Student Number: _

KAMBALA

Mathematics Extension 2

HSC Assessment Task 2

Half-Yearly Examination

March 2008

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Answer all questions in the writing booklets provided. **Start each question in a new booklet.**
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 80
- Attempt Questions 1-5.
- All questions are of equal value.

Question 1 16 marks (Begin a new booklet) **Marks**

(a) Evaluate $|5-2i|$. **1**

(b) If $z=-3-4i$ find $\dfrac{1}{z}$ in the form $a+ib$. **2**

(c) Simplify $\dfrac{1+i^5}{1-i}$. **2**

(d) (i) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. **2**

(e) Let ω be a non-real cube root of unity.

(i) Show that $\omega^2+\omega+1=0$. **2**

(ii) Prove that $b^3+c^3=(b+c)(b+c\omega)(b+c\omega^2)$. **2**

(f) Find a polynomial $P(x)$ with real coefficients having $2i$ and $1-3i$ as zeroes. **3**

Question 2 16 marks (Begin a new booklet) **Marks**

(a) Find $\int \frac{x}{\sqrt{x+1}}dx$ **2**

(b) Solve for x: $\frac{x^2-5x}{4-x} \le -3$ **3**

(c) (i) Sketch the function $f(x)=x^2-c^2$, where c is a positive constant, clearly indicating its vertex and intercepts. **1**

(ii) Hence, without using calculus, draw separate sketches, at least $\frac{1}{3}$ of a page, for each of the following curves. Clearly indicate turning points.

(A) $f(x)=|x^2-c^2|$ **2**

(B) $f(x)=\frac{1}{x^2-c^2}$ **2**

(C) $f(x)=\sqrt{x^2-c^2}$ **2**

(D) $f(x)=(x^2-c^2)^2$ **2**

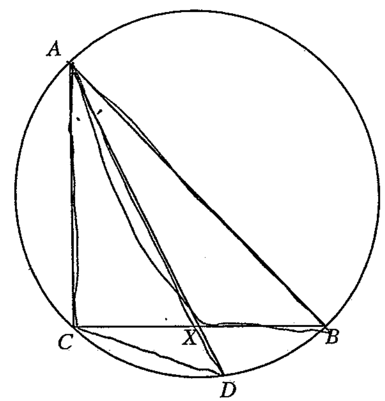
(E) $f(x)=(x^2-c^2)^3$ **2**

Question 3 16 marks (Begin a new booklet) **Marks**

(a) Consider the function $f(x)=x-\ln(1+x^2)$. **3**

Show that $f'(x)\ge 0$ for all values of x.

(b)



ABC is a triangle inscribed in a circle as shown above. The bisector of $\angle BAC$ meets BC in X and the circle at D.

(i) Prove that $\Delta ABX \,|||\, \Delta ADC$ **2**

(ii) Prove that $AB.AC = AD.AX$ **1**

(iii) Prove that $AB.AC = AX^2 + BX.XC$ **2**

Question 3 continues next page

Question 3 continued **Marks**

(c) A cubic equation has roots l, m and n. Given that

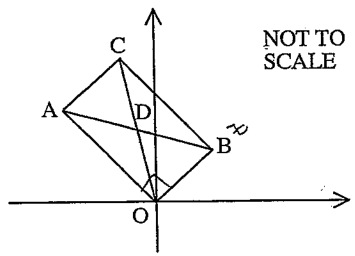
$$l+m+n=-3$$
$$l^2+m^2+n^2=29$$
$$lmn=-6$$

(i) By considering the expansion of $(l+m+n)^2$, show that the cubic equation is given by: **2**

$$x^3+3x^2-10x+6=0$$

(ii) Hence find the values of l, m and n. **3**

(d) OACB is a rectangle where OA = 2OB. D is the point of intersection of the diagonals. The point B represents the complex number z.



Find in terms of z, the complex number represented by:

(i) A **1**

(ii) D **2**

Question 4 16 marks (Begin a new booklet) **Marks**

(a) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using a suitable substitution, solve the equation $8x^3-6x+1=0$. **3**

(b) (i) If x and y are real, prove that $x^2+y^2 \geq 2xy$. **2**

(ii) Hence show that $a^2+b^2+c^2+d^2 \geq 4abcd$. **2**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2=4ay$. F is the focus of the parabola. PQ is the perpendicular from P to the directrix d, of the parabola. The tangent at P to the parabola, cuts the axis of the parabola at the point R.

