KAMBALA CHURCH OF ENGLAND GIRLS' SCHOOL

MATHEMATICS

EXTENSION 1

YEAR 11 PRELIMINARY EXAMINATION

AUGUST 2001

Time Allowed: 2 hours
Reading Time: 5 minutes

INSTRUCTIONS

- Calculators and geometrical instruments may be used.
- Answer all questions on the writing paper provided.
- All questions are of equal value. Start each question on a new page.
- Show all necessary working.
- Marks will be deducted for careless or badly arranged work.

Question 1: **Marks**

(a) Solve the inequality $\frac{x}{2x-1} \le 5$ **3**

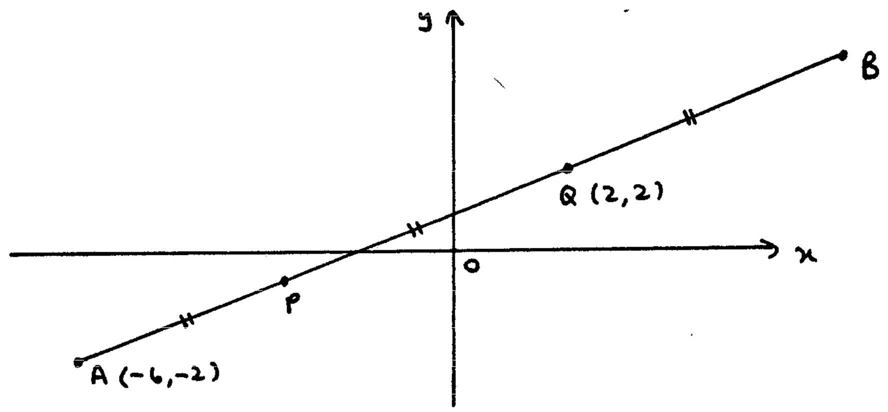
(b) (i) Write down the expansion of $\cos(\alpha - \beta)$ **1**

(ii) Hence find the exact value of $\cos 15^0$ **2**

(c) (i) On the same axes, sketch the curves $y = x^2$ and $y = |x|$ **2**

(ii) Hence, or otherwise, solve $x^2 < |x|$ **1**

(d) The line interval AB is trisected at P and Q.(ie AB:BQ = 3:1). The co-ordinates of A are (-6 , -2) and Q are (2 , 2). Find the co-ordinates of B. **3**



Question 2: (Start a new page)

(a) The equation $x^2 - (1-2k)x + k + 3 = 0$ has consecutive integral roots. Find the values of k. **3**

(b) Find the equation of the line through the point of intersection of the lines $2x + 3y - 7 = 0$ and $x - 2y + 1 = 0$ which is perpendicular to the line $y = 1 - 3x$. **3**

(c) Express $\frac{1 - x^{-1}}{x^{-1} - x^{-2}}$ in its simplest form. **3**

(d) The points (2 , 11) , (1 , 6) and (0 , 5) lie on the parabola $y = ax^2 + bx + c$. Find the values of a, b and c. **3**

Question 3: (Start a new page) **Marks**

(a) If α and β are the roots of the equation $x - 7 + \dfrac{4}{x} = 0$, find the value of: **6**

(i) $\alpha + \beta$ and $\alpha\beta$

(ii) $\alpha^2 + \beta^2$

(iii) $\alpha^3 + \beta^3$

(iv) $\alpha - \beta$

(b) The lines $3x - y + 2 = 0$ and $mx - y - 1 = 0$ intersect at 45^0. Find the possible value(s) of m. **3**

(c) The line $3x + 4y + 7 = 0$ is a tangent to a circle with centre (2 , 1). Find the equation of the circle. **3**

Question 4: (Start a new page)

(a) If $f(x) = x^2 - 2c$ and $F(x) = 3x + c$, find:

(i) $f[F(x)]$ **2**

(ii) the value of c if $f[F(0)] = 0$.

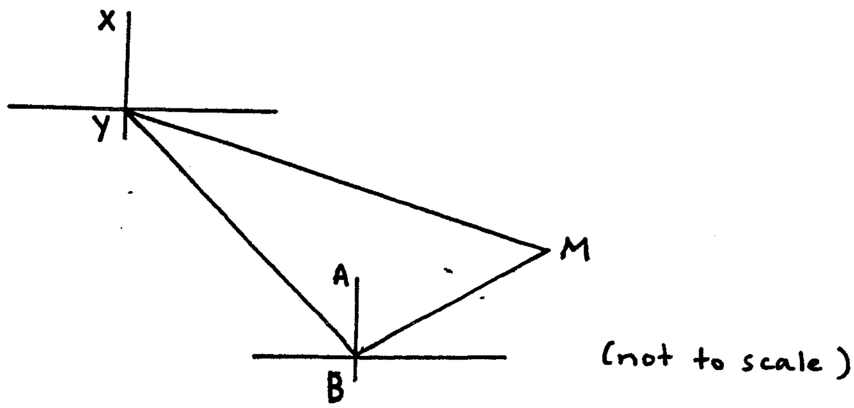
(b) Solve the equation $|x+1|^2 - 4|x+1| - 5 = 0$. **3**

