Student Number:

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Mathematics Extension 2

HSC Assessment Task 3

June 2007

Outcomes Assessed

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E3 uses the relationship between algebraic and geometric representations of conic sections
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
E8 applies further techniques of integration to problems
E9 communicates abstract ideas and relationships using appropriate notation and logical argument

General Instructions

- Reading time – 5 minutes.
- Working time – 75 minutes.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 50

- Attempt Questions 1-3.
- Not all questions are of equal value.

Question 1: **17 marks**

(a) Evaluate $\int_0^1 3xe^{x^2-1}\,dx$ **2**

(b) Find $\int \sin^3 x\,dx$ **2**

(c) Use the substitution $x=\cos\theta$, $0\le\theta\le\pi$, to evaluate $\int_{\frac{1}{2}}^{1}\frac{\sqrt{1-x^2}}{x^2}\,dx$ **4**

(d) Evaluate $\int_0^1 xe^x\,dx$. **3**

(e) (i) If $I_n=\int_0^{\frac{\pi}{2}} x^n \sin x\,dx$ show, using integration by parts, that if n is an integer such that $n\ge 2$, $I_n=n\left(\frac{\pi}{2}\right)^{n-1}-n(n-1)I_{n-2}$ **3**

(ii) Hence find $\int_0^{\frac{\pi}{2}} x^4\sin x\,dx$. **3**

Question 2: (Begin a new page) **16 marks**

(a) Determine the roots of the equation $x^3-7x^2+11x-5=0$ given that it has a double root. **2**

(b) (i) Show that $x-i$ is a factor of $P(x)=x^4+3x^3+6x^2+3x+5$. **1**

(ii) Hence express $P(x)$ as a product of irreducible factors over the field of rational numbers. **2**

(iii) Solve the equation $P(x)=0$ over the field of complex numbers. **2**

(c) Given $\alpha,\ \beta,\ \gamma$ are the roots of the cubic equation $x^3+qx+r=0$, where q and r are constants:

(i) Find $\sum\alpha,\ \sum\alpha\beta$ and $\sum\alpha\beta\gamma$ in terms of q and r. **1**

(ii) Hence prove $(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2=-6q$. **2**

(iii) Find the equation which has roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$. **2**

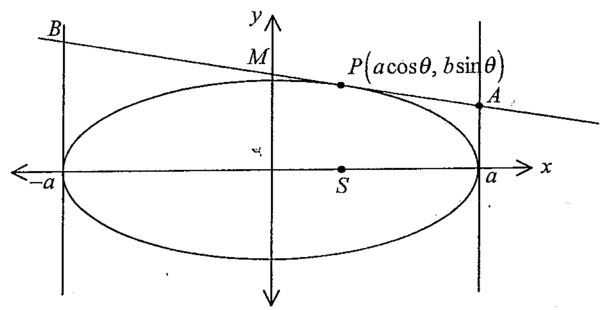
(d) A sequence t_n is defined so that $t_1=2$, $t_2=6$ and $t_{n+2}=6t_{n+1}-5t_n$ for $n\geq 1$. Prove by mathematical induction that $t_n=5^{n-1}+1$ for all positive integral n. **4**

Question 3: (Begin a new page) **17 marks**

(a) Sketch the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ clearly indicating foci and directrices. **3**

(b) The hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ has a chord $y=2x+1$. Find the co-ordinates of the mid-point of the chord. **3**

(c) The point $P(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where S and S' are the foci. The tangent at P intersects the y-axis at M and the lines $x=a$ and $x=-a$ at A and B respectively.



(i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

(ii) Find the co-ordinates of the points A and B. **2**

(iii) Show that $\angle ASB=90°$. **3**

(iv) Show that M is the mid-point of AB. **1**

(v) Hence show that ΔMAS is isosceles. **2**

End of Assessment

Question	Solutions	Marks	Marking Criteria
1 (a)	$\int_0^1 3xe^{x^2-1}\,dx = \left[\frac{3}{2}e^{x^2-1}\right]_0^1$	1	
	$= \frac{3}{2}(e^0 - e^{-1})$		
	$= \frac{3}{2}\left(1 - \frac{1}{e}\right)$	1	
(b)	$\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx$		
	$= \int (\sin x - \sin x\cos^2 x)\,dx$	1	
	$= -\cos x + \frac{1}{3}\cos^3 x + C$	1	
(c)	$x = \cos\theta \Rightarrow \frac{dx}{d\theta} = -\sin\theta$		
	$x = 1 \Rightarrow \theta = 0$	1	
	$x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$		
	$\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2}\,dx = \int_{\frac{\pi}{3}}^0 \frac{\sqrt{1-\cos^2\theta}}{\cos^2\theta} \times -\sin\theta\,d\theta$		
	$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sin^2\theta}}{\cos^2\theta}\sin\theta\,d\theta$	1	$\sqrt{\sin^2\theta} = \lvert\sin\theta\rvert$ $= \sin\theta$ for $0 \le \theta \le \pi$
	$= \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$		
	$= \int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta$	1	
	$= \int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\,d\theta$		
	$= [\tan\theta - \theta]_0^{\frac{\pi}{3}}$		
	$= \sqrt{3} - \frac{\pi}{3}$	1	

