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KAMBALA

MATHEMATICS

EXTENSION 1

YEAR 11 PRELIMINARY EXAMINATION

SEPTEMBER 2003

Time Allowed: 2 hours
Reading Time: 5 minutes

INSTRUCTIONS

- This examination contains 6 questions of equal value. Marks for each question are shown.
- Answer all questions on the writing paper provided. Start each question on a new page.
- Calculators may be used.
- Show all necessary working.
- Marks will be deducted for careless or badly arranged work.

Question 1: ***Start a new page*** ***12 marks***

(a) If $m = \sqrt{2} + \sqrt{3}$, express $m^2 - m^{-2}$ in simplest surd form. [3]

(b) Expand $\left(x^{\frac{1}{3}} - x^{-\frac{1}{3}}\right)^3$ [2]

(c) Find the **exact** value of cosec300°. [2]

(d) Given the points $A(-4, 3)$ and $B(2, -1)$, show that the point which divides the interval AB in the ratio 2 : 3 internally lies on the line $7x + 8y = 0$. [3]

(e) Find $\sum_{n=1}^{5} (-1)^{n+1}.2^n$ [2]

Question 2: ***Start a new page*** ***12 marks***

(a) Find the acute angle between the lines $y = -x$ and $\sqrt{3}y = x$. [3]

(b) Sketch showing all relevant features:

(i) $y = \left| x^2 - 3 \right|$ [2]

(ii) $y = \left| |x| - 3 \right|$ [2]

(c) Solve $\left| \dfrac{3x+1}{2} \right| \le 7$ [2]

(d) If $0^\circ < \theta < 90^\circ$ and $\tan\theta = \dfrac{t+1}{t-1}$, express in terms of t:

(i) $\sin\theta$ [2]

(ii) $\tan(90^\circ - \theta)$ [1]

Question 3: *Start a new page* **12 marks**

(a) Given $f(x)=\dfrac{3x^2}{x^4-16}$:

(i) For what value(s) of x is $f(x)$ undefined? [2]

(ii) Determine whether $f(x)$ is an odd or even function, or neither. [2]

(b) Find all positive values of x for which $\dfrac{6}{x}>x-1$. [3]

(c) Find the values of A, B and C if:

$$2x^2-9x+14\equiv A(x-1)(x-2)+B(x-1)+C$$ [3]

(d) Show that $\tan(\alpha+45^\circ)=\dfrac{1+\tan\alpha}{1-\tan\alpha}$ [2]

Question 4: *Start a new page* **12 marks**

(a) Solve $4\sin x\cos x=\sqrt{3}$ for $-180^\circ\le x\le 180^\circ$. [4]

(b) Can there be an infinite geometric progression with a limiting sum of $\dfrac{5}{8}$ and a first term of 2? Give reasons for your answer. [2]

(c) Find in simplest form, a relation in p, q and r such that the following equation has two equal roots:

$$\left(p^2+q^2\right)x^2+2q(p+r)x+\left(q^2+r^2\right)=0$$ [4]

Question 4 continued

(d) The function $f(x)$ is defined by

$$f(x)=\begin{cases} x^2-4x+4 & \text{for } x<0 \\ 4x+4 & \text{for } 0\le x<2 \\ 16-x^2 & \text{for } x\ge 2 \end{cases}$$

Find the values of:

(i) $f(0)$ [1]

(ii) $f(-a^2)$ [1]

Question 5: *Start a new page* **12 marks**

(a) Show that:

$$\frac{2\sin^3 A+2\cos^3 A}{\sin A+\cos A}=2-\sin 2A \quad (\text{if } \sin A+\cos A\ne 0)$$ [3]

(b) (i) Given that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, prove that

$$\cos 2\theta=1-2\sin^2\theta$$ [2]

(ii) Show that $\sin(x+\theta)=\dfrac{\cos x-\cos(x+2\theta)}{2\sin\theta}$. [3]

[*Hint*: use the results from (i)]

(iii) Using the results from (i) and (ii) and mathematical induction, prove the identity:

$$\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta=\frac{1-\cos(2n\theta)}{2\sin\theta}$$ [4]

