Student ID: ________________

KAMBALA

Mathematics Extension 2

HSC Assessment Task 1

February 2009

Time Allowed: 50 minutes working time

Outcomes Assessed

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E3 uses the relationship between algebraic and geometric representations of complex numbers
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions involving polynomials
E9 communicates abstract ideas and relationships using appropriate notation and logical argument

INSTRUCTIONS

- This task contains 3 questions. Marks for each part question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Board-approved calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks will be allocated to questions involving higher order thinking.

Question 1 *(Start a new page.)* **10 Marks**

(a) If $z = 1 + i$ and $w = 3 + 4i$, find:

(i) $z - w$ **1**

(ii) $\overline{z}w$ **2**

(b) Given the complex numbers $z_1 = -2 + 2i$ and $z_2 = 1 + i\sqrt{3}$:

(i) Evaluate $\frac{z_1}{z_2}$ in the form $a + ib$. **2**

(ii) Write z_1 and z_2 in modulus-argument form and evaluate $\frac{z_1}{z_2}$ in modulus-argument form. **3**

(iii) Hence show that $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$. **2**

Question 2 *(Start a new page.)* **11 Marks**

(a) (i) Find the complex square roots of $5 - 12i$. **2**

(ii) Hence find the roots of the complex equation $x^2 - \sqrt{5}x + 3i = 0$. **2**

(b) On an Argand diagram show the region where the following inequalities hold simultaneously: **2**

$$|z - i| \le 2 \text{ and } 0 \le \arg z \le \frac{\pi}{3}$$

Question 2 continued

(c) (i) Show that $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3. **2**

(ii) Verify that $x = i$ is also a root of $P(x)$. **1**

(iii) Hence factorise $P(x)$ over the complex field. **2**

Question 3 *(Start a new page.)* 9 Marks

(a) If ω is a complex cube root of unity:

(i) Show that the other complex cube root is ω^2. **2**

(ii) Prove that $1 + \omega + \omega^2 = 0$. **1**

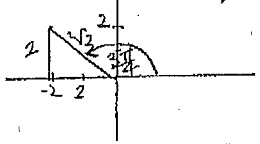
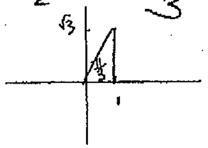
(iii) Prove that $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2} = 1$. **2**

(b) The polynomial $F(x) = x^4 + px^3 + qx^2 + rx + s$ has four zeroes α, β, γ and δ such that the sum of α and β equals the sum of γ and δ.
Let $C = \alpha + \beta = \gamma + \delta$, let $P = \alpha\beta$ and let $Q = \gamma\delta$.

(i) Find p, q, r and s in terms of C, P and Q. **2**

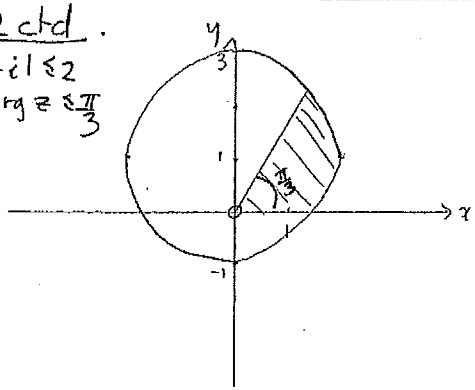
(ii) Show that the coefficients of $F(x)$ satisfy the condition $p^3 + 8r = 4pq$. **2**

