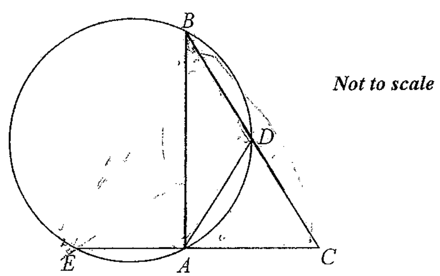
(ii) Determine the period and amplitude of the motion. **3**

(c) (i) In ΔABC, AD bisects $\angle BAC$.



Prove that $\dfrac{BD}{DC}=\dfrac{BA}{AC}$ **4**

(ii)



In the diagram $AB=BC$ and AD bisects $\angle BAC$.

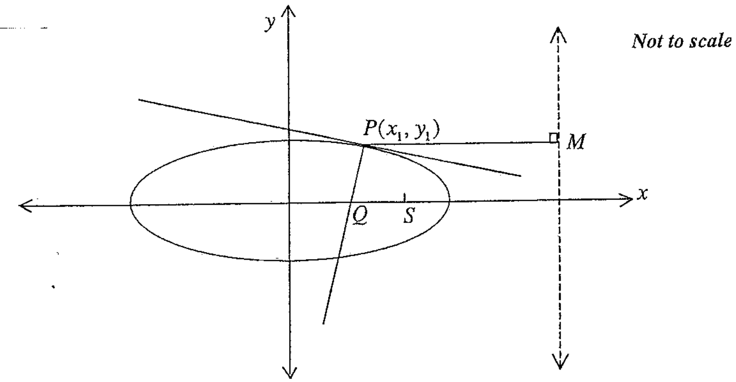
Prove that $BD=CE$. **4**

Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $t=\tan\dfrac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **3**

(b) Consider the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with eccentricity e.



(i) Write down in terms of a and e the foci and equation of the directrices. **1**

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2-b^2$. **2**

(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. **3**

Prove that $QS=e^2PM$.

Question 7 continues on the next page

Marks

QUESTION 7 continued

(c) (i) State why, for $x < 1$, the sum of n terms of **1**

$$1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}$$

(ii) Show that **2**

$$1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$$

(iii) Hence find an expression for $1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots + \frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4. **3**

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use De Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Using $\cos 3\theta = \frac{1}{2}$ deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$. **1**

(iii) Use this result to solve the equation $8x^3 - 6x - 1 = 0$ in terms of $\cos\theta$. **2**

(iv) Hence prove that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. **2**

(b) (i) If $f(x)$, $g(x)$ and $h(x)$ are distinct non-negative continuous functions of x in the interval $a \le x \le b$ and $f(x) < g(x) < h(x)$, explain why **1**

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$$

(ii) By considering the interval $0 < x < 1$ as an inequality, use algebra to show that **3**

$$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$

(iii) Deduce that $\frac{1}{2}\int_0^1 x(1-x)^3\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,dx < \int_0^1 x(1-x)^3\,dx$ **1**

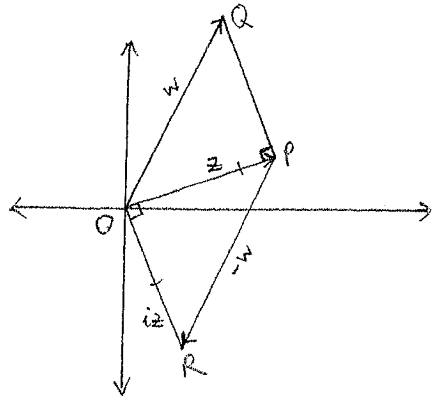
(iv) Given that $\int_0^1 \frac{x(1-x)^3}{1+x}\,dx = \frac{67}{12} - 8\ln 2$, deduce that $\frac{83}{120} < \ln 2 < \frac{667}{960}$ **3**

