Student ID: _

KAMBALA

Mathematics Extension 1

HSC Assessment Task 1

February 2008

Calculus, Harder Curve Sketching and Further Trigonometry

Time Allowed: 45 minutes working time

Outcomes Assessed

H5 applies appropriate techniques from the study of calculus and trigonometry
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H8 uses techniques of integration to calculate areas and volumes
PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

INSTRUCTIONS

- This examination contains 2 questions of 15 marks each. Marks for each question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks will be awarded for questions involving higher order thinking.

Question 1 (15 Marks) *(Start a new page.)* **Marks**

(a) Find the obtuse angle between the lines whose equations are: **3**

$$y = x - 2$$
$$y = -3x + 5$$

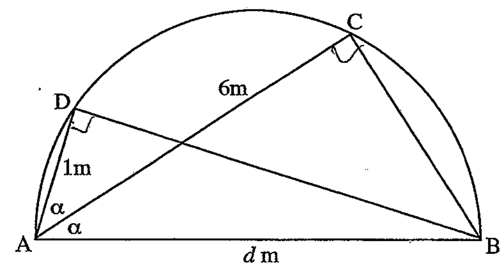
(b) By writing $\sin(-15^\circ)$ in the form $\sin(A-B)$, find the exact value of $\sin(-15^\circ)$. **2**

(c) If $t = \tan\frac{\theta}{2}$, simplify $\frac{1-\cos\theta}{1+\cos\theta}$ and write in simplest form in terms of t. **2**

(d) (i) Using the expansion for $\cos(\alpha+\beta)$, prove

$$\cos 2\alpha = 2\cos^2\alpha - 1$$ **2**

(ii) The figure below shows a semicircle with diameter d metres. **3**
AC = 6 metres, AD = 1 metre and $\angle BAC = \angle CAD = \alpha$.
$\angle ADB = \angle ACB = 90^\circ$.



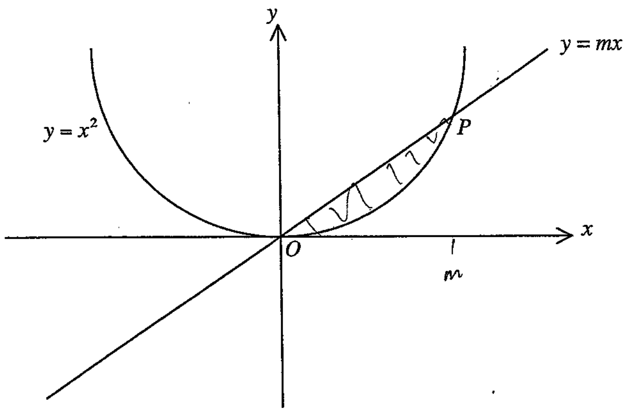
Write expressions for $\cos\alpha$ and $\cos 2\alpha$ and, using part (i), find the value(s) of d.

(e) Prove the identity $\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$ **3**

Question 2 (15 Marks) *(Start a new page.)* **Marks**

(a) Integrate $\int 2x(1-2x)^4\,dx$ using the substitution $u=1-2x$. **2**

(b) The diagram shows the curve $y=x^2$ and the line $y=mx$, where $m>0$, intersecting at the origin, O, and the point, P.



(i) Find the co-ordinates of P. **2**

(ii) The region enclosed by the interval OP and the arc OP is rotated about the x-axis. **3**

Show that the volume of the solid of revolution thus formed is $\dfrac{\pi m^5}{30}$ cubic units.

Question 2 is continued next page

Question 2 continued

(c) Consider the curve $y=\dfrac{x^2-3}{x+2}$.

(i) Find all intercepts and the equation of the vertical intercepts. **2**

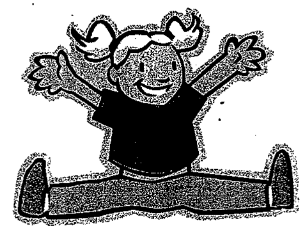
(ii) Find and determine the nature of the stationary points. **3**

(iii) Show that $(x-2)+\dfrac{1}{x+2}=\dfrac{x^2-3}{x+2}$. **1**

(iv) Hence, or otherwise, find the equation(s) of any non-vertical asymptotes. **1**

(v) Sketch the curve showing all of the above features (you may assume there are no points of inflexion). **1**

End of Assessment Task



Q1.

(a) $y = x - 2$ Let θ = obtuse $\angle$

$y = -3x + 5$ & α = acute $\angle$.

$m_1 = 1$

$m_2 = -3$.

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{1+3}{1-3}\right|$$

$$= \left|-\frac{4}{2}\right|$$

$$= 2$$

$\therefore \alpha = 63°26'$ (n.')

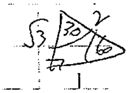
$\therefore \theta = 180 - 63°26'$

$= 116°34'$. (n.') no! (to nearest minute).