End of Assessment Task

Qn	Solutions	Marks	Comments
	Kambala 2009 Ext#2 HSC Task 1		
	Question 1		
(a)	$z = 1+i$ and $w = 3+4i$		
	(i) $z - w$		
	$= (1+i) - (3+4i)$		
	$= -2-3i$	1	
	(ii) $\bar{z}w$		
	$= (1-i)(3+4i)$		
	$= 3+4i-3i+4$		
	$= 7+i$	2	
(b)	(i) $z_1 = -2+2i$ and $z_2 = 1+i\sqrt{3}$		
	$\frac{z_1}{z_2} = \frac{-2+2i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$	1	
	$= \frac{-2+i\,2\sqrt{3}+2i+2\sqrt{3}}{1+3}$		
	$= \frac{2\sqrt{3}-2+i(2\sqrt{3}+2)}{4}$		
	$= \frac{\sqrt{3}-1}{2} + i\frac{(\sqrt{3}+1)}{2}$	1	
	(ii) $z_1 = 2\sqrt{2}\,(\text{cis}\,\frac{3\pi}{4})$ $\quad z_2 = 2\,\text{cis}\,\frac{\pi}{3}$	2	
	 		
	$\frac{z_1}{z_2} = \frac{2\sqrt{2}\,(\text{cis}\,\frac{3\pi}{4})}{2\,\text{cis}(\frac{\pi}{3})}$		
	$= \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$		
	$= \sqrt{2}\,\text{cis}\,\frac{5\pi}{12}$	1	
	(iii) from i) and ii)		
	$\frac{\sqrt{3}-1}{2} + i\left(\frac{\sqrt{3}+1}{2}\right) = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$		
	$\tan\frac{5\pi}{12} = \frac{y}{x} = \frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}$		
	$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$		
	$= \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$	2	

Qn	Solutions	Marks	Comments
	Question 2		
(a)	(i) $\sqrt{5-12i} = x+iy$		
	$5-12i = x^2+2ixy-y^2$		
	$5-12i = x^2-y^2+i.2xy$		
	$x^2-y^2 = 5$ ①		
	$2xy = -12$ ②		
	$y = -\frac{6}{x}$		
	Sub in ①		
	$x^2 - \frac{36}{x^2} = 5$		
	$x^4 - 36 = 5x^2$		
	$x^4 - 5x^2 - 36 = 0$		
	$(x^2-9)(x^2+4) = 0$		
	$x = \pm 3$, $x^2 = -4$ no soln as x real.		
	$y = -2, 2$		
	∴ Complex square roots are $3-2i, -3+2i$	2	
	(ii) $x^2 - \sqrt{5}x + 3i = 0$		
	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$		
	$= \frac{\sqrt{5} \pm \sqrt{5-4.1.3i}}{2}$		
	$= \frac{\sqrt{5} \pm \sqrt{5-12i}}{2}$ (using i))	1	
	$= \frac{\sqrt{5} \pm (3-2i)}{2}$ or $\frac{\sqrt{5} \pm (-3+2i)}{2}$		
	$= \frac{\sqrt{5}+3}{2} - i$ $\quad \frac{\sqrt{5}-3}{2} + i$		
	or $\frac{\sqrt{5}-3}{2} + i$ $\quad$ or $\frac{\sqrt{5}+3}{2} - i$		
	ie $\left(\frac{\sqrt{5}+3}{2}\right) - i$ or $\left(\frac{\sqrt{5}-3}{2}\right) + i$	1	

Qn	Solutions	Marks	Comments
	Q2 ctd.		
(b)	$\lvert z-i\rvert \le 2$		
	$0 \le \arg z \le \frac{\pi}{3}$		



Qn	Solutions	Marks	Comments
(c)(i)	$P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$		
	if $x=1$ is a root of multiplicity 3 of $P(x)$		
	then it is a root of multiplicity 2 of $P'(x)$		
	and a root of multiplicity 1 of $P''(x)$	1	
	(Multiple roots theorem)		
	$P'(x) = 5x^4 - 12x^3 + 12x^2 - 8x + 3$		
	$P''(x) = 20x^3 - 36x^2 + 24x - 8$		
	$P''(1) = 20 - 36 + 24 - 8 = 0$		
	Since $P(x)$ has at most 5 distinct roots		
	it only has 1 root of multiplicity 3		
	$\therefore x = 1$ is the root of multiplicity 3 of $P(x)$	1	
(ii)	$P(i) = i^5 - 3i^4 + 4i^3 - 4i^2 + 3i - 1$	1	
	$= i^2 \cdot i^2 \cdot i - 3 + 4i^2 \cdot i + 4 + 3i - 1$		
	$= 0$		
	$\therefore x = i$ is also a root of $P(x)$.		
(iii)	Since $P(x)$ has real coefficients then		
	roots occur in conjugate pairs.		
	$\therefore x = -i$ is also a root of $P(x)$.	1	
	$\therefore P(x) = (x-1)^3 (x+i)(x-i)$.	1	