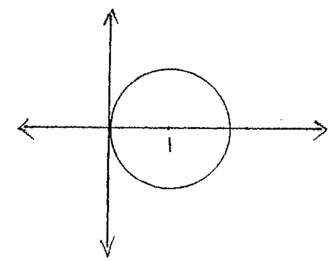
End of Assessment

KAMBALA

Question	Solutions	Marks	Marking Criteria
1(a)	$\int \frac{x^2+6}{x^2+4}\,dx = \int 1 + \frac{2}{x^2+4}\,dx$ $= x + \tan^{-1}\frac{x}{2} + c$		
(b) (i)	$A(x+1)^2 + B(x+1)(2x-1) + C(2x-1) = 9$ $x=-1 \Rightarrow -3C = 9\ ;\ C=-3$ $x=\frac{1}{2} \Rightarrow \frac{9}{4}A = 9\ ;\ A=4$ $x=0 \Rightarrow A-B-C=9\ ;\ B=-2$ $\frac{9}{(2x-1)(x+1)^2} = \frac{4}{2x-1} - \frac{2}{x+1} - \frac{3}{(x+1)^2}$		
(ii)	$\int_1^2 \frac{9}{(2x-1)(x+1)^2}\,dx$ $= \int_1^2 \left(\frac{4}{2x-1} - \frac{2}{x+1} - \frac{3}{(x+1)^2}\right)dx$ $= \left[2\ln(2x-1) - 2\ln(x+1) + \frac{3}{(x+1)}\right]_1^2$ $= 2\ln 3 - 2\ln 3 + 1 - 2\ln 1 + 2\ln 2 - \frac{3}{2}$ $= 2\ln 2 - \frac{1}{2}$		
(c)	$\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$ $\quad u = x\ \ \frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = \sec^2 x\ \ v = \tan x$ $= \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx$ $= \frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$ $= \frac{\pi}{4} - \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{4} + \ln(\cos\frac{\pi}{4}) - \ln(\cos 0)$ $= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} - \ln 1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$		

Question	Solutions	Marks	Marking Criteria
1(d)	$\int \frac{x^n}{e^x}\,dx$ $\quad u = x^n\ \ \frac{du}{dx} = nx^{n-1}$ $\quad \frac{dv}{dx} = e^{-x}\ \ v = -e^{-x}$ $= -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx$ $= n\int \frac{x^{n-1}}{e^x}\,dx - \frac{x^n}{e^x}$ $\int \frac{x^3}{e^x}\,dx = 3\int \frac{x^2}{e^x}\,dx - \frac{x^3}{e^x}$ $= 3\left(2\int \frac{x}{e^x}\,dx - \frac{x^2}{e^x}\right) - \frac{x^3}{e^x}$ $= 6\left(\int e^{-x}\,dx - \frac{x}{e^x}\right) - \frac{3x^2}{e^x} - \frac{x^3}{e^x}$ $= \frac{-6}{e^x} - \frac{6x}{e^x} - \frac{3x^2}{e^x} - \frac{x^3}{e^x}$		

Question	Solutions	Marks	Marking Criteria
2(a) (i)	$z = 1+\sqrt{3}\,i \qquad \lvert z\rvert = \sqrt{1^2+(\sqrt{3})^2} = 2$ $z = 2\operatorname{cis}\frac{\pi}{3} \qquad \arg z = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$		
(ii)	$z^7 = 2^7 \operatorname{cis}\frac{7\pi}{3}$ $= 128\operatorname{cis}\frac{\pi}{3}$ $z^7 - 64z = 128\operatorname{cis}\frac{\pi}{3} - 64\times 2\operatorname{cis}\frac{\pi}{3}$ $= 0$		
(b) (i)	$z^4 = 1 \Rightarrow z^2 = \pm 1$ $z = 1, -1, i, -i$		
(ii)	$z^4 = (z-1)^4$ $\left(\frac{z}{z-1}\right)^4 = 1$ $\frac{z}{z-1} = 1 \Rightarrow z = z-1$; no soln $\frac{z}{z-1} = -1 \Rightarrow z = -z+1$; $z = \frac{1}{2}$ $\frac{z}{z-1} = i \Rightarrow z = iz - i$; $z = \frac{i}{i-1}$ $\frac{z}{z-1} = -i \Rightarrow z = -iz+i$; $z = \frac{i}{1+i}$		
(c) (i)			