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Question	Solutions	Marks	Marking Criteria
1 (d)	$\int_0^1 xe^x\,dx$ $\quad u = x \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 1 \quad v = e^x$	1	
	$= [xe^x]_0^1 - \int_0^1 e^x\,dx$		
	$= e^1 - 0 - [e^x]_0^1$	1	
	$= e^1 - (e^1 - e^0)$		
	$= 1$	1	
(e) (i)	$\int_0^{\frac{\pi}{2}} x^n \sin x\,dx$ $\quad u = x^n \quad \frac{dv}{dx} = \sin x \quad \frac{du}{dx} = nx^{n-1} \quad v = -\cos x$	1	
	$= [-x^n\cos x]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx$ $\quad u = x^{n-1} \quad \frac{dv}{dx} = \cos x \quad \frac{du}{dx} = (n-1)x^{n-2} \quad v = \sin x$		
	$= \left[-\left(\frac{\pi}{2}\right)^n\cos\frac{\pi}{2} - 0\right] + n\left([x^{n-1}\sin x]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\,dx\right)$	1	
	$= n\left(\left(\frac{\pi}{2}\right)^{n-1}\sin\frac{\pi}{2} - 0 - (n-1)I_{n-2}\right)$	1	
	$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$		
(ii)	$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 4\times 3\,I_2$		
	$I_2 = 2\left(\frac{\pi}{2}\right)^1 - 2\times 1\times I_0$	1	
	$I_0 = \int_0^{\frac{\pi}{2}} \sin x\,dx$		
	$= [-\cos x]_0^{\frac{\pi}{2}}$	1	
	$= 1$		
	$\int_0^{\frac{\pi}{4}} x^4 \sin x\,dx =$		
	$= 4\left(\frac{\pi}{2}\right)^3 - 12\left(2\left(\frac{\pi}{2}\right) - 2\times 1\times 1\right)$		
	$= \frac{\pi^3}{2} - 12\pi + 24$	1	

Question	Solutions	Marks	Marking Criteria
2 (a)	$P(x)=x^3-7x^2+11x-5$		
	$P'(x)=3x^2-14x+11$		
	If a double root $P(\alpha)=P'(\alpha)=0$		
	$3x^2-14x+11=(3x-11)(x-1)=0$		
	$x=\frac{11}{3},1$	1	
	test $P(1)=1-7+11-5=0$		
	$\therefore x=1$ is a double root		
	$P(x)=(x-1)^2(x-5)$	1	
	$\therefore$ roots are $1,1,5$		
(b)	(i) Factor thm $P(i)=0$		
	$P(i)=(i)^4+3(i)^3+6(i)^2+3i+5$		
	$=1-3i-6+3i+5$	1	
	$=0$		
	$\therefore (x-i)$ is a factor		
	(ii) $(x-i)$ factor $\therefore (x+i)$ is also a factor		
	$(x-i)(x+i)=x^2+1$	1	
	$P(x)=(x^2+1)(x^2+3x+5)$	1	
	(iii) $P(x)=0$		
	$x^2+1=0$ $\quad x^2+3x+5=0$		
	$x=\pm i$ $\quad x=\frac{-3\pm\sqrt{-11}}{2}$	1	1 for solving each quadratic
	$=\frac{-3\pm i\sqrt{11}}{2}$	1	

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Question	Solutions	Marks	Marking Criteria
2 (c)	(i) $\sum\alpha=0$		
	$\sum\alpha\beta=q$	1	
	$\sum\alpha\beta\gamma=-r$		
	(ii) $(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2$		
	$=\beta^2-2\beta\gamma+\gamma^2+\gamma^2-2\gamma\alpha+\alpha^2+\alpha^2-2\alpha\beta+\beta^2$		
	$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\alpha\gamma+\beta\gamma)$	1	
	$=2\left((\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\right)-2q$		
	$=2(0-2q)-2q$	1	
	$=-6q$		
	(iii) roots $\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}$		
	ie replace x by $2x$	1	
	eq. $(2x)^3+q(2x)+r=0$		
	$8x^3+2qx+r=0$	1	
(d)	Show true for $n=1,2,3$		
	$n=1$ $\quad t_1=5^0+1=2$		
	$n=2$ $\quad t_2=5^1+1=6$		
	$n=3$ $\quad t_3=5^2+1=26$		
	and $t_3=6t_2-5t_1$		
	$=6\times6-5\times2$	1	
	$=26$		
	$\therefore$ true for $n=1,2,3$		
	Assume true for $n=k$ + $n=k+1$		
	ie $t_k=5^{k-1}+1$		
	$t_{k+1}=5^{(k+1)-1}+1=5^k+1$		