Question 6: *Start a new page* *12 marks*

(a) For a certain sequence the sum of the first n terms, S_n, is given by

$$S_n = 102n - 2n^2$$

(i) By looking at partial sums or otherwise, determine what type of sequence this is. [2]

(ii) Find an expression for T_n, the nth term of the sequence. [1]

(iii) How many positive terms of the sequence must be added to give a sum of 460? [3]

(b) For what values of k does the equation $x^2 + (k-1)x - 2k - 1 = 0$ have:

(i) roots which are equal in magnitude but opposite in sign [2]

(ii) one root equal to -3 [2]

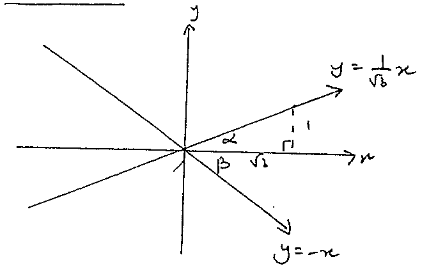
(c) Without using a calculator, simplify $\dfrac{1-2\sin^2 80}{2\cos^2 10 - 1}$ [2]

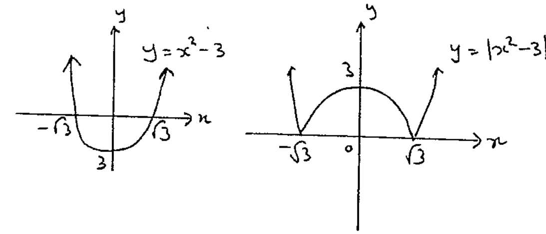
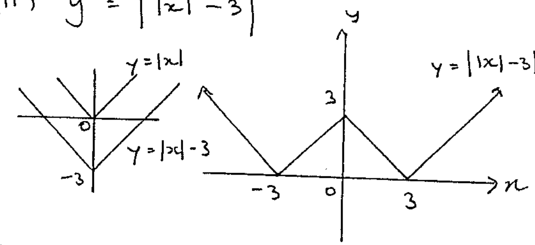
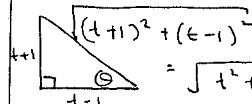
END OF EXAMINATION

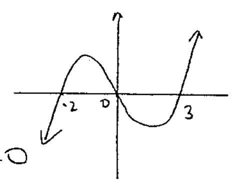
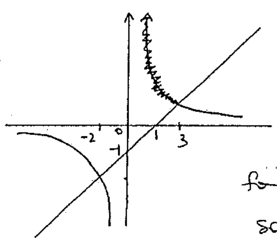
YR11 EXT1 2003 PRELIM EXAM

Qn	Solutions	Marks	Comments+Criteria
1	(a) $m^2 - m^{-2} = (\sqrt{2}+\sqrt{3})^2 - \dfrac{1}{(\sqrt{2}+\sqrt{3})^2}$		
	$(\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3$		
	$= 5 + 2\sqrt{6}$	1	expanding
	$\dfrac{1}{(\sqrt{2}+\sqrt{3})^2} = \dfrac{1}{5+2\sqrt{6}} \cdot \dfrac{5-2\sqrt{6}}{5-2\sqrt{6}}$	1	rationalising
	$= \dfrac{5-2\sqrt{6}}{25-4.6}$		
	$= 5-2\sqrt{6}$		
	$\therefore m^2 - m^{-2} = 5+2\sqrt{6} - (5-2\sqrt{6})$		
	$= 4\sqrt{6}$	1 /3	
	(b) $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^3$		
	$(a-b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$		
	$(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^3 = x^{\frac{3}{2}} - 3x.x^{-\frac{1}{2}} + 3x^{\frac{1}{2}}x^{-1} - x^{-\frac{3}{2}}$	1	
	$= x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} - x^{-\frac{3}{2}}$	1 /2	
	(c) $\operatorname{cosec} 300° = \dfrac{1}{-\sin 60°}$	1	-1mk no neg. sign
	$= -\dfrac{2}{\sqrt{3}}$	1 /2	