(i) Show that the tangent at the point P to the parabola has equation **2**

$$px-y-ap^2=0$$

(ii) Show that PR and QF bisect each other. **3**

(iii) Show that $PR \perp QF$. **2**

(iv) What type of quadrilateral is $PQRF$? Give reasons for your answer. **2**

Question 5 **16 marks** **(Begin a new booklet)** **Marks**

(a) Let $y = uv$ be the product of u and v, where u and v are functions of x.

(i) Show that $y'' = uv'' + 2u'v' + u''v$. **2**

(ii) Find similar expressions for y''', y'''' and y'''''. **2**

(iii) Hence or otherwise, find and simplify $\frac{d^5}{dx^5}\left((1-x^2)e^{-x}\right)$. **2**

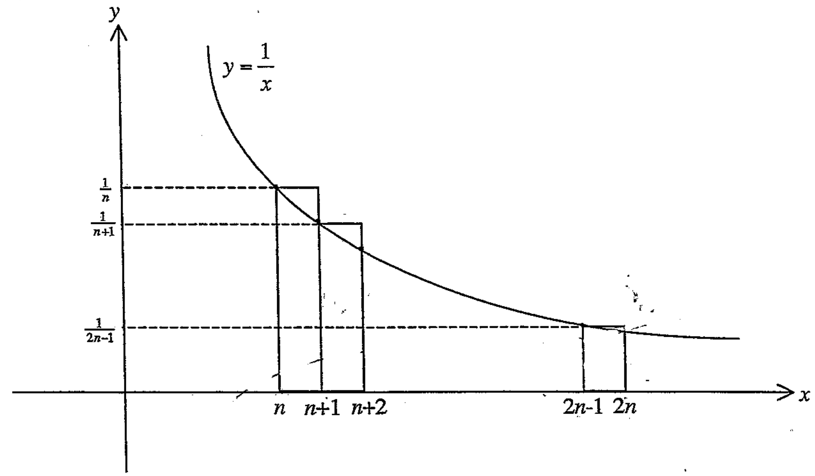
Question 5 continues next page

Question 5 continued **Marks**

(b) For all integers $n \geq 1$, $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}$.

(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1}$ **1**

(ii)



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \leq x \leq 2n$.

Using the diagram, show that $t_n + \frac{1}{2n} > \ln 2$. **3**

(iii) For all integers $n \geq 1$, let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$.

Using Mathematical Induction, prove that $s_n = t_n$. **4**

(iv) Hence find, to three decimal places, the value of:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{9999} - \frac{1}{10000}$$ **2**

End of Examination

Qn	Solutions	Marks	Comments
	Kambala Extn 2 Half-Yearly Exam 2008		
	Question 1		
(a)	$\lvert 5-2i \rvert = \sqrt{5^2+2^2} = \sqrt{29}$	1	
(b)	$z = -3-4i$		
	$\frac{1}{z} = \frac{1}{-3-4i} \times \frac{-3+4i}{-3+4i}$	1	
	$= \frac{-3+4i}{9+16}$		
	$= -\frac{3}{25} + \frac{4}{25}i$	1	
(c)	$\frac{1+i^5}{1-i}$ $i^4=1$, $i^5=i$		
	$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$	1	
	$= \frac{1-1+2i}{1+1}$		
	$= \frac{2i}{2}$		
	$= i$	1	
(d)	(i) $z = \frac{-1+i}{\sqrt{3}+i}$		
	$(-1+i) = \sqrt{2}\ \mathrm{cis}\ \frac{3\pi}{4}$	1	
	$(\sqrt{3}+i) = 2\ \mathrm{cis}\ \frac{\pi}{6}$		
	$z = \frac{\sqrt{2}\ \mathrm{cis}\ \frac{3\pi}{4}}{2\ \mathrm{cis}\ \frac{\pi}{6}}$		
	$= \frac{\sqrt{2}}{2}\ \mathrm{cis}\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$		
	$= \frac{\sqrt{2}}{2}\ \mathrm{cis}\left(\frac{7\pi}{12}\right)$	1	
(ii)	$\frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = \frac{-1+i}{\sqrt{3}+i}$		
	$\frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-\sqrt{3}+i+\sqrt{3}i+1}{3+1} = \frac{1-\sqrt{3}+i(1+\sqrt{3})}{4}$	1	
	Equating real parts		
	$\frac{\sqrt{2}}{2}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{4}$		