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Question	Solutions	Marks	Marking Criteria
2(d) c'td	Show true for $n=k+2$ $t_{k+2}=6t_{k+1}-5t_k$	1	
	$=6(5^k+1)-5(5^{k-1}+1)$ $=6\times5^k+6-5^k-5$ $=5\times5^k+1$ $=5^{k+1}+1$ $t_{k+2}=5^{(k+2)-1}+1$ $=5^{k+1}+1$	1	
	$\therefore$ true for $n=k+2$ <u>if</u> true for $n=k$ <u>and</u> $n=k+1$ Since true for $n=1,2,3$ true for $n=3+1=4$ & thus true for $n=4+1=5$ & so on for all positive integral n.	1	

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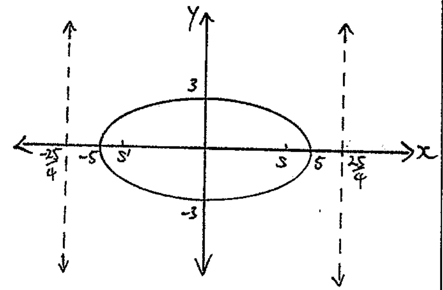
Question	Solutions	Marks	Marking Criteria
3(a)	$\frac{x^2}{25}+\frac{y^2}{9}=1$ $a=5,\ b=3$ $b^2=a^2(1-e^2)\Rightarrow\frac{9}{25}=1-e^2$ $e^2=\frac{16}{25}$ $e=\frac{4}{5}$	1	
	foci $(\pm ae,0)=(\pm4,0)$		
	eq directrices $x=\pm\frac{a}{e}=\pm\frac{25}{4}$	1	
		1	
(b)	$\frac{x^2}{4}-\frac{y^2}{9}=1$ $y=2x+1$ $9x^2-4(2x+1)^2=36$ $9x^2-4(4x^2+4x+1)=36$ $-7x^2-16x-4=36$ $7x^2+16x+40=0$	1	
	solve for $x\Rightarrow$ roots x_1 & x_2 midpt. $=\frac{x_1+x_2}{2}$ $\frac{\text{sum of roots}}{2}$ $x=-\frac{16}{7}/2$ $x=\frac{-8}{7}$	1	
	$y=2\times\frac{-8}{7}+1=\frac{-9}{7}$ midpt. of chord $\left(-\frac{8}{7},-\frac{9}{7}\right)$	1	

Question	Solutions	Marks	Marking Criteria
3 (c)	(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$		
	differentiate $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$		
	$\frac{dy}{dx} = \frac{-2x}{a^2} \Big/ \frac{2y}{b^2}$		
	$= \frac{-b^2x}{a^2y}$		
	at P $\Rightarrow m_T = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$	1	
	$= \frac{-b\cos\theta}{a\sin\theta}$		
	eq. tangent		
	$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\ )$		
	$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$	1	
	$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$		
	$\frac{bx\cos\theta}{ab} + \frac{ay\sin\theta}{ab} = \frac{ab}{ab}$		
	$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$	1	
	(ii) at A $x = a$		
	$\frac{a\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$		
	$\frac{y\sin\theta}{b} = 1 - \cos\theta$		
	$y = \frac{b(1-\cos\theta)}{\sin\theta}$		
	$A\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$	1	
	$B\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$	1	

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Question	Solutions	Marks	Marking Criteria
3 (c) c'td	(iii) If $\angle ASB = 90°$		
	$m_{AS} \times m_{SB} = -1$		
	$S(ae, 0)$ NB		
	$m_{AS} = \frac{b(1-\cos\theta)}{\sin\theta} \Big/ (a - ae)$		
	$= \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$	1	
	$m_{SB} \ \frac{-b(1+\cos\theta)}{a(1+e)\sin\theta}$		
	$m_{AS} \times m_{SB} = \frac{b(1-\cos\theta)}{a(1-e)\sin\theta} \times \frac{-b(1+\cos\theta)}{a(1+e)\sin\theta}$		
	$= \frac{-b^2(1-\cos^2\theta)}{a^2(1-e^2)\sin^2\theta}$	1	
	$= \frac{-b^2 \cancel{\sin^2\theta}}{\cancel{a^2} \times \frac{b^2}{\cancel{a^2}} \cancel{\sin^2\theta}}$		
	$= -1$	1	
	$AS \perp SB$ ie $\angle ASB = 90°$		
(iv)	M is pt. on tangent & y-axis		
	ie $x = 0$, $y = \frac{b}{\sin\theta}$		
	midpt AB $= \left(\frac{a-a}{2}, \frac{\frac{b(1-\cos\theta)}{\sin\theta} + \frac{b(1+\cos\theta)}{\sin\theta}}{2}\right)$		
	$= \left(0, \frac{\cancel{2}b}{\cancel{2}\sin\theta}\right)$	1	
	ie M is the midpt of AB		
(v)	M is midpt AB and $\angle ASB = 90°$		
	∴ AB is diameter of circle thro' A, S, B (∠ in semi-circle is right ∠)	1	
	∴ M is the centre & MS = MA radii ie ΔMAS isosceles	1	