(c) For what value(s) of k does the equation $kx^2 - 4x + (k+3) = 0$ have real roots? **3**

Question 4 (continued): **Marks**

(d) A man, M, is standing in a horizontal plane which also contains a tower, AB, which is 10 metres high, and a building, XY, which is 95 metres high. The man is 210 metres from the tower and his bearing from it is 042^0T. His bearing from the building is 110^0T. The bearing of the tower from the building is 150^0T. **4**

(i) Copy and complete the diagram showing all the information.



(ii) Find, to the nearest degree, the angle of elevation from the top of the tower to the top of the building.

Question 5: (Start a new page)

(a) Solve the equation $2\sin^2\theta + 5\sin\theta + 2 = 0$ for $0^0 \le \theta \le 360^0$. **2**

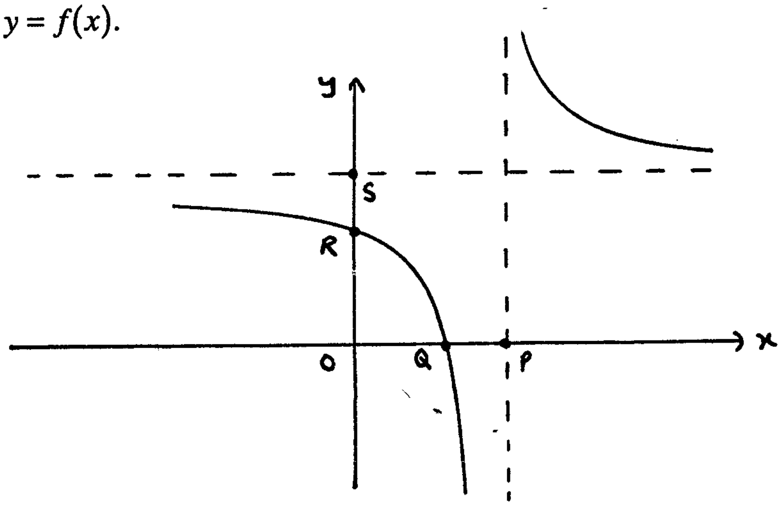
(b) Consider the line $3x - 4y + 5 = 0$. Find, correct to the nearest degree, the angle of inclination made with the positive x-axis. **2**

(c) A triangle has sides of length p, $(p+1)$ and $(p+2)$ units. Find the cosine of the largest angle in terms of p in simplest form. **2**

(d) Solve for x: $2x - 5 = \sqrt{x-2}$ **2**

Question 5 (continued): **Marks**

(e) Suppose a, b and c are positive real numbers and let $f(x)$ be the function defined by $f(x) = \dfrac{ax-b}{x-c}$. The diagram below shows the graph of $y = f(x)$. **4**



Find, in terms of a, b and c, the co-ordinates of:

(i) P

(ii) Q

(iii) R

(iv) S

Question 6: (Start a new page)

(a) Simplify $\dfrac{24^{x+1} \times 8^{-1}}{6^{2x}}$, writing your answer in the form $2^a \times 3^b$ **2**

(b) Show that the equation $mx^2 + (2m+n)x + 2n = 0$ has rational roots for all rational values of m and n. **2**

Marks

Question 6 (continued):

(c) (i) Use the perpendicular distance formula to show that A (-2 , 3) is equidistant from the two lines $x-3y+1=0$ and $3x+y-7=0$. **4**

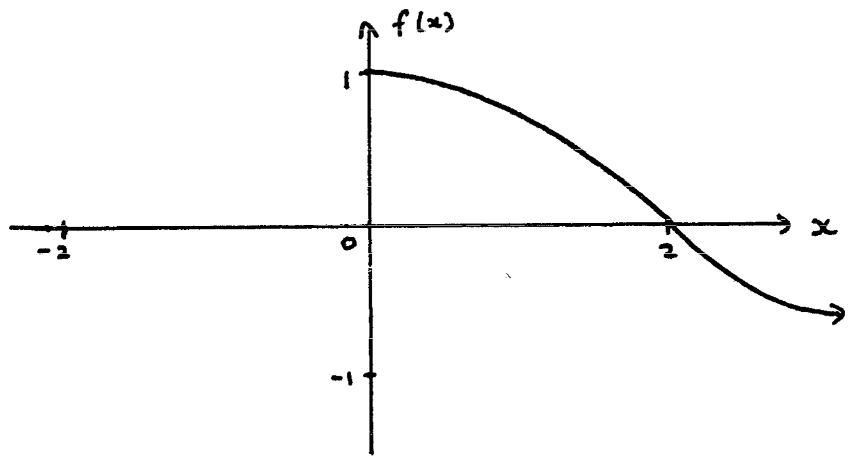
(ii) Hence find the equation of the line through A that bisects the angle between the two lines, without finding their point of intersection.