Qn	Solutions	Marks	Comments+Criteria
1 ctd	(d) $A(-4,3)$ $B(2,-1)$		
	$m:n = 2:3$		
	$P = \left(\dfrac{nx_1 + mx_2}{m+n}, \dfrac{ny_1+my_2}{m+n}\right)$		
	$= \left(\dfrac{3.-4 + 2.2}{5}, \dfrac{3.3 + 2.-1}{5}\right)$	1	
	$= \left(\dfrac{-8}{5}, \dfrac{7}{5}\right)$		
	$= \left(-1\frac{3}{5}, 1\frac{2}{5}\right)$	1	
	$7x + 8y = 0$:		
	LHS $= 7.-\dfrac{8}{5} + 8.\dfrac{7}{5}$	1	
	$= -\dfrac{56}{5} + \dfrac{56}{5}$		
	$= 0$		
	$=$ RHS		
	$\therefore$ P lies on line	/3	
	(e) $\sum_{n=1}^{5} (-1)^{n+1}.2^n$		
	$= 1.2 - 2^2 + 2^3 - 2^4 + 2^5$	1	
	$= 2 - 4 + 8 - 16 + 32$		
	$= 22$	1 /2	

Qn	Solutions	Marks	Comments+Criteria
2	(a) $y = -x$ $\quad \sqrt{3}y = x$ $\therefore y = \frac{1}{\sqrt{3}}x$		
	method 1  $\beta = 45^\circ$ $\quad \tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = 30^\circ$ $\therefore$ acute angle is $30+45 = 75^\circ$	1 (α) 1 (β) 1 /3	
OR	method 2 $m_1 = -1 \qquad m_2 = \frac{1}{\sqrt{3}}$ $\tan\theta = \left\| \frac{-1 - \frac{1}{\sqrt{3}}}{1 + -1 \cdot \frac{1}{\sqrt{3}}} \right\|$ $= \left\| \frac{-\sqrt{3}-1}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}-1} \right\|$ $= \left\| \frac{-\sqrt{3}-1}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} \right\|$ $= \left\| \frac{-3-2\sqrt{3}-1}{3-1} \right\|$ $= \left\| \frac{-4-2\sqrt{3}}{2} \right\|$ $\tan\theta = \|-2-\sqrt{3}\|$ $= 2+\sqrt{3}$ $\theta = 75^\circ$	1 1 1	

Qn	Solutions	Marks	Comments+Criteria
2 ctd	(b) (i) $y = \|x^2-3\|$  	1 1 /2	curve intercepts
	(ii) $y = \|\,\|x\|-3\,\|$  	1 1 /2	straight line graph intercepts
	(c) $\left\| \frac{3x+1}{2} \right\| \le 7$ $-7 \le \frac{3x+1}{2} \le 7$ $-14 \le 3x+1 \le 14$ $-15 \le 3x \le 13$ $-5 \le x \le \frac{13}{3} = 4\frac{1}{3}$	1 1 /2	
	(d) $0 < \theta < 90^\circ$, $\tan\theta = \frac{t+1}{t-1}$  $= \sqrt{t^2+2t+1+t^2-2t+1}$ $= \sqrt{2t^2+2}$ (i) $\sin\theta = \frac{t+1}{\sqrt{2t^2+2}}$ (ii) $\tan(90-\theta) = \frac{t-1}{t+1}$	1 1 1 /3	

Qn	Solutions	Marks	Comments+Criteria
3	(a) (i) $x^4 - 16 \neq 0$ method 1: $x^4 \neq 16$ $x \neq \pm \sqrt[4]{16}$ $\therefore x \neq \pm 2$	 1 1 ½	1 mk $x=2$ only
	method 2: $x^4 - 16 = 0$ $(x^2-4)(x^2+4) = 0$ $(x-2)(x+2)(x^2+4) = 0$ $x = 2, -2$ $\quad x^2 = -4$ no soln.	 1 1	
	(ii) $f(x) = \dfrac{3x^2}{x^4-16}$ $f(-x) = \dfrac{3(-x)^2}{(-x)^4-16}$ $= \dfrac{3x^2}{x^4-16}$ $= f(x)$ $\quad \therefore$ even fn	 1 1	
	(b) method 1: $\frac{6}{x} > x-1$ $6x > x^3 - x^2$ $x^3 - x^2 - 6x < 0$ $x(x^2-x-6) < 0$ $x(x-3)(x+2) < 0$ for pos. x values, $0 < x < 3$ 	 1 1 1	2 mks for $x<-2$, $0<x<3$.
	method 2:  $\frac{6}{x} = x-1$ $6 = x^2 - x$ $x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x = 3, -2$ for $x > 0$, soln is $0 < x < 3$	 1 1 1	

