Student ID: ____________

KAMBALA

Mathematics Extension 2

HSC Assessment Task 1

November 2006

Time Allowed: 1 hour

Outcomes Assessed

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E3 uses the relationship between algebraic and geometric representations of complex numbers

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

E9 communicates abstract ideas and relationships using appropriate notation and logical argument

INSTRUCTIONS

- This task contains 3 questions of 12 marks each. Marks for each part question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.

Question 1 *(Start a new page.)* **12 Marks**

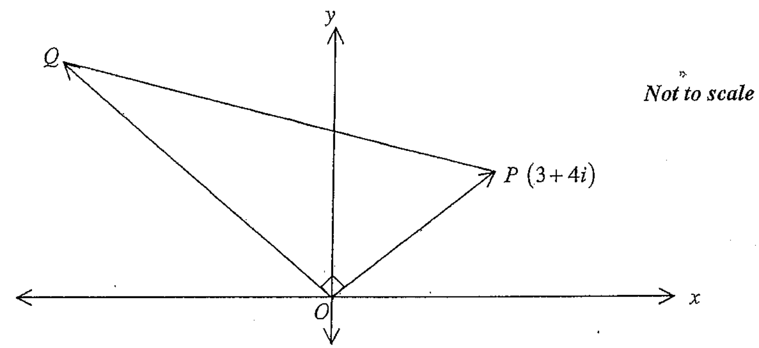
(a) For the complex number $z = 2 - i$, express the following in the form $x + iy$ where x and y are real:

(i) $\overline{z}$ **1**

(ii) $\dfrac{1}{z}$ **1**

(iii) $(1-i)z$ **2**

(b)



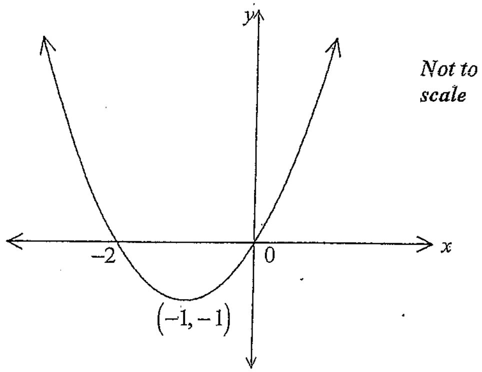
The triangle POQ above is right-angled at O. The length of OQ is twice that of OP. Given that $\overrightarrow{OP}$ represents the complex number $3+4i$, determine the complex number represented by $\overrightarrow{OQ}$. **2**

(c) Find u and v, where u and v are real, if $(u - iv)^2 = 5 + 12i$. **3**

(d) By factorising $z^3 + 8$, or otherwise, find all solutions of $z^3 + 8 = 0$. Express any complex roots in the form $r(\cos\theta + i\sin\theta)$ where r is real and θ is in radians. **3**

Question 2 *(Start a new page.)* 12 Marks

(a) Let $f(x) = x(x+2)$. The graph of $y = f(x)$ is drawn below.



On separate diagrams, and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x) + 2$ 1

(ii) $y = |f(x)|$ 2

(iii) $y = \dfrac{1}{f(x)}$ 2

(b) Let $f(x) = \dfrac{x^2}{(x+2)(x-2)}$ for $x \neq 0$

(i) Show that $f(x) = 1 + \dfrac{4}{x^2 - 4}$ 1

(ii) Find the equations of all asymptotes to $y = f(x)$. 2

(iii) Using part (i) find any stationary points on the curve and determine their nature. 2

(iv) Sketch the curve $y = f(x)$. Clearly indicate any asymptotes and intercepts. 2

Question 3 *(Start a new page.)* 12 Marks

(a) Write $z = 5(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ in Cartesian form. 1

(b) Given z is a complex number such that $5z + \overline{z} = 12 + 8i$, find z. 2
Let $z = x + iy$ where x and y are real.

(c) (i) Sketch $|z - 2| = 1$, and describe the locus geometrically. 2

(ii) Considering z as a point on the locus indicate on your sketch the points that give the greatest values of $|z|$ and of $\arg(z)$. Hence find the greatest value of $|z|$ and $\arg(z)$. 2

(d) (i) On the same diagram sketch the graphs of $y = |x+2|$ and $y = x^2$. 1

(ii) Show that the graphs intersect at $x = -1$ and $x = 2$. 2
Hence solve $|x+2| > x^2$ for all x.