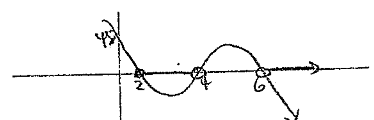
$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{4} \times \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

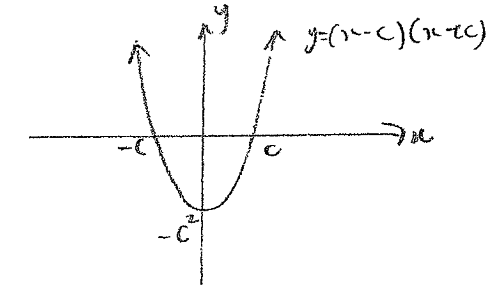
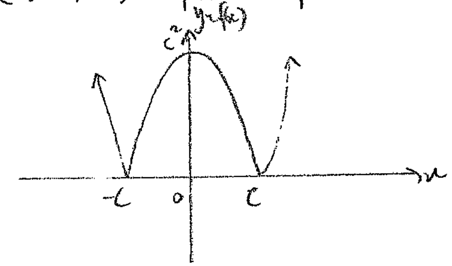
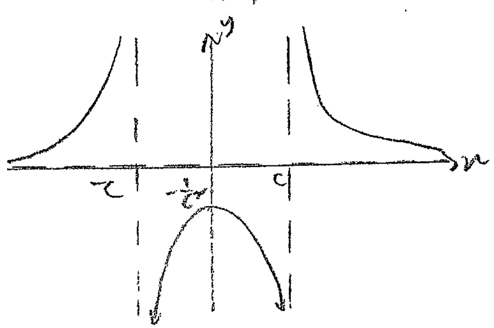
$= \frac{\sqrt{2}(1-\sqrt{3})}{4}$ or $\frac{\sqrt{2}-\sqrt{6}}{4}$ 1