3

14/15

(b) $\sin(-15) = \sin(30 - 45)$



$= \sin 30 \cos 45 - \cos 30 \sin 45$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$ ✓2

(c) $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1 - \left(\frac{1-t^2}{1+t^2}\right)}{1 + \left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \frac{1+t^2-1+t^2}{1+t^2+1-t^2}$$

$$= \frac{\cancel{2}t^2}{\cancel{2}} = t^2.$$ ✓2

(d) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

RTP: $\cos 2\alpha = 2\cos^2\alpha - 1$

LHS $= \cancel{\cos\alpha\cos\beta}\ \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$.

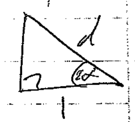
$= \cos^2\alpha - \sin^2\alpha$.

$= \cos^2\alpha - (1 - \cos^2\alpha)$.

$= \cos^2\alpha - 1 + \cos^2\alpha$

$= 2\cos^2\alpha - 1$. ✓2

(ii) $\cos\alpha = \frac{6}{d}$ $\quad d\cos\alpha = 1$ $\quad \cancel{d\cos\alpha = 6}$.

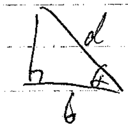


$\cos 2\alpha = \frac{\cancel{6}\,1}{d}$.

$\cancel{\cos 2\alpha = 2(\cos\alpha)^2 - 1}$

$\cancel{= 2\left(\frac{1}{d}\right)^2 - 1}$

$\cancel{= \frac{2}{d^2} - 1}$



$\cancel{\cos 2\alpha = 2\left(\frac{6}{d}\right)^2 - 1}$

$\cancel{= 2 \times \frac{36}{d^2} - 1}$

$\cancel{= \frac{72}{d^2} - 1}$

$\cancel{\therefore \frac{2}{d^2} - 1 = \frac{72}{d^2} - 1}$.

$\cancel{\frac{2}{d^2} = \frac{72}{d^2}}$.

$\cos 2\alpha = 2\cos^2\alpha - 1$

$= 2\left(\frac{6}{d}\right)^2 - 1$

$\frac{1}{d} = \frac{72}{d^2} - 1$ $\qquad \frac{d^2}{d} = 72 - d^2$.

$d = 72 - d^2$

$d^2 + d - 72 = 0$

$(d-8)(d+9) = 0$

$\therefore d = 8, -9$ ✓ 2

but $d > 0$ since d is a length

$\therefore d = 8m$

(e) ~~$\cos x - \cos 2x$~~

RTP $\dfrac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$

LHS $= \dfrac{\cos x - (2\cos^2 x - 1)}{2\cos x \sin x + \sin x}$

$= \dfrac{\cos x - 2\cos^2 x + 1}{2\cos x \sin x + \sin x}$

$= \dfrac{\cos x (1 - 2\cos x) + 1}{\sin x (2\cos x + 1)}$

LHS $= \dfrac{\cos x - 2\cos^2 x + 1}{\sin x (2\cos x + 1)}$

$= \dfrac{\cancel{\cos x} - 2\cos^2 x + \cos x + 1}{\sin x (2\cos x + 1)}$

$= \dfrac{(-2\cos x - 1)(\cos x - 1)}{\sin x (2\cos x + 1)}$

$= \dfrac{-\cancel{(2\cos x + 1)}(\cos x - 1)}{\sin x \cancel{(2\cos x + 1)}}$

$= \dfrac{1 - \cos x}{\sin x}$

$=$ RHS. ✓ 3

RHS $= \dfrac{1}{\sin x} - \dfrac{1}{\tan x}$

$= \dfrac{\tan x - \sin x}{\sin x \tan x}$

$= \dfrac{\frac{\sin x}{\cos x} - \sin x}{\sin x \times \frac{\sin x}{\cos x}}$

$= \dfrac{\frac{\sin x - \sin x \cos x}{\cos x}}{\frac{\sin^2 x}{\cos x}}$

$= \dfrac{(\sin x - \sin x \cos x)\cancel{\cos x}}{(\sin^2 x)(\cancel{\cos x})}$

$= \dfrac{\cancel{\sin x}(1 - \cos x)}{\sin^2 x}$

$= \dfrac{1 - \cos x}{\sin x}$

$$\frac{2+5+8}{15} = \frac{15}{15}$$ Well done! :)

Q2.