(d) Graph the region $|2x+3y|\le 6$.

(e) Sketch the function $f(x)=\dfrac{1}{x-8}$ showing any x-intercepts, y-intercepts or asymptotes. **2**

Question 7: (Start a new page)

(a) Part of the graph of the function $y=f(x)$ is shown below. **3**



Draw three neat copies of this graph and label them A, B and C. Complete the graphs of $y=f(x)$ on each sketch so that:

(i) Graph A shows $y=f(x)$ is an even function.

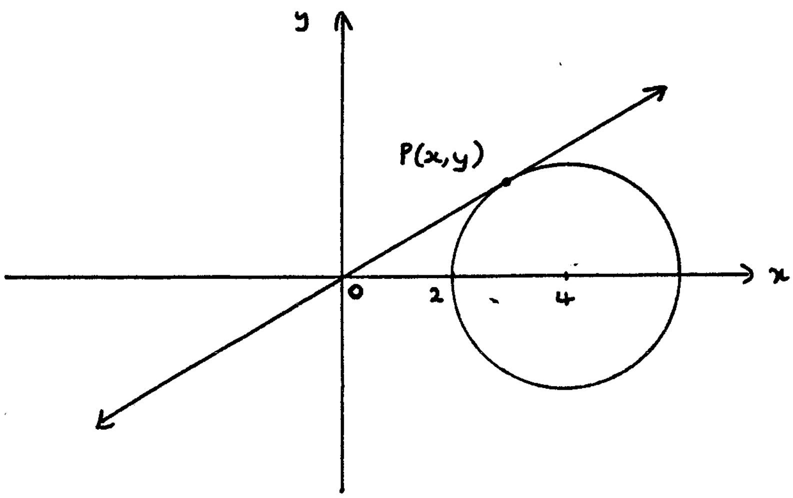
(ii) Graph B shows $y=f(x)$ is an odd function.

(iii) Graph C shows $y=f(x)$ is neither odd nor even.

Question 7 (continued): **Marks**

(b) If $x^2 - 4px + 3p - 2 = 0$ find the value of p given that the product of the roots is three times the sum. **2**

(c) (i) Write down the equation of the circle in the diagram. **7**



(ii) Write down the equation of the line through the origin with gradient m.

(iii) Show that the x co-ordinate of P, the point of intersection between the line and the circle, satisfies the equation $(m^2 + 1)x^2 - 8x + 12 = 0$.

(iv) Hence find the value of m and the co-ordinates of P.

END OF EXAMINATION

Stephanie Simi

8/2

KAMBALA Maths 3U August 2001

1) A) $\dfrac{x}{2x-1} \le 5$

$\dfrac{x}{2x-1} - 5 \le 0$

$\dfrac{x-5(2x-1)}{2x-1} \le 0$

$\dfrac{x-10x+5}{2x-1} \le 0$

$\dfrac{-9x+5}{2x-1} \le 0$

$-9x \le -5$

$9x \ge 5$

$x \ge \dfrac{5}{9}$

Consider $\dfrac{+}{-}$ & $\dfrac{-}{+}$ cases. Try again.

B) i.) $\cos(\alpha-\beta)$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta$ ✓

ii.) $\cos 15° = \cos(45-30)$

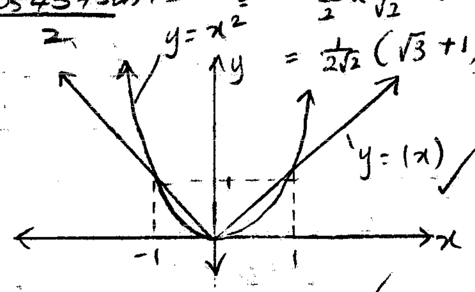
$\cos 45\cos 30 + \sin 45 \sin 30$

$= \dfrac{\sqrt{3}\cos 45}{2} + \dfrac{\sin 45}{2}$ $\quad \cos 45 = \frac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3}\cos 45 + \sin 45}{2} = \dfrac{\sqrt{3}}{2}\times\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\times\dfrac{1}{\sqrt{2}}$

$= \dfrac{1}{2\sqrt{2}}(\sqrt{3}+1)$

c) i.)