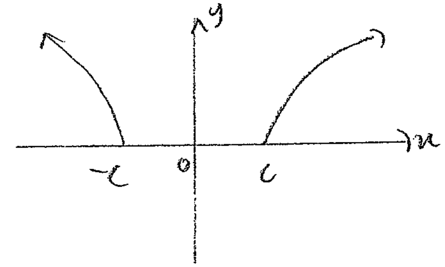
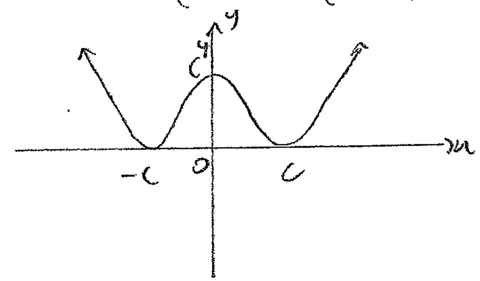
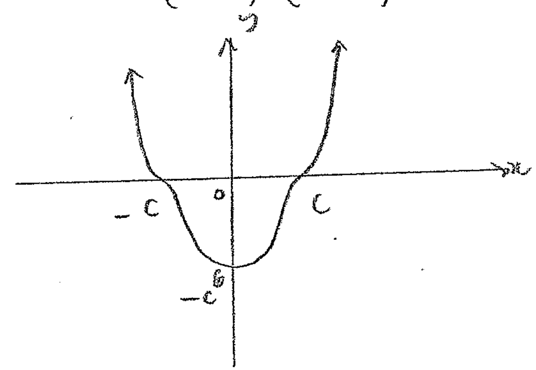
Qn	Solutions	Marks	Comments
	Question 1 ctd.		
(e)	(i) If ω is a non-real cube root of unity,		
	Method I		
	$\omega^3 = 1$		
	$\omega^3 - 1 = 0$		
	$(\omega-1)(\omega^2+\omega+1) = 0$	1	
	$\therefore \omega = 1$ or $\omega^2+\omega+1=0$		
	but ω is a non-real cube root of unity	1	
	$\omega \neq 1$ so $\omega^2+\omega+1=0$		
	Method II		
	$z^3 = 1$		
	let $z = 1\ \mathrm{cis}\ \theta$		
	$\mathrm{cis}\ 3\theta = \mathrm{cis}\ 0$		
	$3\theta = 0 + 2k\pi$		
	$\theta = \frac{2k\pi}{3}$		
	when $k=0$ $z_1 = \mathrm{cis}\ 0 = 1$		
	$k=1$ $z_2 = \mathrm{cis}\ \frac{2\pi}{3} = \omega$	1	
	$k=2$ $z_3 = \mathrm{cis}\ \frac{4\pi}{3} = \omega^2$		
	For $z^3-1=0$ Sum roots $=0$	1	
	$\therefore 1+\omega+\omega^2 = 0$		
	Method III		
	$\omega^3 = 1$		
	$(\omega^2)^3 = (\omega^3)^2 = 1$		
	$\therefore \omega^2$ is also a root		
	$1^3 = 1$ and 1 is obviously a root		
	$\therefore \omega^2, \omega$ and 1 are cube root of unity		
	For $z^3-1=0$ sum of roots $=0$		
	$\therefore \omega^2+\omega+1 = 0$		
(ii)	$RHS = (b+c)(b+c\omega)(b+c\omega^2)$ $\omega^3=1$		
	$= (b+c)(b^2 + bc\omega^2 + bc\omega + c^2\omega^3)$ $\omega^2+\omega+1=0$	1	
	$= (b+c)(b^2 - bc + c^2)$ $\therefore \omega^2+\omega = -1$		
	$= b^3+c^3 = LHS$ since $x^3+y^3 = (x+y)(x^2-xy+y^2)$	1	
	Method II (Similar)		
	LHS $b^3+c^3 = (b+c)(b^2-bc+c^2)$		
	RHS $= (b+c)(b+c\omega)(b+c\omega^2)$		
	$(b+c\omega)(b+c\omega^2) = b^2 + bc\omega^2 + bc\omega + c^2\omega^3$		
	$= b^2 + bc(\omega^2+\omega) + c^2$		
	$= b^2 - bc + c^2$		

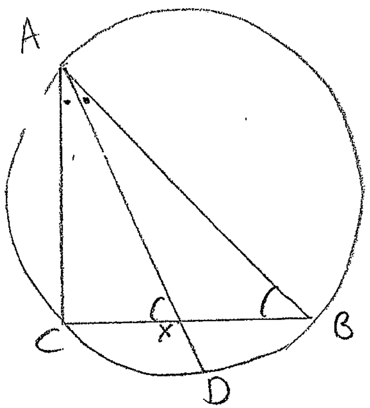
$\therefore LHS = RHS$

Qn	Solutions	Marks	Comments
(f)	Question 1 ctd. $P(x) = (z-2i)(z-(1-3i))$ $(z+2i)(z-(1+3i))$ If $z=2i$ and $1-3i$ are zeroes then $-2i$ and $1+3i$ are also zeroes as $P(x)$ has real coefficients. $(z-2i)(z+2i) = z^2+4$ $(z-(1-3i))(z-(1+3i))$ $= z^2-(1-3i+1+3i)z+1+9$ $= z^2-2z+10$ $P(x) = (z^2+4)(z^2-2z+10)$ $= z^4-2z^3+10z^2$ $+4z^2-8z+40$ $= z^4-2z^3+14z^2-8z+40$ $\therefore P(x)$ could be $P(x) = x^4-2x^3+14x^2-8x+40$	1 1 1	

Qn	Solutions	Marks	Comments+Criteria
2	(a) $\int \frac{x\,dx}{\sqrt{x+1}}$ Let $u = x+1$ $\therefore du = dx$ and $x = u-1$ $\int \frac{x}{\sqrt{x+1}}\,dx$ $= \int \frac{u-1}{\sqrt{u}}\,.du$ $= \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,.dx$ $= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$ $= \frac{2u^{\frac{3}{2}}}{3} + 2u^{\frac{1}{2}} + C$ $= \frac{2}{3}\sqrt{(x+1)^3} + 2\sqrt{x+1} + C$		
	(b) $\frac{x^2-5x}{4-x} \le -3$ $x(x-5)(4-x) \le -3(4-x)^2$ $x(x-5)(4-x)+3(4-x)^2 \le 0$ $(4-x)(x(x-5)+3(4-x)) \le 0$ $(4-x)(x^2-5x+12-3x) \le 0$ $(4-x)(x^2-8x+12) \le 0$ $(4-x)(x-6)(x-2) \le 0$. $x=2, 6, 4$. if $x=0$, $y=4\cdot-6\cdot-2$  $\therefore -2 \le x < 4$ and $x \ge 6$		