Qn	Solutions	Marks	Comments+Criteria
3 Ctd	(c) $2x^2 - 9x + 14 \equiv A(x-1)(x-2) + B(x-1) + C$ method 1 $2x^2-9x+14 \equiv A(x^2-3x+2) + Bx - B + C$ $\equiv Ax^2 - 3Ax + 2A + Bx - B + C$ $= Ax^2 + (B-3A)x + 2A - B + C$ $\therefore A = 2$ $\quad B - 3A = -9$ $B - 6 = -9$ $B = -3$ $2A - B + C = 14$ $4 + 3 + C = 14$ $\therefore C = 7$	 1 1 1	expanding/equating coefficients method of simult. eqns solns
	Method 2 If $x=1$: $2 - 9 + 14 = C$ $\quad \therefore C = 7$ If $x=0$: $14 = 2A - B + C$ $14 = 2A + 3 + 7$ $2A = 4$ $A = 2$ If $x=2$: $8 - 18 + 14 = B + C$ $4 = B + 7$ $\therefore B = -3$	1 1 1	substitutions of x values solving solutions
	(d) $\tan(\alpha+45) = \dfrac{\tan\alpha + \tan 45}{1 - \tan\alpha \tan 45}$ $= \dfrac{\tan\alpha + 1}{1 - \tan\alpha . 1}$ $= \dfrac{1+\tan\alpha}{1-\tan\alpha}$	1 1 /2	

Qn	Solutions	Marks	Comments+Criteria
4	(a) $4\sin x\cos x = \sqrt{3}$ $\quad -180 \le x \le 180$ $2.\,2\sin x\cos x = \sqrt{3}$ $2\sin 2x = \sqrt{3}$ $\sin 2x = \frac{\sqrt{3}}{2}$	1	2 mks finding the angles only
	$\therefore$ basic angle $2x = 60°$	1	
	S | A, T | C $2x = 60°, 120°, 420°, 480°, -300°, -240°$ $x = 30°, 60°, 210°, 240°, -150°, -120°$	1	
	for $-180 \le x \le 180°$: $x = -150°, -120°, 30, 60$	1 /4	
	(b) $\frac{2}{1-r} = \frac{5}{8}$	1	
	$16 = 5 - 5r$ $5r = -11$ $r = -\frac{11}{5}$ $= -2\frac{1}{5}$ $\lvert r\rvert \not< 1 \quad \therefore$ NO	1 /2	

Qn	Solutions	Marks	Comments+Criteria
4 ctd	(c) $(p^2+q^2)x^2 + 2q(p+r)x + (q^2+r^2) = 0$ equal roots: $b^2 - 4ac = 0$		
	$[2q(p+r)]^2 - 4(p^2+q^2)(q^2+r^2) = 0$	1	using $\Delta = 0$
	$4q^2(p+r)^2 - 4(p^2+q^2)(q^2+r^2) = 0$	1	$b^2 - 4ac$.
	$\therefore q^2(p+r)^2 - (p^2+q^2)(q^2+r^2) = 0$ $q^2(p^2+2pr+r^2) - (p^2q^2 + p^2r^2 + q^4 + q^2r^2) = 0$	1	working or correct $b^2 - 4ac$
	$\cancel{p^2q^2} + 2prq^2 + \cancel{q^2r^2} - \cancel{p^2q^2} - p^2r^2 - q^4 - \cancel{q^2r^2} = 0$ $2prq^2 - p^2r^2 - q^4 = 0$ $q^4 - 2prq^2 + p^2r^2 = 0$		
	$(q^2 - pr)(q^2 - pr) = 0$	1	factorising and
	$\therefore q^2 = pr$ $q = \pm\sqrt{pr}$	1	simplest form
	(d) (i) $f(0) = 4.0 + 4$ $= 4$	1	
	(ii) $f(-a^2) = (-a^2)^2 - 4(-a^2) + 4$ $= a^4 + 4a^2 + 4$	1	