✓

ii) $x^2 < |x| \quad \underline{-1 < x < 1}$ ✓

D) $k:l$ is ratio.

Point $= \dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}$

INTERNALLY

Point $= \dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}$

EXTERNALLY

or could use this formula

D) $AP = PQ = QB$

$\therefore P$ is midpt. of AQ.

$P = \left(\left(\dfrac{-6+2}{2}\right), \left(\dfrac{-2+2}{2}\right)\right)$

$= \underline{(-2, 0)}$

Q is mdpt. of PB. Let $B = (x,y)$.

$\dfrac{-2+x}{2} = 2 \qquad \dfrac{0+y}{2} = 2$ ✓

$-2+x = 4 \qquad \dfrac{y}{2} = 2 ; \underline{y=4}$

$\underline{x=6}$

$\therefore \underline{B = (6,4)}$ ✓

2) A) $P(x) = x^2 - (1-2k)x + (k+3) = 0$

Let roots be α and $(\alpha+1)$

$\alpha+\alpha+1 = 1-2k$

$2\alpha+1 = 1-2k$

$2\alpha = -2k$

$\alpha = -k \; ; \; \underline{k=-\alpha}$ (1)

$\alpha(\alpha+1) = +(k+3) \qquad \frac{c}{a}$

$-k(-k+1) = +(k+3)$

$k^2 - k + k - 3 = 0$

$k^2 - 3 = 0$

$k^2 = +3$ ✗ N/S ?

$\therefore \underline{k = -\alpha}$.

$k = \pm\sqrt{3}$.

B) $2x+3y-7=0$ ①

$x-2y+1=0$;

$2x-4y+2=0$ ②

Solve ① and ② simult

① - ② $= 2x+3y-7-2x+4y-2$

$7y-9=0$

$7y=9$; $y=\frac{9}{7}$

$x-2(\frac{9}{7})+1=0$

$x=\frac{11}{7}$

∴ pt of intersection is $(\frac{11}{7}, \frac{9}{7})$

$y=1-3x$

$m_1=-3$

gradient, m_2 of line through $(\frac{11}{7}, \frac{9}{7})$ is perpendicular to $y=1-3x$.

∴ $m_2=\frac{1}{3}$ ($m_1m_2=-1$)

$y-y_1=m(x-x_1)$

(sub in values of pt. and gradient value)

$y-\frac{9}{7}=\frac{1}{3}(x-\frac{11}{7})$

$3(y-\frac{9}{7})=x-\frac{11}{7}$

$3y-\frac{27}{7}-x+\frac{11}{7}=0$

$21y-27-7x+11=0$

$21y-7x-16=0$

∴ $7x-21y+16=0$

c) $\dfrac{1-x^{-1}}{x^{-1}-x^{-2}} = \dfrac{1-\frac{1}{x}}{\frac{1}{x}-\frac{1}{x^2}}$

$= \dfrac{\frac{x-1}{x}}{\frac{x-1}{x^2}}$; $= \dfrac{\cancel{x-1}}{\cancel{x}} \times \dfrac{x^{\cancel{2}}}{\cancel{x-1}}$

$= \boxed{x}$

D) if $(2,11)$, $(1,6)$, $(0,5)$ lie on $y=ax^2+bx+c$, then

① $11=4a+2b+c$

② $6=a+b+c$

③ $5=c$

Sub ③ into ① and ②

$11=4a+2b+5$.

$4a+2b=6$

$2a+b=3$ ... ④

$6=a+b+5$

$a+b=1$

$b=1-a$ ⑤

Sub ⑤ into ④

$2a+1-a=3$

$a+1=3$; $a=2$

$b=1-2=-1$

$b=-1$

$c=5$

②

3) A) $p(x) = x - 7 + \frac{4}{x} = 0$

$\frac{x^2-7x+4}{x} = 0$

$x^2 - 7x + 4 = 0$

α and β are roots.

i.) $\alpha+\beta = \frac{-(-7)}{1} = 7$ ✓

$\alpha\beta = 4$ ✓

ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 7^2 - 2.4$
$= 49 - 8 = 41$ ✓

iii) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3(\alpha^2\beta+\alpha\beta^2)$
$= (\alpha+\beta)^3 - 3(\alpha\beta(\alpha+\beta))$ ✓