(iii) For what values of b will $|x+2| = x^2 + b$ have no solution? 2

End of Assessment Task

Question	Solutions	Marks	Marking Criteria

1.(a) $z = 2 - i$

(i) $\bar{z} = 2 + i$

(ii) $\frac{1}{z} = \frac{1}{2\,i} \times \frac{2+i}{2+i}$

$= \frac{2+i}{4+1}$

$= \frac{2+i}{5}$

(iii) $(1-i)z$

$= (1-i)(2-i)$

$= 2 - 3i - 1$

$= 1 - 3i$

(b) $\vec{OQ} = 2\,\text{cis}\frac{\pi}{2} \times \vec{OP}$

$= 2i \times (3+4i)$

$= 6i + 8i^2$

$= -8 + 6i$

(c) $(u-iv)^2 = 5 + 12i$

$(u^2 - v^2) - (2uv)i = 5 + 12i$

$\left.\begin{array}{l} u^2 - v^2 = 5 \\ 2uv = -12 \end{array}\right\}$

$uv = -6$

$v = -\frac{6}{u}$

$u^2 + \frac{36}{u^2} = 5$

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Question	Solutions	Marks	Marking Criteria

1(c) c'td

$u^4 - 5u^2 + 36 = 0$

$(u^2 - 9)(u^2 + 4) = 0$

$u^2 = 9 \qquad u^2 = -4$

$u = \pm 3 \qquad$ no soln.

$u = 3, \quad v = -2$

$u = -3, \quad v = 2$

(d) $z^3 = -8$

$z^3 = 8\,\text{cis}\,\pi$

$= 8\,\text{cis}(\pi + 2k\pi)$

$z = 2\,\text{cis}\frac{(2k+1)\pi}{3}$, where k is an integer

$k = 0: \quad z = 2\,\text{cis}\frac{\pi}{3}$

$k = 1: \quad z = 2\,\text{cis}\,\pi = -2.$

$k = 2: \quad z = 2\,\text{cis}\frac{5\pi}{3}$

OR

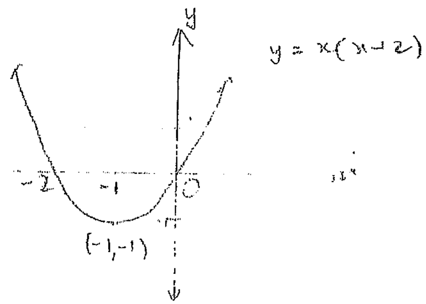
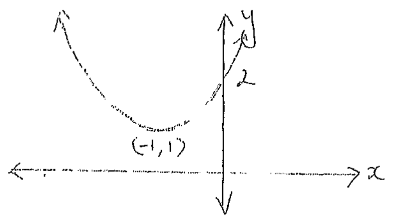
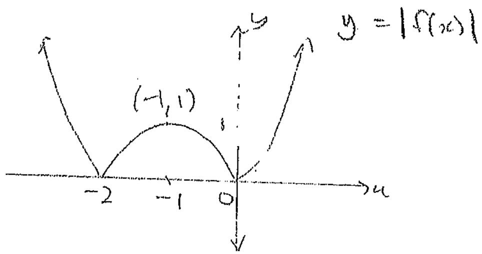
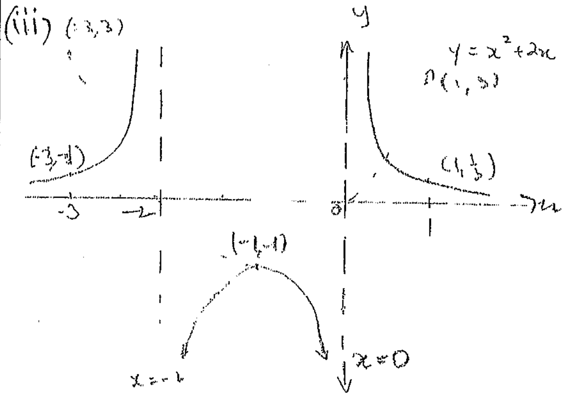
$z^3 + 8 = (z+2)(z^2 - 2z + 4)$

$z = -2, \quad \frac{2 \pm \sqrt{-12}}{2}$

$z = -2, \quad 1 \pm \sqrt{3}\,i$

$z = 2\,\text{cis}\frac{\pi}{3}, \; 2\,\text{cis}\,\pi,$

$2\,\text{cis}\frac{5\pi}{3}$

Qn	Solutions	Marks	Comments+Criteria
2	(a) $f(x) = x(x+2)$ 	1	
	(i) 		
	(ii) 	2	
	(iii) 	2	