Question	Solutions	Marks	Marking Criteria
2(c) (i) c'td	In ΔPOR $\quad OP = OR \quad$ given $\vec{OR} = iz$ $\angle POR = \frac{\pi}{2} \quad \lvert iz\rvert = \lvert z\rvert$ iz - rotation of z through $\frac{\pi}{2}$ $\therefore \Delta POR$ is right-angled isosceles		
(ii)	$\angle OPQ = \frac{\pi}{2}$ $\angle OPR = \frac{\pi}{4}$ $\therefore \angle QPR = \frac{3\pi}{4}$ $= 135°$		
(d)	$\lvert z\rvert^2 = z + \bar{z}$ $\left(\sqrt{x^2+y^2}\right)^2 = x+iy + x-iy$ $x^2+y^2 = 2x$ $x^2 - 2x + 1 + y^2 = 1$ $(x-1)^2 + y^2 = 1$ circle centre $(1,0)$ radius 1 		

Question	Solutions	Marks	Marking Criteria

3 (a) (i)

$D: -2 \le x \le 2$

$f(-x) = -x\sqrt{4-(-x)^2}$

$= -x\sqrt{4-x^2}$

$= -f(x)$

$\therefore f(x)$ is an odd function

(ii)

$f(x) = x(4-x^2)^{\frac{1}{2}}$

$f'(x) = x \times \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \times -2x + (4-x^2)^{\frac{1}{2}}$

$= \dfrac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2}$

$= \dfrac{-x^2+4-x^2}{\sqrt{4-x^2}}$

$= \dfrac{4-2x^2}{\sqrt{4-x^2}}$

Stat pt. $f'(x) = 0 \Rightarrow 4-2x^2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

Test

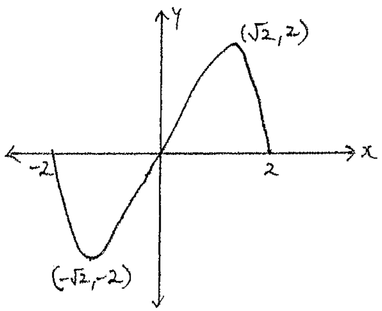
x	$-\sqrt{2}^-$	$-\sqrt{2}$	$-\sqrt{2}^+$
$f'(x)$	$-$	0	$+$

x	$\sqrt{2}^-$	$\sqrt{2}$	$\sqrt{2}^+$
$f'(x)$	$+$	0	$-$

minimum at $(-\sqrt{2}, -2)$

maximum at $(\sqrt{2}, 2)$

(iii)



endpts $(-2,0)$ & $(2,0)$

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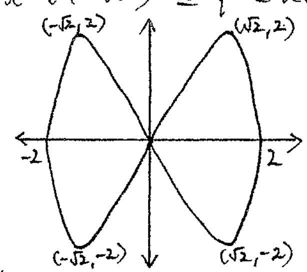
Question	Solutions	Marks	Marking Criteria

3 (a) (iv)

1. $y^2 = x^2(4-x^2)$ ie $y = \pm x\sqrt{4-x^2}$



2. $y = \dfrac{1}{f(x)}$

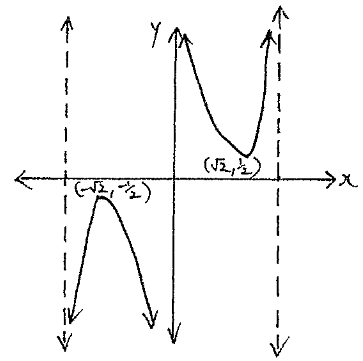
as $f(x) \to 0$

$\dfrac{1}{f(x)} \to \infty$

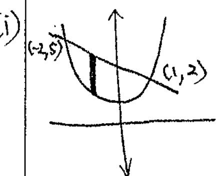
turning pts

$(-\sqrt{2}, -\frac{1}{2})$

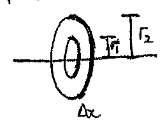
and $(\sqrt{2}, \frac{1}{2})$



(b) (i)



slice perp. to x-axis, width Δx



Area of slice $A = \pi(r_2^2 - r_1^2)$

$= \pi\left((3-x)^2 - (x^2+1)^2\right)$

$= \pi(9-6x+x^2-x^4-2x^2-1)$

$= \pi(8-6x-x^2-x^4)$

(ii)