Qn	Solutions	Marks	Comments+Criteria
2 ctd	(c) (i) $f(x) = x^2 - c^2$, $c > 0$ 		
	(ii) (A) $f(x) = \lvert x^2 - c^2 \rvert$ 		
	(B) $f(x) = \dfrac{1}{x^2-c^2} = \dfrac{1}{(x-c)(x+c)}$ $x \neq \pm c$  if $x = 0$, $f(x) = \dfrac{1}{-c^2}$		

Qn	Solutions	Marks	Comments+Criteria
2 ctd	(c) ctd (C) $f(x) = \sqrt{x^2-c^2}$ 		
	(D) $f(x) = (x^2-c^2)^2$ $= (x^2-c^2)(x^2-c^2)$ $= (x-c)^2(x+c)^2$ 		
	(E) $f(x) = (x^2-c^2)^3$ $= (x-c)^3(x+c)^3$ 		

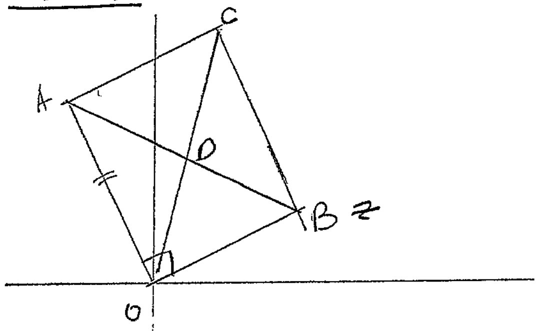
Qn	Solutions	Marks	Comments+Criteria
3	(a) $f(x) = x - \ln(1+x^2)$ $f'(x) = 1 - \frac{2x}{1+x^2}$ $= \frac{1+x^2-2x}{1+x^2}$ $= \frac{x^2-2x+1}{1+x^2}$ $= \frac{(x-1)^2}{1+x^2}$ Since $x^2 > 0$, then $1+x^2 > 0$ $(x-1)^2 > 0$ $\therefore f'(x) > 0 \ \forall x$		
	(b)  (i) Prove $\triangle ABX \parallel\!\mid \triangle ADC$ In $\triangle ABX$ and $\triangle ADC$: AD is bisector of $\angle BAC$ (data) $\therefore \angle CAX = \angle BAX$. $\angle AXC = \angle ABX$ (angles on same arc =) $\therefore \angle ACX = \angle AXB$ (angle sum of $\triangle$) ie $\triangle ABX \parallel\!\mid \triangle ADC$ (equiangular		

Qn	Solutions	Marks	Comments+Criteria
3 ctd	(b) ctd (ii) Prove $AB.AC = AD.AX$ Since $\triangle ABX \parallel\!\mid \triangle ADC$ from (i), then corresponding sides are in proportion. ie $\frac{AB}{AX} = \frac{AD}{AC}$ $\therefore AB.AC = AD.AX$		
	(iii) Prove $AB.AC = AX^2 + BX.XC$ From (ii), $AB.AC = AD.AX$ $= (AX + XD).AX$ $= AX^2 + XD.AX$ $= AX^2 + BX.XC$ since $AX.XD = BX.XC$ by intercept theorem		