$= 7^3 - 3(4(7))$

$= 343 - 3\times 28$
$= 343 - 84$ ✓
$= 259$

iv) $\alpha-\beta$
$(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$
$= (\alpha+\beta)^2 - 4\alpha\beta$
$= 7^2 - 4(4)$ ✓
$= 49 - 16$
$= 33$

$\boxed{\alpha - \beta = \pm\sqrt{33}}$ ✓

B) $3x - y + 2 = 0$
$y - 3x - 2 = 0$
$y = 3x + 2$
$\therefore m_1 = 3$ ① ✓

$mx - y - 1 = 0$
$y - mx + 1 = 0$
$y = mx - 1$
$m_2 = m$ ② ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan 45° = \left|\frac{3-m}{1+3m}\right|$

$\frac{3-m}{1+3m} = 1$	$\frac{3-m}{1+3m} = -1$
$3 - m = 1 + 3m$	$3 - m = -1 - 3m$
$1 + 3m - 3 + m = 0$	$3 - m + 1 + 3m = 0$
$4m - 2 = 0$	$4 + 2m = 0$
$4m = 2$	$m = -2$ ✓
$m = \frac{1}{2}$ ✓	

c) To find r, find perpendicular dist from $3x + 4y + 7 = 0$ to $(2,1)$

$\perp$ Dist $= \frac{|ax_1 + by_1 + c_1|}{\sqrt{a^2+b^2}}$
$= \frac{|3\times 2 + 4\times 1 + 7|}{\sqrt{3^2+4^2}}$
$= \left|\frac{6+4+7}{5}\right|$ ✓

$= \frac{17}{5}$ units

$\therefore r = \frac{17}{5}$ units

$(x-2)^2 + (y-1)^2 = \left(\frac{17}{5}\right)^2$ ✓

$$x^2-4x+4+y^2-2y+1=\frac{289}{25}$$

$$x^2-4x+y^2-2y+5=\frac{289}{25}$$

$$25x^2+25y^2-100x-50y+125=289$$

$$\therefore 25x^2+25y^2-100x-50y-164=0$$

4A) i) $f[F(x)]$ =

$$=(3x+c)^2-2c$$

$$=9x^2+6xc+c^2-2c$$

ii) If $f[F(0)]=0$

$$F(0)=3(0)+c$$
$$=c$$
$$f(c)=c^2-2c$$
$$c^2-2c=0$$
$$c(c-2)=0$$
$$c=0 \text{ or } 2$$

B) $|x+1|^2-4|x+1|-5=0$

① $x^2+2x+1-4x-4-5=0$

$$x^2-2x-8=0$$
$$(x-4)(x+2)=0$$
$$x=4 \text{ or } -2$$

② $x^2+2x+1+4x+4-5=0$

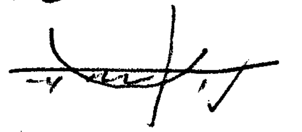
$$x^2+6x=0$$
$$x(x+6)=0$$
$$x=0 \text{ or } -6$$

$\therefore x=4,-2,0 \text{ or } -6$

Test the results $x\neq 0,\ x\neq -2$

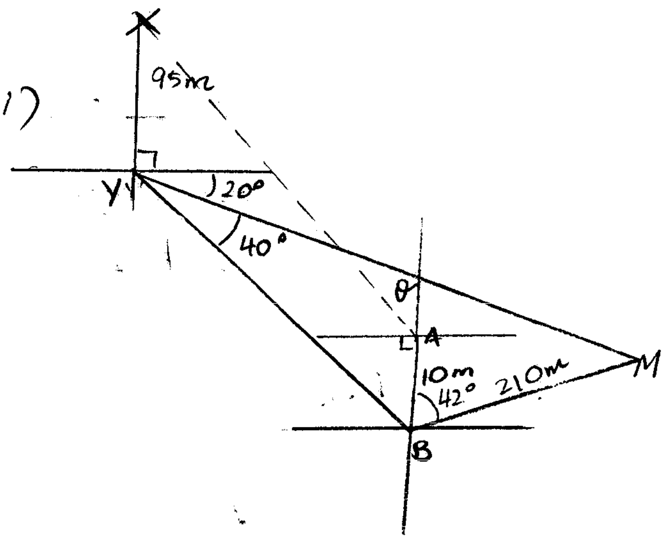
c) Real roots, $\Delta \geq 0$

$$kx^2-4x+(k+3)=0$$
$$b^2-4ac\geq 0$$
$$16-4k(k+3)\geq 0$$
$$16-4k^2-12k\geq 0$$
$$4k^2+12k-16\leq 0$$
$$k^2+3k-4\leq 0$$
$$(k+4)(k-1)\leq 0$$
$$k=-4 \text{ or } 1$$



$$\therefore -4\leq k\leq 1$$

4) i)