Volume of slice $\Delta V = \pi(8-6x-x^2-x^4)\Delta x$

Volume $V = \lim_{\Delta x \to 0} \sum_{x=-2}^{1} \pi(8-6x-x^2-x^4)\Delta x$

$= \pi\int_{-2}^{1}(8-6x-x^2-x^4)\,dx$

$= \pi\left[8x-3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}$

$= \dfrac{117\pi}{5}\ u^3$

Question	Solutions	Marks	Marking Criteria

4 (a)

zeros $1+i$ and $-1+i$ $\therefore$ also has zeros $1-i$ and $-1-i$

$$P(x) = (x-(1+i))(x-(1-i))(x-(-1+i))(x-(-1-i))$$
$$= (x^2-2x+2)(x^2+2x+2)$$
$$= x^4+4$$

(b)(i)

$$\cos(A-B) - \cos(A+B)$$
$$= \cos A\cos B + \sin A \sin B - (\cos A\cos B - \sin A\sin B)$$
$$= 2\sin A\sin B$$

(ii)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 5x \sin x \, dx$$
$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cos 4x - \cos 6x)\, dx$$
$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{6}\sin 6x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left(\frac{1}{4}\sin\frac{4\pi}{3} - \frac{1}{6}\sin 2\pi - \frac{1}{4}\sin\pi + \frac{1}{6}\sin\frac{3\pi}{2}\right)$$
$$= \frac{1}{2}\left(-\frac{\sqrt{3}}{8} - \frac{1}{6}\right)$$
$$= -\frac{\sqrt{3}}{16} - \frac{1}{12}$$

(c)

$\angle BXC = 90°$ diagonals of a rhombus bisect at right-angles

$\angle BPC = 90°$ given $CP \perp AB$

Interval BC subtends equal angles $\angle BPC$ and $\angle BXC$ on the same side of it $\therefore$ endpts B and C and the points P and X are concyclic

B, P, X, C concyclic

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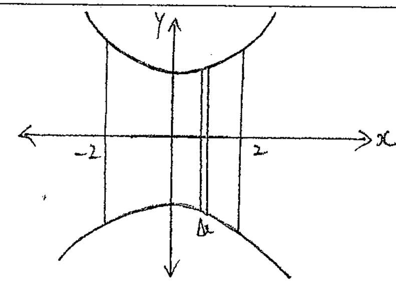
Question	Solutions	Marks	Marking Criteria

4 (d)



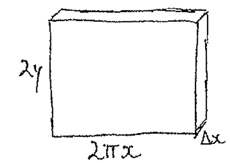
(i)

Take a slice parallel to y-axis, width Δx rotate about y-axis to form a cylindrical shell

$$\frac{y^2}{9} - \frac{x^2}{4} = 1 \Rightarrow y = \frac{3}{2}\sqrt{4+x^2}$$

shell ht. $2y = 3\sqrt{4+x^2}$

shell cross-section $A = \pi\left((x+\Delta x)^2 - x^2\right)$
$$= \pi\left((x+\Delta x)+x\right)\left((x+\Delta x)-x\right)$$
$$= \pi(2x+\Delta x)\Delta x$$
$$= 2\pi x \Delta x \quad (\Delta x^2 \text{ term can be ignored})$$



radius $= x$
height $= 2y$
$= 3\sqrt{4+x^2}$

Volume of shell $\Delta V = 2\pi x\Delta x \times 3\sqrt{4+x^2}$
$$= 6\pi x\sqrt{4+x^2}\,\Delta x$$

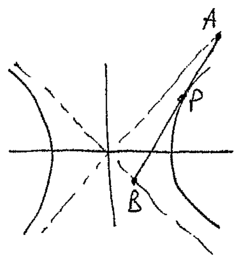
Volume $V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 6\pi x\sqrt{4+x^2}\,\Delta x$

(ii)

$$= 6\pi\int_0^2 x\sqrt{4+x^2}\,dx$$
$$= 6\pi\left[\frac{1}{3}(4+x^2)^{\frac{3}{2}}\right]_0^2$$
$$= 2\pi\left(8^{3/2} - 4^{3/2}\right)$$
$$= 16\pi(2\sqrt{2}-1)\ u^3$$

OR by subst. let $u = 4+x^2$ $\quad x=2\ u=8$
$\frac{du}{dx} = 2x$ $\quad x=0\ u=4$

$$V = \frac{6\pi}{2}\int_0^2 \sqrt{4+x^2}\,2x\,dx = 3\pi\int_4^8 \sqrt{u}\,du$$
etc.