Qn	Solutions	Marks	Comments+Criteria
3	(c) $l+m+n=-3=\Sigma\alpha$ cubic $l^2+m^2+n^2=29$ $\Sigma\alpha\beta=?$ $lmn=-6$ $\Sigma\alpha\beta\gamma$ (i) $(l+m+n)^2$ $=l^2+m^2+n^2+lm+ln+ml+mn+nl+nm$ $=l^2+m^2+n^2+2(lm+mn+nl)$ $(-3)^2=29+2(lm+mn+nl)$ $9=29+2(lm+mn+nl)$ $lm+mn+nl=\frac{-20}{2}$ $=-10$ cubic is given by $x^3-(\text{sum})x^2+(\text{product }\alpha\beta)x-\text{product}=0$ $\therefore x^3+3x^2-10x+6=0$ as req'd		
	(ii) $x^3+3x^2-10x+6=0=P(x)$. $P(1)=0$. $\therefore x-1$ is a factor. $x-1\,\overline{)\,x^3+3x^2-10x+6}$ quotient x^2+4x-6 x^3-x^2 $4x^2-10x$ $4x^2-4x$ $-6x+6$ $-6x+6$		

Qn	Solutions	Marks	Comments+Criteria
3 ctd	(c) ctd $\therefore P(x)=(x-1)(x^2+4x-6)$ $x=1$ a root, $x=\frac{-4\pm\sqrt{16-4.1.-6}}{2.1}$ $=\frac{-4\pm\sqrt{16+24}}{2}$ $=\frac{-4\pm\sqrt{40}}{2}$ $=\frac{-4\pm2\sqrt{10}}{2}$ $x=-2\pm\sqrt{10}$ $\therefore l,m,n$ are $1,\ -2\pm\sqrt{10}$.		

Q3 (d)



(i) let z be $r(\cos\theta + i\sin\theta)$

$\vec{OB} = z$

$OA = 2OB$

$\vec{OA} = 2zi$ (rotation anticlockwise by $\frac{\pi}{2}$)

$\therefore A$ is $2iz$ 1

(ii) $\vec{OC} = \vec{OB} + \vec{BC}$

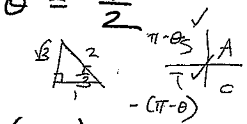
$= z + 2iz$ 1

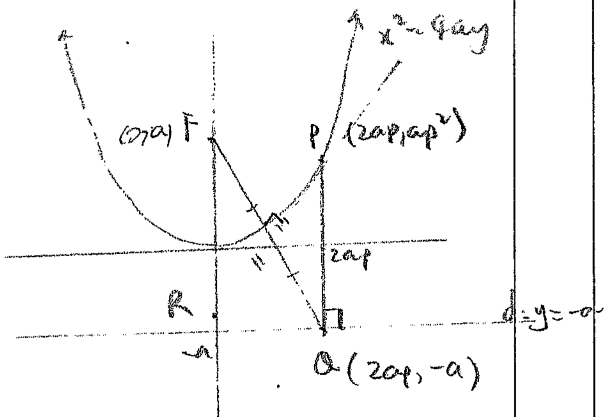
$= (1+2i)z$

$\therefore \vec{OD} = \frac{1}{2}(1+2i)z$

$\therefore D$ is $\frac{1}{2}(1+2i)z$ 1

or $\left(\frac{1}{2}+i\right)z$

Qn	Solutions	Marks	Comments
	Question 4		
(a)	$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$		
	If $8x^3 - 6x + 1 = 0$		
	$8x^3 - 6x = -1$		
	$4x^3 - 3x = -\frac{1}{2}$		
	Looking for 3 solutions (unique)		
	Let $x = \cos\theta$		
	$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$	1	
	$\cos 3\theta = -\frac{1}{2}$ 		
	$3\theta = \pi - \frac{\pi}{3}, \quad -(\pi - \frac{\pi}{3})$		
	$= \frac{2\pi}{3} + 2k\pi, \quad \frac{-2\pi}{3} + 2k\pi$		
	$3\theta = \frac{6k\pi \pm 2\pi}{3} = \frac{2\pi(3k\pm1)}{3}$		
	$\theta = \frac{2\pi(3k\pm1)}{9}$		
	when $k=0$ $\theta = \frac{\pm 2\pi}{9}$ $\therefore \boxed{x = \cos\frac{2\pi}{9}}$		
	$x = \cos(-\frac{2\pi}{9}) = \cos\frac{2\pi}{9}$		
	when $k=1$ $\theta = \frac{2\pi\times4}{9} = \frac{8\pi}{9}$ $\therefore \boxed{x = \cos\frac{8\pi}{9}}$		
	$\theta = \frac{2\pi\times2}{9} = \frac{4\pi}{9}$ $\therefore \boxed{x = \cos\frac{4\pi}{9}}$		
	when $k=2$ $\theta = \frac{2\pi\times7}{9} = \frac{14\pi}{9} = \frac{-4\pi}{9}$ $x = \cos(\frac{-4\pi}{9}) = \cos\frac{4\pi}{9}$		
	$\theta = \frac{2\pi\times5}{9} = \frac{10\pi}{9} = \frac{-8\pi}{9}$ $x = \cos(\frac{-8\pi}{9}) = \cos\frac{8\pi}{9}$		
	$\therefore x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$		