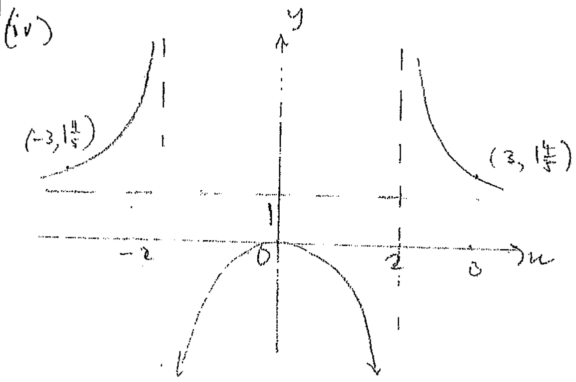
Qn	Solutions	Marks	Comments+Criteria
2 ctd	(b) $f(x) = \dfrac{x^2}{(x+2)(x-2)}$ for $x \neq 0$		
	(i) $f(x) = \dfrac{x^2}{x^2-4}$ $= \dfrac{x^2-4+4}{x^2-4}$ $= 1 + \dfrac{4}{x^2-4}$	1	
	(ii) $\lim\limits_{x\to\infty}\left(1+\dfrac{4}{x^2-4}\right) = 1+0$ $= 1$ $\therefore y=1$ is horizontal asymptote.	1	
	$x=2$ and $x=-2$ are vertical asymptotes. $, x=0$	1	
	(iii) $f(x) = 1 + 4(x^2-4)^{-1}$ $f'(x) = -4(x^2-4)^{-2}.2x$ $= -8x(x^2-4)^{-2}$ $= \dfrac{-8x}{(x^2-4)^2}$	OR 1	$f'(x) = \dfrac{2x(x^2-4)-2x.x^2}{(x^2-4)^2}$ $= \dfrac{2x^3-8x-2x^3}{(x^2-4)^2}$ $= \dfrac{-8x}{(x^2-4)^2}$
	$f''(x) = \dfrac{-8(x^2-4)^2 - 2(x^2-4).2x.-8x}{(x^2-4)^2}$ $= \dfrac{-8(x^2-4)+32x^2}{(x^2-4)}$ $= \dfrac{16x^2+32}{x^2-4}$ Stat pts at: $\dfrac{-8x}{(x^2-4)^2} = 0$ $-8x = 0$ $x=0, y=0$		

2		Marks	Comments+Criteria
(iii)	$(0,0)$:		

x	-1	0	1
$P'(x)$	$\frac{8}{9}$	0	$\frac{-8}{9}$
	/	—	\

$\therefore$ max. at $(0,0)$. — 1

(iv)



$y = 1 + \frac{4}{x^2-4}$

$f(-x) = 1 + \frac{4}{(-x)^2-4}$

$= 1 + \frac{4}{x^2-4}$

$= f(x)$ even — 2

Question	Solutions	Marks	Marking Criteria

3(a)

$z = 5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 5\left(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$

$z = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$

(b)

$5z + \bar{z} = 12 + 8i$

$5(x+iy) + (x-iy) = 12+8i$

$5x + 5iy + x - iy = 12+8i$

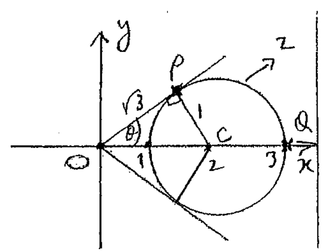
$6x + i(4y) = 12+8i$

$6x = 12 \qquad 4y = 8$

$x = 2 \qquad y = 2$

$\therefore z = 2+2i$

(c) (i)



locus is a circle

centre $(2,0)$ radius 1

(ii)

$|z|_{MAX} = |\vec{OQ}|$

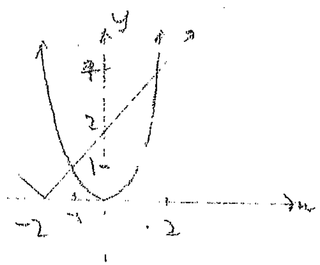
$= \boxed{3}$

$\arg z_{MAX} = \theta$

$= \tan^{-1}(1/\sqrt{3})$

$= \boxed{\pi/6}$

Qn	Solutions	Marks	Comments+Criteria
3 cld	(d) (i) $y = \lvert x+2 \rvert$ and $y = x^2$		



(ii)
$x^2 = x + 2$

$x^2 - x - 2 = 0$

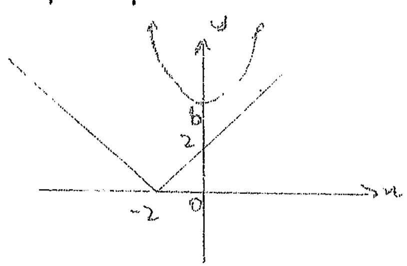
$(x-2)(x+1) = 0$

$x = 2, -1$

$\lvert x+2 \rvert > x^2$

$-1 < x < 2$

(iii) $\lvert x+2 \rvert = x^2 + b$ have no solns



$x + 2 = x^2 + b$

$x^2 - x + b - 2 = 0$

$\Delta = b^2 - 4ac$

$= 1 - 4.1.(b-2)$

$= 1 - 4b + 8$

$= 9 - 4b$

For no solns, $\Delta < 0$

ie $9 - 4b < 0$

$-4b < -9$

$4b > 9$

$b > \frac{9}{4} = 2\frac{1}{4}$. ie $b > 2\frac{1}{4}$.