Question	Solutions	Marks	Marking Criteria
5 (a)	$(x-1)^2$ a factor ie $(x-1)$ double root $\therefore Q(1) = Q'(1) = 0$ $Q(x) = Ax^n + Bx^{n-2} + 6$ $Q'(x) = nAx^{n-1} + (n-2)Bx^{n-3}$ $Q(1) = A + B + 6 = 0 \Rightarrow B = -A-6$ $Q'(1) = nA + (n-2)B = 0$ $nA + (n-2)(-A-6) = 0$ $nA - nA + 2A - 6n + 12 = 0$ $2A = 6n - 12$ $A = 3n - 6$ $B = -(3n-6) - 6 = -3n$		
(b) (i)	$\frac{x^2}{16} - \frac{y^2}{9} = 1$ asymptotes $y = \pm \frac{bx}{a}$ $a=4, b=3$ $y = \pm \frac{3x}{4}$ differentiate implicitly $\frac{x}{8} - \frac{2y}{9}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{9x}{16y} \Rightarrow m_T = \frac{3\sqrt{2}}{4}$ eq. tangent $y - 3 = \frac{3\sqrt{2}}{4}(x - 4\sqrt{2})$ $4y - 12 = 3\sqrt{2}x - 24$ $3\sqrt{2}x - 4y - 12 = 0$ $y = \pm\frac{3x}{4} \Rightarrow 3\sqrt{2}x \mp 3x - 12 = 0$ A: $3(\sqrt{2}-1)x = 12$; $x = \frac{4}{\sqrt{2}-1}$ $y = \frac{3}{\sqrt{2}-1}$ B: $3(\sqrt{2}+1)x = 12$; $x = \frac{4}{\sqrt{2}+1}$ $y = \frac{-3}{\sqrt{2}+1}$ A $\left(\frac{4}{\sqrt{2}-1}, \frac{3}{\sqrt{2}-1}\right)$ B $\left(\frac{4}{\sqrt{2}+1}, \frac{-3}{\sqrt{2}+1}\right)$ midpt $= \left(\frac{1}{2}\left(\frac{4}{\sqrt{2}-1} + \frac{4}{\sqrt{2}+1}\right), \frac{1}{2}\left(\frac{3}{\sqrt{2}-1} + \frac{-3}{\sqrt{2}+1}\right)\right)$		