Qn	Solutions	Marks	Comments+Criteria
4	(b) (i) prove $x^2+y^2 \geq 2xy$ Now $(x-y)^2 \geq 0 \quad \forall\ x,y$ real $\therefore x^2-2xy+y^2 \geq 0$ i.e. $x^2+y^2 \geq 2xy$		
	(ii) Hence show $a^2+b^2+c^2+d^2 \geq 4abcd$ using (i): $a^2+b^2 \geq 2ab$ ① $c^2+d^2 \geq 2cd$ ② ① + ②: $a^2+b^2+c^2+d^2 \geq 2ab+2cd$ Let $a' = a^2$, $b' = b^2$ etc $a^4+b^4+c^4+d^4 \geq 2a^2b^2+2c^2d^2$ $\geq 2((ab)^2+(cd)^2)$ $\geq 2(ab)(cd)$ $\geq 4abcd$		Let $a=ab$, $b=cd$ OR $ab+cd \geq 2abcd$
4	(c) 		

Qn	Solutions	Marks	Comments+Criteria
4	(c) ctd. (i) $P(2ap, ap^2)$ $\quad y = \frac{1}{4a}x^2$ $\frac{dy}{dx} = \frac{1}{2a}x$ $m = \frac{2ap}{2a} = p$. $y - ap^2 = p\cdot(x-2ap)$ $y-ap^2 = px - 2ap^2$ $\therefore px - ap^2 - y = 0$.		
	(ii) Show PR and QF bisect. R lies on tangent and axis of parabola. $\therefore$ at $x=0$, $0-ap^2-y=0$ $\therefore y=-ap^2$. R is $(0,-ap^2)$. midpt of PR $= \left(\frac{2ap+0}{2}, \frac{-ap^2+ap^2}{2}\right)$ $=(ap,0)$ midpt of QF $= \left(\frac{2ap+0}{2}, \frac{-a+a}{2}\right)$ $=(ap,0)$ since midpt of PR = midpt of QF then PR and QF must bisect		

Qn	Solutions	Marks	Comments+Criteria
4	(c) ctd		

(iii) Show $PR \perp OF$

$$M_{PR} = \frac{ap^2 + ap^2}{2ap - 0} = \frac{2ap^2}{2ap} = p \quad \text{(already shown in (i)!)}$$

$$M_{OF} = \frac{-a - a}{2ap - 0} = \frac{-2a}{2ap} = -\frac{1}{p}$$

$$M_{PR} \times M_{OF} = p \times -\frac{1}{p} = -1$$

$\therefore PR \perp OF$

(iv) PQRF?

Diagonals bisect @ right angles, $\therefore$ PQRF is a rhombus.

$FR \parallel PQ$ since PQ is perp. to directrix, a horizontal line.

Qn	Solutions	Marks	Comments+Criteria
5	(a) $y = uv$		

(i)
$$y' = u'v + v'u$$
$$y'' = u''v + v'u' + v''u + u'v'$$
$$= uv'' + \underline{2u'v'} + u''v$$

(ii) find y''', y'''', y'''''

$$y''' = \underline{u'v''} + v'''u \boxed{+ 2u''v'} + \underline{v''.2u'} + u'''v \boxed{+ v'u''}$$
$$= uv''' + \underline{3u'v''} + \underline{3u''v'} + u'''v$$

$$y'''' = \underline{u'v'''} + v''''u + 3u''v'' + \underline{v'''.3u'} + \underline{3u'''.v'} + v''3u'' + u''''v + \underline{v'u'''}$$
$$= uv'''' + \underline{6u''v''} + \underline{4u'v'''} + \underline{4u'''v'} + u''''v$$

$$y''''' = u'v'''' + \underline{v'''''u} + 6u'''v'' + v'''.6u'' + 4u''v''' + v''''.4u' + 4u''''v' + v''.4u''' + u'''''v + v'u''''$$
$$= uv''''' + 5u'v'''' + 10u''v''' + 10u'''v'' + 5u''''v' + u'''''v$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(iii) $\frac{d^5}{dx^5}\left(\underbrace{(1-x^2)}_{u}\underbrace{e^{-x}}_{v}\right)$