(a) $\int 2x\,(1-2x)^4\,dx$

$u = 1-2x$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$-du = 2\,dx$

$2x = 1-u$

$x = \frac{1-u}{2}$

$= -\int \frac{1-u}{2}\,(u)^4\,du$ ✓

$= -\int \frac{u^4 - u^5}{2}\,du$

$= -\int \frac{u^4}{2} - \frac{u^5}{2}\,du$ ✓

$= -\frac{1}{2}\int u^4 - u^5\,du$ ✓

$= -\frac{1}{2}\left(\frac{u^5}{5} - \frac{u^6}{6}\right) + C$

$= -\frac{1}{2}\left(\frac{(1-2x)^5}{5} - \frac{(1-2x)^6}{6}\right) + C$

$= -\frac{(1-2x)^5}{10} + \frac{(1-2x)^6}{12} + C$ ✓

2/2

(b) (i) $y = mx$ & $y = x^2$

$\therefore mx = x^2$

$x^2 - mx = 0$ ✓

$x(x-m) = 0$

$\therefore x = 0,\ x = m$

@ $x = m$, $y = m^2$

$\therefore P(m, m^2)$ ✓

2/2

(ii) $V = \pi \int_0^m (mx)^2 - (x^2)^2\,dx$ ✓

$= \pi \int_0^m m^2x^2 - x^4\,dx$ ✓

$= \pi \left[\frac{m^2x^3}{3} - \frac{x^5}{5}\right]_0^m$ ✓

$= \pi \left(\frac{m^2m^3}{3} - \frac{m^5}{5}\right) = \left(\frac{m^5}{3} - \frac{m^5}{5}\right)\pi = \left(\frac{5m^5 - 3m^5}{15}\right)\pi$

$= \frac{2m^5}{15}\pi$ ✓

3/3

(c) $y = \frac{x^2-3}{x+2}$

(i) when $x = 0$, $y = -\frac{3}{2}$

when $y = 0$, $0 = x^2 - 3$

$3 = x^2$

$x = \pm\sqrt{3}$ ✓ ✓

$\therefore$ intercepts are $(0, -\frac{3}{2})$ & $(\sqrt{3}, 0)$ & $(-\sqrt{3}, 0)$.

$x \neq -2$

$\therefore$ vertical asymptote is $x = -2$. ✓

2/2

(ii) $y = \frac{x^2-3}{x+2}$

$y' = \frac{u'v - v'u}{v^2}$

$= \frac{2x(x+2) - 1(x^2-3)}{(x+2)^2}$

$= \frac{2x^2 + 4x - x^2 + 3}{(x+2)^2}$ ✓

$= \frac{x^2 + 4x + 3}{(x+2)^2}$

$\therefore$ stat pts occur when $y' = 0$.

$\therefore x^2 + 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1,\ x = 3$

when $x = 1$, $y = \frac{1-3}{1+2} = -\frac{2}{3}$

when $x = 3$, $y = \frac{9-3}{3+2} = \frac{6}{5}$

1/2

x	$\frac{1}{2}$	1	$\frac{3}{2}$
y'	0.84	0	0.9

x	2	3	4
y'	$\frac{15}{16}$	0	$\frac{35}{36}$

$(x+3)(x+1) = 0.$

$\therefore x = -3$ & $x = -1.$

When $x = -3$, $y = \frac{9-3}{-1}$

$= -6$

When $x = -1$, $y = \frac{1-3}{-1+2}$

$= -2.$

$\therefore$ ~~$3p$~~ $(-3,-6)$ & $(-1,-2)$ are stat pts

x	$-3\frac{1}{2}$	-3	$-2\frac{1}{2}$
y'	0.55	6	-13

better shape an asymptote

x	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$
y'	-3	0	0.55

$\therefore (-3,-6)$ is max t.p & $(-1,-2)$ is min t.p

3/3.

RTP

(iii) $(x-2) + \frac{1}{(x+2)} = \frac{x^2-3}{x+2}$

$LHS = \frac{(x-2)(x+2)+1}{x+2}$

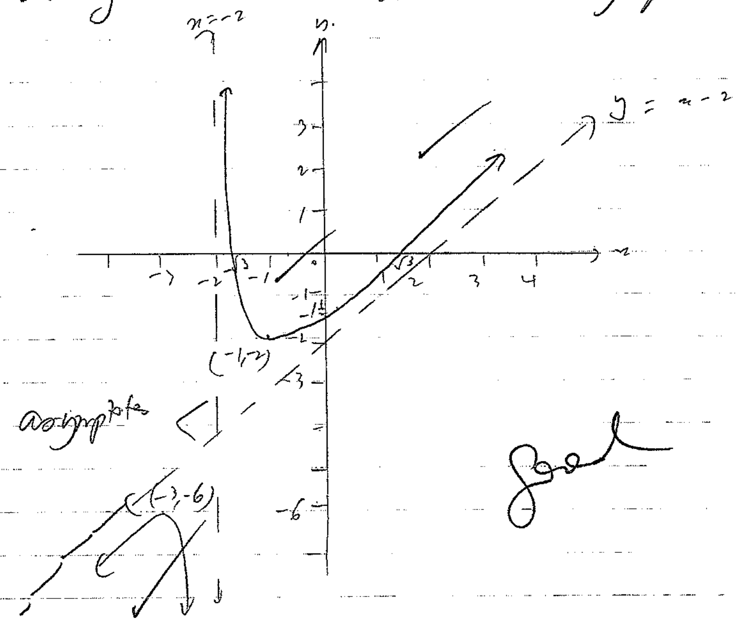
$= \frac{x^2-4+1}{x+2}$

$= \frac{x^2-3}{x+2}$

$= RHS.$

(iv) $\therefore y = x-2$ is also an asymptote.

(v)



Good