Question	Solutions	Marks	Marking Criteria
5 (b) (i) c'td	midpt $= \left(\frac{1}{2}(8\sqrt{2}), \frac{1}{2}(6)\right) = (4\sqrt{2}, 3) = P$		
(ii)	$AB = \sqrt{\left(\frac{4}{\sqrt{2}-1} - \frac{4}{\sqrt{2}+1}\right)^2 + \left(\frac{3}{\sqrt{2}-1} - \frac{-3}{\sqrt{2}+1}\right)^2}$ $= \sqrt{8^2 + (6\sqrt{2})^2}$ $= \sqrt{136}$ $= 2\sqrt{34}$		
(c) (i)	In ΔACD $\tan \angle ADC = \frac{g+a}{y}$ In ΔBCD $\tan \angle BDC = \frac{a}{y}$ $\theta = \angle ADC - \angle BDC$ $\tan\theta = \tan(\angle ADC - \angle BDC)$ $= \frac{\tan\angle ADC - \tan\angle BDC}{1 + \tan\angle ADC \tan\angle BDC}$ $= \frac{\frac{g+a}{y} - \frac{a}{y}}{1 + \frac{g+a}{y} \times \frac{a}{y}}$ $= \frac{g}{y} \Big/ \frac{y^2 + a(g+a)}{y^2}$ $= \frac{gy}{a(g+a)+y^2}$		
(ii)	$\theta = \tan^{-1}\left(\frac{gy}{a(g+a)+y^2}\right)$ θ acute $\frac{d\theta}{dy} = \frac{1}{1 + \left(\frac{gy}{a(g+a)+y^2}\right)^2} \times \frac{g(a(g+a)+y^2) - gy \times 2y}{(a(g+a)+y^2)^2}$ $= \frac{1}{\frac{(a(g+a)+y^2)^2 + (gy)^2}{\cancel{(a(g+a)+y^2)^2}}} \times \frac{ga(g+a) - gy^2}{\cancel{(a(g+a)+y^2)^2}}$ $= \frac{g(a(g+a) - y^2)}{(a(g+a)+y^2)^2 + g^2y^2}$ max. when $\frac{d\theta}{dy} = 0 \Rightarrow y^2 = a(g+a)$ $y = \sqrt{a(g+a)}$ ($y > 0$ as it's a length) test: $y < \sqrt{a(g+a)}$ $\frac{d\theta}{dy} > 0$ $y > \sqrt{a(g+a)}$ $\frac{d\theta}{dy} < 0$ / max \\ Maximum θ when $y = \sqrt{a(g+a)}$		OR implicit diff. $\frac{d\theta}{dy} = \frac{g(a(g+a) - y^2)}{(a(g+a)+y^2)^2} \times \cos^2\theta$ $\frac{d\theta}{dy} = 0$ $\cos^2\theta = 0$ $\theta = 90°$ impossible $\therefore a(g+a) - y^2 = 0$ etc.

Question	Solutions	Marks	Marking Criteria

6 (a)

If roots are $\alpha^2, \beta^2, \gamma^2$ replace x by $\sqrt{x}$

$(\sqrt{x})^3 + 2(\sqrt{x})^2 - 2(\sqrt{x}) + 3 = 0$

$x\sqrt{x} + 2x - 2\sqrt{x} + 3 = 0$

$(2\sqrt{x} - x\sqrt{x}) = (2x+3)$

$\therefore 4x - 4x^2 + x^3 = 4x^2 + 12x + 9$

$x^3 - 8x^2 - 8x - 9 = 0$

(b) (i)

$x = \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t$

$= -9x$ which is of the form

$\ddot{x} = -n^2 x$ $\therefore$ motion is simple harmonic

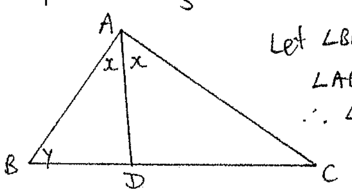
(ii)

$x = \sqrt{3}\cos 3t - \sin 3t = 2\cos\left(3t + \frac{\pi}{6}\right)$

amplitude 2

period $\frac{2\pi}{3}$

(c) (i)



Let $\angle BAD = \angle DAC = x$

$\angle ABD = y$

$\therefore \angle ADB = 180 - (x+y)$

$\angle ADC = x + y$

In ΔABD

$\frac{BD}{\sin x} = \frac{AB}{\sin(180-(x+y))}$

$\sin(180-(x+y)) = \sin(x+y)$

$\therefore \frac{BD}{AB} = \frac{\sin x}{\sin(x+y)}$

In ΔADC

$\frac{DC}{\sin x} = \frac{AC}{\sin(x+y)}$

$\therefore \frac{DC}{AC} = \frac{\sin x}{\sin(x+y)}$

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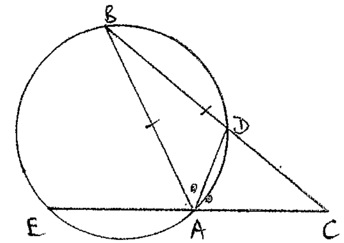
Question	Solutions	Marks	Marking Criteria

6 (c) (i) c'td

$\therefore \frac{BD}{AB} = \frac{DC}{AC}$

ie $\frac{BD}{DC} = \frac{AB}{AC}$

(ii)