$u = 1-x^2$, $\frac{du}{dx} = u' = -2x$, $u'' = -2$, $u^3 = u^4 = u^5 = 0$

$y^5 = uv^5 + 5u^1v^4 + 10u^2v^3 + 10u^3v^2 + 5u^4v^1 + u^5v$

$v = e^{-x}$, $v' = -e^{-x}$, $v'' = e^{-x}$, $v''' = -e^{-x}$, $v^4 = e^{-x}$, $v^5 = -e^{-x}$

$$\therefore \frac{d^5}{dx^5}\left((1-x^2)e^{-x}\right)$$
$$= (1-x^2)(-e^{-x}) + 5(-2x)e^{-x} + 10(-2)(-e^{-x}) + 10.0\ldots + 5\times 0\ldots + 0$$
$$= -e^{-x}(1-x^2 + 10x - 20)$$
$$= -e^{-x}(-x^2 + 10x - 19) = e^{-x}(x^2 - 10x + 19)$$

Qn	Solutions	Marks	Comments+Criteria

5 (b) $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$

(i) $\underbrace{t_n + \frac{1}{2n}} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n} + \underbrace{\frac{1}{2n}}$

$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{\cancel{2}}{2n}$

$= \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}$

as required

(ii) $\int_n^{2n} \frac{1}{x}\,dx \approx n\times\frac{1}{n} + n\times\frac{1}{n+1} + n\times\frac{1}{n+2} + \dots + n\times\frac{1}{2n-1}$

$= 1.\left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}\right) = \left(t_n + \frac{1}{2n}\right)$

also $\int_n^{2n} \frac{1}{x}\,dx = \left[\ln x\right]_n^{2n}$

$= \ln 2n - \ln n$

$= \ln 2 + \ln n - \ln n$

$= \ln 2$

From diagram, area of rectangle is an approximation yielding an area greater than exact area.

$\therefore \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}\right) > \ln 2$

ie $\left(t_n + \frac{1}{2n}\right) > \ln 2$

Qn	Solutions	Marks	Comments+Criteria

5 (iii) $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n}$

Prove $S_n = t_n$.

$t_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$

RTP:

$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$ for $n \geq 1$.

test $n=1$:

LHS $= 1 - \frac{1}{2}$ RHS $= \frac{1}{1+1}$

$= \frac{1}{2}$ $= \frac{1}{2} =$ LHS

$\therefore$ true for $n=1$.

Assume true for $n=k$ ie $S_k = t_k$.

ie $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k-1} + \frac{1}{2k}$

Prove true for $n = k+1$ ie $S_{k+1} = t_{k+1}$.

ie RTP:

$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)} = \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \dots + \frac{1}{2(k+1)}$

ie $1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k+1} - \frac{1}{2k+2} = \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+2}$

LHS $= 1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2k+1} - \frac{1}{2k+2}$

$= \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2k}} + \frac{1}{2k+1} - \frac{1}{2k+2}$

$= \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k-1} + \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2}$

$= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{2k+2 - (k+1)}{(2k+2)(k+1)}$

$= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{\cancel{k+1}}{(2k+2)\cancel{(k+1)}}$

$= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{1}{2k+2}$

$=$ RHS

Qn	Solutions	Marks	Comments+Criteria
5	(iv) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{9999}-\frac{1}{10000}$ $S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n-1}-\frac{1}{2n}$ So $2n=10000$ $n=5000$ $t_n=S_n \therefore t_{5000}=S_{5000}$ $t_n+\frac{1}{2n}>\ln 2$ $t_{5000}+\frac{1}{10000}>\ln 2$ $t_{5000}>\ln 2-\frac{1}{10000}$ $>0.693047\dots$ i.e. $t_{5000} \doteqdot 0.693$		