Qn	Solutions	Marks	Comments+Criteria
5	(a) LHS $= \dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$		
	$= \dfrac{2(\sin^3 A + \cos^3 A)}{\sin A + \cos A}$		
	$= \dfrac{2(\cancel{\sin A + \cos A})(\sin^2 A - \sin A\cos A + \cos^2 A)}{\cancel{\sin A + \cos A}}$	1	factorising (a^3+b^3)
	$= 2(\sin^2 A + \cos^2 A - \sin A\cos A)$		
	$= 2(1 - \sin A\cos A)$	1	$\sin^2 A + \cos^2 A = 1$
	$= 2 - 2\sin A\cos A$		
	$= 2 - \sin 2A$	1 /3	
	(b) (i) $\cos 2\theta = \cos(\theta + \theta)$		
	$= \cos\theta\cos\theta - \sin\theta\sin\theta$		
	$= \cos^2\theta - \sin^2\theta$	1	
	$= (1 - \sin^2\theta) - \sin^2\theta$		
	$= 1 - 2\sin^2\theta$	1 /2	
	(ii)		
	LHS $= \sin(x + \theta)$		
	$= \sin x\cos\theta + \cos x\sin\theta$	ctd	

Qn	Solutions	Marks	Comments+Criteria
5 ctd	(ii) ctd		
	RHS $= \dfrac{\cos x - \cos(x + 2\theta)}{2\sin\theta}$		
	$= \dfrac{\cos x - [\cos x\cos 2\theta - \sin x\sin 2\theta]}{2\sin\theta}$		
	$= \dfrac{\cos x - \cos x(1 - 2\sin^2\theta) + \sin x\sin 2\theta}{2\sin\theta}$	1	
	$= \dfrac{\cos x - \cos x + 2\cos x\sin^2\theta + \sin x\sin 2\theta}{2\sin\theta}$		
	$= \dfrac{2\cos x\sin^2\theta + \sin x(2\sin\theta\cos\theta)}{2\sin\theta}$	1	
	$= \dfrac{\cancel{2\sin\theta}(\cos x\sin\theta + \sin x\cos\theta)}{\cancel{2\sin\theta}}$	1	
	$= \cos x\sin\theta + \sin x\cos\theta$	/3	
	= RHS		

Qn	Solutions	Marks	Comments+Criteria
5 ctd	(iii) $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta$		
	Prove $= \dfrac{1-\cos(2n\theta)}{2\sin\theta}$		
	$n=1$: LHS $= \sin\theta$	1	induction process
	RHS $= \dfrac{1-\cos 2\theta}{2\sin\theta} = \dfrac{2\sin^2\theta}{2\sin\theta} = \sin\theta$ = LHS		
	Assume true for $n=k$:		
	ie $\sin\theta + \sin 3\theta + \ldots + \sin(2k-1)\theta = \dfrac{1-\cos(2k\theta)}{2\sin\theta}$ (A)		
	Prove true for $n=k+1$:		
	ie Prove $\sin\theta + \sin 3\theta + \ldots + \sin(2k-1)\theta + \sin(2k+1)\theta = \dfrac{1-\cos(2(k+1))\theta}{2\sin\theta}$		
	LHS $= \dfrac{1-\cos(2k\theta)}{2\sin\theta} + \sin(2k+1)\theta$ using (A)	1	using (A)
	$= \dfrac{1-\cos(2k\theta)}{2\sin\theta} + \theta\left(\sin(2k+1)\right)$		
	$= \dfrac{1-\cos(2k\theta)}{2\sin\theta} + \theta\left(\dfrac{\cos 2k - \cos(2k+2)}{2\sin\theta}\right)$ using (ii)	1	using (ii)
	$= \dfrac{1-\cos 2k\theta + \cos 2k\theta - \cos 2(k+1)\theta}{2\sin\theta}$		
	$= \dfrac{1-\cos 2(k+1)\theta}{2\sin\theta}$	1	proof /4

$=$ RHS

$\therefore$ If true for $n=k$, then true for $n=k+1$ etc.