From (i) $\frac{BD}{DC} = \frac{AB}{AC}$

since AD bisects $\angle BAC$

$EC \times CA = BC \times CD$ products of intercepts of intersecting chords

$\therefore \frac{AC}{DC} = \frac{BC}{EC}$

$\frac{AC}{DC} = \frac{AB}{EC}$ given $AB = BC$

ie $\frac{AB}{AC} = \frac{EC}{DC}$

$\therefore \frac{BD}{DC} = \frac{EC}{DC}$

ie $BD = EC$

Question	Solutions	Marks	Marking Criteria

7 (a)

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ $\quad t=\tan\frac{x}{2}$ $\quad \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2}$

$\displaystyle = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$ $\quad \frac{dx}{dt}=\frac{2}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$

$\displaystyle = \int_0^1 \frac{2}{2(1+t^2)+1-t^2}\,dt$

$\displaystyle = \int_0^1 \frac{2}{3+t^2}\,dt$

$\displaystyle = \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$\displaystyle = \frac{\pi}{3\sqrt{3}}$

(b) (i) foci $(\pm ae, 0)$ directrices $x = \pm\frac{a}{e}$

(ii) $\displaystyle \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\displaystyle \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

at $P(x_1,y_1)$ $\displaystyle m_T = \frac{-b^2x_1}{a^2y_1}$

$\displaystyle \therefore m_N = \frac{a^2y_1}{b^2x_1}$

equation $\displaystyle y-y_1 = \frac{a^2y_1}{b^2x_1}(x-x_1)$

$b^2x_1y - b^2x_1y_1 = a^2xy_1 - a^2x_1y_1$

$a^2x_1y_1 - b^2x_1y_1 = a^2xy_1 - b^2x_1y$

$\displaystyle a^2-b^2 = \frac{a^2x}{x_1} - \frac{b^2y}{y_1}$

(iii) x-int. $\displaystyle a^2-b^2 = \frac{a^2x}{x_1}$

$\displaystyle x = x_1 - \frac{b^2x_1}{a^2}$

$\displaystyle Q\left(x_1 - \frac{b^2x_1}{a^2}, 0\right)$ $\quad S(ae, 0)$

$\displaystyle QS = \left|x_1 - \frac{b^2x_1}{a^2} - ae\right|$ $\quad \frac{b^2}{a^2}=1-e^2$

$= |x_1 - x_1 + e^2x_1 - ae|$

$= |e^2x_1 - ae|$

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Question	Solutions	Marks	Marking Criteria

7 (b) (iii) c'td

$P(x_1, y_1)$ $\quad M\left(\frac{a}{e}, y_1\right)$

$\displaystyle e^2PM = e^2\left(x_1 - \frac{a}{e}\right)$

$= e^2x_1 - ae$

$= QS$

(c) (i) $1 + x + x^2 + x^3 + \dots + x^{n-1}$

geom series $a=1$, $r=x$, n terms

$\displaystyle \therefore S_n = \frac{1(1-x^n)}{1-x} = \frac{1-x^n}{1-x}$

(ii) differentiate

LHS $= 1 + 2x + 3x^2 + \dots + (n-1)x^{n-2}$

$\displaystyle \text{RHS} = \frac{(1-x)\times -nx^{n-1} - (1-x^n)\times -1}{(1-x)^2}$

$\displaystyle = \frac{-nx^{n-1}+nx^n+1-x^n}{(1-x)^2}$

$\displaystyle = \frac{(n-1)x^n - nx^{n-1}+1}{(1-x)^2}$

$\displaystyle \therefore 1+2x+3x^2+\dots+(n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1}+1}{(1-x)^2}$

(iii) let $x=\frac{1}{2}$

$\displaystyle \text{LHS} = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \dots + \frac{n-1}{2^{n-2}}$

$\displaystyle \text{RHS} = \frac{\frac{(n-1)}{2^n} - \frac{n}{2^{n-1}} + 1}{\left(\frac{1}{2}\right)^2}$

$\displaystyle = 4\left(1 + \frac{n}{2^n} - \frac{1}{2^n} - \frac{n}{2^{n-1}}\right)$

$\displaystyle = 4\left(1 + \frac{n}{2}\times\frac{1}{2^{n-1}} - \frac{1}{2}\times\frac{1}{2^{n-1}} - n\times\frac{1}{2^{n-1}}\right)$