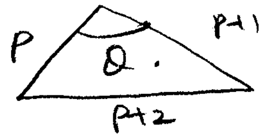
ii) $\theta=90°-42°$ (right $\angle=90°$)
$=48°$

$\delta=90°-60°$ (right $\angle=90°$)
$=30°$

In Δ YBS, $\alpha=180°-30°-90°$ ($\angle$ sum of $\Delta=180°$)
$=60°$

$\therefore \beta=90°-60°$ (right $\angle=90°$)
$=30°$

3



c) Using cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$ ✓

Largest $\angle$ is opp. largest side, which is $\underline{p+2}$. $\therefore$ $\underline{a = p+2}$

$$\cos A = \frac{(p+1)^2 + p^2 - (p+2)^2}{2p(p+1)}$$ ✓

$$= \frac{p^2+2p+1+p^2-p^2-4p-4}{2p^2+2p}$$

$$= \frac{p^2-2p-3}{2p(p+1)}$$

$$= \frac{(p-3)\cancel{(p+1)}}{2p\cancel{(p+1)}}$$

$$= \frac{p-3}{2p}$$ ✓

$$\boxed{\cos A = \frac{p-3}{2p}}$$

D) $2x - 5 = \sqrt{x-2}$

$(2x-5)^2 = x-2$

$4x^2 - 20x + 25 - x + 2 = 0$

$4x^2 - 21x + 27 = 0$ ✓

$(4x-9)(x-3) = 0$

$\underline{x = \frac{9}{4} \text{ or } 3}$

Test the results.

$x \neq \frac{9}{4}$

5) A) $2\sin^2\theta + 5\sin\theta + 2 = 0 \quad 0° < \theta < 360°$

$(2\sin\theta + 1)(\sin\theta + 2) = 0$ ✓

① $\sin\theta = -\frac{1}{2}$ $\quad$ S | A, T | C

(Acute $\theta = 30°$)

$\underline{\theta = 210°, 330°}$ ✓

② $\sin\theta = -2$ ∴ No solution

B) $3x - 4y + 5 = 0$

$4y = 5 + 3x$

$y = \frac{3}{4}x + \frac{5}{4}$

Gradient $= \frac{3}{4}$ ✓

$\tan\theta = \frac{3}{4}$

$\theta = 36°52'$ ✓

$\underline{= 37° \text{ (to nearest °)}}$

E) i.) $f(x) = \frac{ax-b}{x-c}$

$x - c \neq 0$

$x \neq c \therefore \underline{P = (c, 0)}$ ✓

ii.) At Q, $f(x) = 0$

$0 = \frac{ax-b}{x-c}$

$ax - b = 0$

$ax = b; \ x = \frac{b}{a}$

$\therefore \underline{Q = (\frac{b}{a}, 0)}$ ✓

iii.) At R, $x = 0$

$f(0) = \frac{-b}{-c} = \underline{\frac{b}{c}}$

$\therefore \underline{R(0, \frac{b}{c})}$ ✓

iv.) S is the y-asymptote.

$f(x) = \frac{ax-b}{x-c}$

$= \frac{a(\cancel{x-c})}{\cancel{x-c}} + \frac{ac-b}{x-c}$

$= a + a\frac{c-b}{x-c}$

$\therefore \underline{S = a}$

$\underline{S = (0, a)}$ ✓

6) $\frac{24^{x+1} \times 8^{-1}}{6^{2x}}$

$= 24^x \cdot 24 \cdot 8^{-1}$

$= \frac{\cancel{6^x} \cdot 4^x \cdot 6 \cdot 4 \cdot 8^{-1}}{\cancel{6^{2x}}}$

$= \frac{4^x \cdot \cancel{24}^3 \cdot \frac{1}{\cancel{8}}}{6^x}$

$= \frac{3 \cdot 4^x}{6^x}$ ✓

$= \frac{\cancel{3} \cdot 2^x \cdot \cancel{2^x}}{\cancel{2^x} \cdot 3^x}$

$= \frac{2^x}{3^{x-1}}$

$= 2^x \times (3^{x-1})^{-1}$

$\boxed{= 2^x \times 3^{1-x}}$ ✓

B) For rational roots, $\Delta > 0$ & $-b \pm \sqrt{b^2 - 4ac}$ is a PERFECT SQUARE

$mx^2 + (2m+n)x + 2n = 0$.

$(2m+n)^2 - 4m \times 2n > 0$

$4m^2 + 4mn + n^2 - 8mn > 0$.