Qn	Solutions	Marks	Comments+Criteria
6	(a) $S_n = 102n - 2n^2$		0 mk no reason
	(i) $S_1 = 102.1 - 2.1 = 100$ $\quad T_1 = 100$ (1)		1mk for S_1, S_2, S_3 values
	$S_2 = 102.2 - 2.4 = 204 - 8 = 196$ $\quad \therefore T_1 + T_2 = 196$ (2)		
	$S_3 = 102.3 - 2.9 = 306 - 18 = 288$ $\quad \therefore T_1 + T_2 + T_3 = 288$ (3)	1	simultaneous eqns
	sub (1) into (2): $100 + T_2 = 196$, $T_2 = 96$		
	$\therefore 100 + 96 + T_3 = 288$, $T_3 = 288 - 196 = 92$		
	$T_3 - T_2 = 92 - 96 = -4$	1	common diff.
	$T_2 - T_1 = 96 - 100 = -4$		
	$\therefore$ sequence is an A.P. where $a = 100$, $d = -4$	/2	
	(ii) $T_n = a + (n-1)d = 100 + (n-1).-4 = 100 - 4n + 4 = 104 - 4n$	1	
	OR $S_n - S_{n-1} = 102n - 2n^2 - \left[102(n-1) - 2(n-1)^2\right]$	(1/1)	
	$= 102n - 2n^2 - \left[102n - 102 - 2(n^2 - 2n + 1)\right]$		
	$= \cancel{102n} - \cancel{2n^2} - \cancel{102n} + 102 + \cancel{2n^2} - 4n + 2$		
	$= 104 - 4n$	1	

Qn	Solutions	Marks	Comments+Criteria
6 ctd	(a) ctd		
	(iii) $102n - 2n^2 = 460$		
	$51n - n^2 = 230$		
	$n^2 - 51n + 230 = 0$	1	equation
	$n = \dfrac{51 \pm \sqrt{(-51)^2 - 4.1.230}}{2}$		
	$= \dfrac{51 \pm \sqrt{2601 - 920}}{2}$		
	$= \dfrac{51 \pm \sqrt{1681}}{2}$	1	Solving
	$= \dfrac{51 \pm 41}{2}$		
	$n = \dfrac{51+41}{2}, \dfrac{51-41}{2}$		
	$= \dfrac{92}{2}, \dfrac{10}{2}$		
	$= 46, 5$		
	$\therefore$ two solutions,	1	two solutions
	5 or 46 terms	/3	
	$T_5 = 84$ but $T_{46} = -80$		
	$\therefore$ 5 terms must be added	1	bonus mark

Qn	Solutions	Marks	Comments+Criteria
6 ctd	(b) $x^2 + (k-1)x - 2k - 1 = 0$		
	$a = 1, \ b = k-1, \ c = -2k-1$		
	(i) Let roots be $\alpha, -\alpha$.		
	$\alpha + -\alpha = \dfrac{-(k-1)}{1}$	1	
	$0 = -k+1$		
	$\therefore k = 1$	1/2	
	(ii) Let roots be $\alpha, -3$		method 2:
			let $x = -3$
	$\alpha - 3 = -k+1$		$9 + (k-1).-3 - 2k - 1 = 0$ 1
	$\alpha = -k+4$ (1)		$9 - 3k + 3 - 2k - 1 = 0$
	$-3\alpha = -2k-1$		$-5k = -11$
	$\therefore 3\alpha = 2k+1$ (2)	1	$k = \dfrac{11}{5}$ 1
	sub (1) into (2): $3(4-k) = 2k+1$		
	$12 - 3k = 2k+1$		
	$-5k = -11$		
	$k = \dfrac{11}{5}$		
	$= 2\frac{1}{5}$	1/2	
	(c) $\dfrac{1 - 2\sin^2 80}{2\cos^2 10 - 1}$		1mk $\cos 10 = \cos 80$
	method 2: $\dfrac{1 - 2\cos^2 10}{2\cos^2 10 - 1}$		1mk simplification
	$= \dfrac{\cos 2(80)}{\cos 2(10)}$	1	
	$= \dfrac{\cos 160}{\cos 20}$		
	$= \dfrac{-\cos 20}{\cos 20}$	1	
	$= -1$	/2	