$\displaystyle = 4\left(1 - \frac{1}{2^{n-1}}\left(\frac{n}{2}+\frac{1}{2}\right)\right)$

$\displaystyle = 4\left(1 - \frac{1}{2^{n-2}}(n+1)\right)$

$\displaystyle 1+1+\frac{3}{4}+\frac{4}{8}+\dots+\frac{n-1}{2^{n-2}} = 4\left(1-\frac{1}{2^{n-2}}(n+1)\right)$

since $\frac{n+1}{2^{n-2}} > 0$ $(n>0)$

$\displaystyle 1+1+\frac{3}{4}+\frac{4}{8}+\dots+\frac{n-1}{2^{n-2}} < 4$

Question	Solutions	Marks	Marking Criteria

8 (a) (i)

$(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Also $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Equate coefficients

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

(ii)

$\cos 3\theta = \frac{1}{2}$ $\therefore \frac{1}{2} = 4\cos^3\theta - 3\cos\theta$

$1 = 8\cos^3\theta - 6\cos\theta$

$8\cos^3\theta - 6\cos\theta - 1 = 0$

$\therefore 8x^3 - 6x - 1 = 0$ $\quad x = \cos\theta$

(iii)

Solve $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{9} + \frac{2k\pi}{3}$ $\quad k = 0, 1, 2, \ldots$

$\theta = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9}, \ldots$

Solutions $x = \cos\frac{\pi}{9}$

$x = \cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

$x = \cos\frac{13\pi}{9} = -\cos\frac{4\pi}{9}$

(iv)

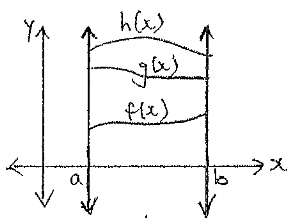
$8x^3 - 6x - 1 = 0$

sum of roots $= 0$

$\therefore \cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = 0$

ie $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$

(b) (i)



$\int_a^b f(x)\,dx$ gives area between the curve, the x-axis + the lines $x=a$ and $x=b$.

Similarly for $\int_a^b g(x)\,dx$ and $\int_a^b h(x)\,dx$

For $a \le x \le b$

$f(x) < g(x) < h(x)$

$\therefore$ areas increasing also

ie $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$

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Question	Solutions	Marks	Marking Criteria

8 (b) (ii) c'td

$0 < x < 1$

$1 < x+1 < 2$

$\frac{1}{2} < \frac{1}{x+1} < 1$

x and $(1-x)^3$ positive for $0<x<1$

$\therefore \frac{x(1-x)^3}{2} < \frac{x(1-x)^3}{x+1} < x(1-x)^3$

(iii)

All 3 functions in (ii) are distinct, non-negative continuous functions for $0 < x < 1$

$\therefore$ using (i)

$\int_0^1 \frac{x(1-x)^3}{2}\,dx < \int_0^1 \frac{x(1-x)^3}{x+1}\,dx < \int_0^1 x(1-x)^3\,dx$

ie $\frac{1}{2}\int_0^1 x(1-x)^3\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,dx < \int_0^1 x(1-x)^3\,dx$

(iv)

$\int_0^1 x(1-x)^3\,dx$ $\qquad$ let $u = 1-x$ $\frac{du}{dx} = -1$

$x = 1-u$

$x=0$ $u=1$

$x=1$ $u=0$

$= -\int_1^0 (1-u)u^3\,du$

$= \left[\frac{u^4}{4} - \frac{u^5}{5}\right]_0^1$

$= \frac{1}{20}$

$\therefore \frac{1}{2}\int_0^1 x(1-x)^3\,dx = \frac{1}{40}$

Given $\int_0^1 \frac{x(1-x)^3}{1+x}\,dx = \frac{67}{12} - 8\ln 2$

$\frac{1}{40} < \frac{67}{12} - 8\ln 2 < \frac{1}{20}$

$\frac{-667}{120} < -8\ln 2 < \frac{-83}{15}$

$\therefore \frac{83}{15} < 8\ln 2 < \frac{667}{120}$

$\therefore \frac{83}{120} < \ln 2 < \frac{667}{960}$