$4m^2 - 4mn + n^2 > 0$

$(2m-n)(2m-n) > 0$

$\underline{(2m-n)^2 > 0}$ ✓

$\therefore$ since Δ is <u>BOTH</u> a perfect square <u>AND</u> is greater than 0,

then it has rational roots for all rational values of m and n.

c) i) Perpend. dist = $\frac{|ax_1+by_1+c_1|}{\sqrt{a^2+b^2}}$

① $\left|\frac{1\times-2-3\times3+1}{\sqrt{1+9}}\right|$

$=\left|\frac{-2-9+1}{\sqrt{10}}\right|$

$=\frac{10}{\sqrt{10}} = \frac{10\sqrt{10}}{10} = \sqrt{10}$ units

② $\left|\frac{3\times-2+1\times3-7}{\sqrt{9+1}}\right|$

$=\frac{|-6+3-7|}{\sqrt{10}}$

$=\frac{10}{\sqrt{10}} = \sqrt{10}$ units

∴ A is equidistant from $x-3y+1=0$ & $3x+y-7=0$

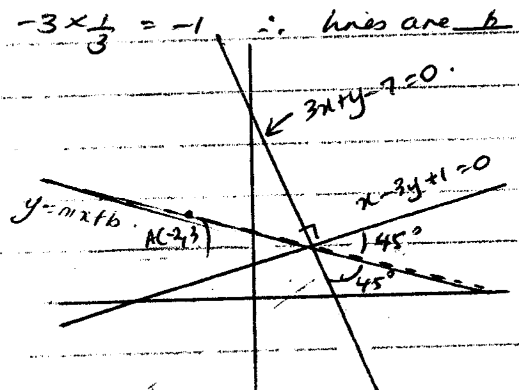
ii) $x-3y+1=0$

$y=\frac{x+1}{3}$; $m_1=\frac{1}{3}$ ①

$3x+y-7=0$

$y=7-3x$; $m_2=-3$ ②

$-3\times\frac{1}{3}=-1$ ∴ lines are ⊥



$\tan 45 = \left|\frac{\frac{1}{3}-m}{1+\frac{m}{3}}\right|$

$1=\left|\frac{1-3m}{3}\times\frac{3}{3+m}\right|$

$1=|1-3m|$

D) $|2x+3y|\le 6$

$-6\le 2x+3y\le 6$

① $-6\le 2x+3y$

$2x+3y\ge -6$

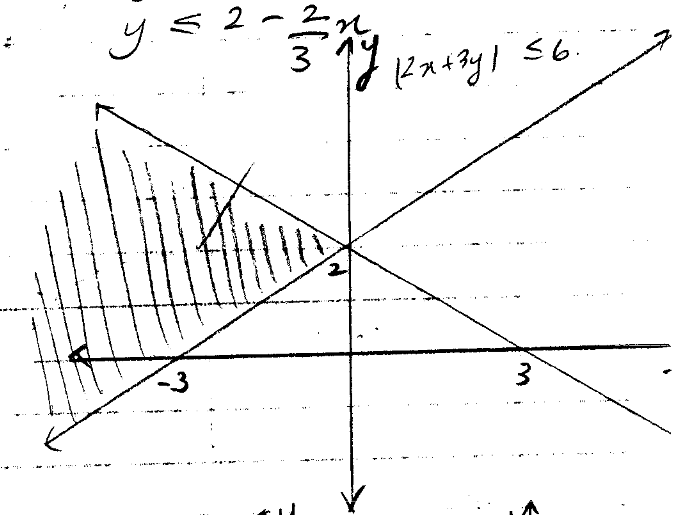
$3y\ge -6-2x$

$y\ge -2-\frac{2}{3}x$

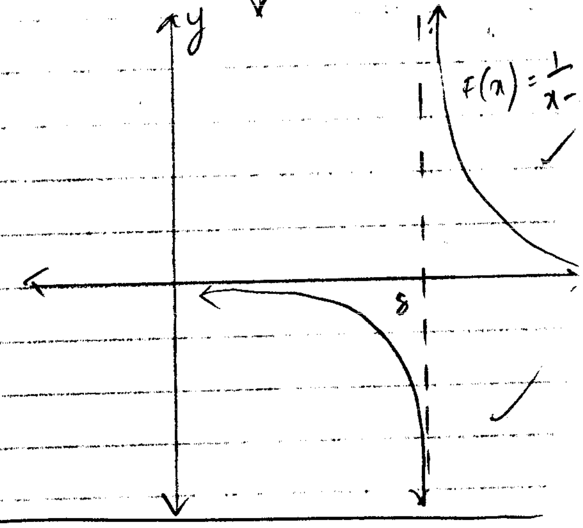
② $2x+3y\le 6$

$3y\le 6-2x$

$y\le 2-\frac{2}{3}x$



E)



① $1=\frac{1-3m}{3+m}$; $3+m=1-3m$

$3+4m=1$

$4m=-2$; $m_1=-\frac{1}{2}$

② $-1=\frac{1-3m}{3+m}$; $-3-m=1-3m$

$-3+2m=1$

$2m=4$; $m_2=2$

But since line has negative gradient

Now, to find eqt of line, sub in A coord + grad. value

$y = mx + b.$

$3 = -\frac{1}{2}(-2) + b; \quad 3 = 1 + b; \quad \underline{\underline{b = 2}}$

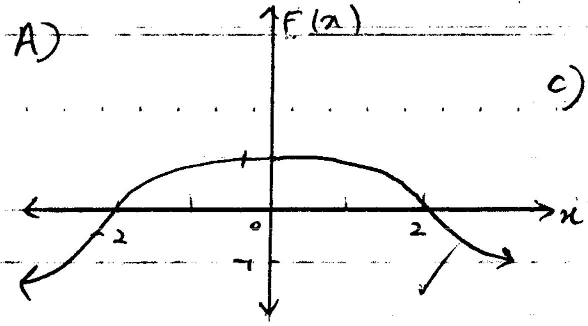
$\therefore \; y = -\frac{1}{2}x + 2$

$2y = -x + 4$

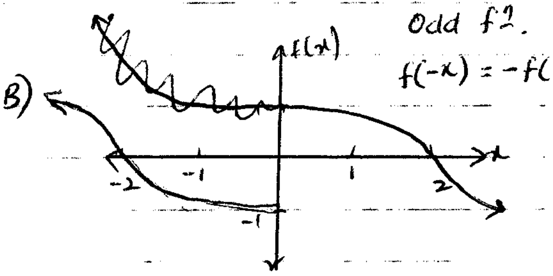
$\boxed{2y + x - 4 = 0}$

7) A)

A)

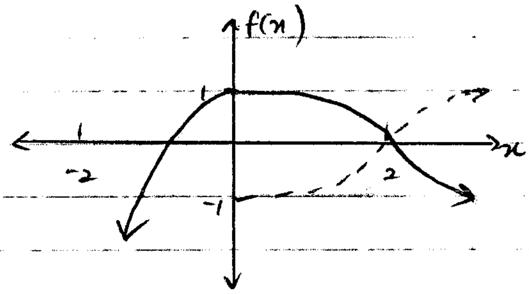


B)



Odd f2.
$f(-x) = -f(x)$

C)



B) $s(x) = x^2 - 4px + 3p - 2 = 0$

Let roots be :

α and β

We are told that $\alpha\beta = 3(\alpha+\beta)$ ✓

$\alpha+\beta = 4p$

$\alpha\beta = 3p - 2$

$3p - 2 = 3(4p)$

$3p - 2 = 12p$

$3p - 2 - 12p = 0$ ✓

$-9p - 2 = 0$

$-9p = 2$

$9p = -2$

$p = \frac{-2}{9}$ ✓

c) i) $(x-4)^2 + y^2 = 2^2$ ✓

$x^2 - 8x + 16 + y^2 = 4$

$x^2 + y^2 - 8x + 12 = 0$ ✓

ii) $y = mx$ ✓ ; $y^2 = m^2x^2$

iii) sub $y^2 = m^2x^2$ into eq C.

$x^2 + m^2x^2 - 8x + 12 = 0$

$x^2(1+m^2) - 8x + 12 = 0$ ✓

iv) ~~the line passes through the origin.~~

~~$x^2(1+m^2) - 8x + 12 = 0$~~

~~(sub $x = 0$ into eqC)~~

Since the line is TANGENT to the circle, there is only ONE pt. of intersection,

$\therefore \Delta = 0$.

$64 - 4(m^2+1)12 = 0$

$64 - 48(m^2+1) = 0$

$64 - 48m^2 - 48 = 0$ ✓

$16 - 48m^2 = 0$

$48m^2 = 16$

$m^2 = \frac{1}{3}$

$m = \pm\frac{1}{\sqrt{3}}$ ✓

Sub back into eqC.

$\left(\left(\frac{1}{\sqrt{3}}\right)^2 + 1\right)x^2 - 8x + 12 = 0$

$\frac{4}{3}x^2 - 8x + 12 = 0$

$4x^2 - 24x + 36 = 0$

$(2x - 6)^2 = 0$ ✓

$2x = 6$; $x = 3$ if $x = 3$,

$y = \frac{1}{\sqrt{3}} \times 3 = \frac{3}{\sqrt{3}}$ $\therefore P = \left(3, \frac{3}{\sqrt{3}}\right)$