Qn	Solutions	Marks	Comments
	Question 3		
(a)	$\omega^3 = 1$		
(i)	let $z^3 = 1$		
	$z = \cos\theta + i\sin\theta$		
	$\cos 3\theta + i \sin 3\theta = 1 \operatorname{cis} 0$		
	$\lvert r\rvert = 1 \quad 3\theta = 0 + 2k\pi$		
	$\theta = \frac{2k\pi}{3}$		
	when $k=0 \quad z_1 = 1$		
	$k=1 \quad z_2 = \operatorname{cis}\frac{2\pi}{3} = \omega$	1	
	$k=2 \quad z_3 = \operatorname{cis}\frac{4\pi}{3} = \left(\operatorname{cis}\frac{2\pi}{3}\right)^2 = \omega^2$	1	
(ii)	$z^3 - 1 = 0$		
	Roots are $1, \omega, \omega^2$		
	Sum of roots $= 1 + \omega + \omega^2 = 0$	1	
(iii)	$LHS = \frac{1}{1+\omega} + \frac{1}{1+\omega^2}$		
	$= \frac{1+\omega^2 + 1+\omega}{(1+\omega)(1+\omega^2)}$	1	
	$= \frac{1+\omega+\omega^2 + 1}{1+\omega+\omega^2+\omega^3}$ but $\omega^3 = 1$, $1+\omega+\omega^2 = 0$		
	$= \frac{0+1}{0+1}$		
	$= 1 = RHS$	1	
	Alternative		
(ii)	ω is a complex root		
	$\therefore \omega^3 = 1$		
	$\omega^3 - 1 = 0$		
	$(\omega - 1)(\omega^2 + \omega + 1) = 0$		
	but $\omega - 1 \ne 0$ because ω is a complex root.		
	$\therefore 1 + \omega + \omega^2 = 0$ as required.		

Qn	Solutions	Marks	Comments
(b) (i)	Q3 ctd. $F(x) = x^4 + px^3 + qx^2 + rx + s$ $C = \alpha + \beta = \gamma + \delta$ $P = \alpha\beta$ $Q = \gamma\delta$ $\alpha + \beta + \gamma + \delta = \frac{-b}{a}$ $2C = -p$ (1) ie $p = -2C$ $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$ $P + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + Q = q$ $P + \alpha(\gamma + \delta) + \beta(\gamma + \delta) + Q = q$ $P + \alpha C + \beta C + Q = q$ $P + C^2 + Q = q$ (2) $\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -\frac{d}{a}$ $P\gamma + P\delta + \beta Q + \alpha Q = -r$ $PC + CQ = -r$ (3) ie $r = -(PC + CQ)$ $\alpha\beta\gamma\delta = \frac{e}{a}$ $PQ = s$ (4)	2	
(ii)	WTS $p^3 + 8r = 4pq$ Using (1)-(4) above $LHS = (-2C)^3 + 8(-C(P+Q))$ $= -8C^3 - 8PC - 8CQ$	1	
	$RHS = 4(-2C)(P + C^2 + Q)$ $= -8C(P + C^2 + Q)$ $= -8PC - 8C^3 - 8CQ$ $= -8C^3 - 8PC - 8CQ$ $= LHS$ $\therefore p^3 + 8r = 4pq$	1	

Qn	Solutions	Marks	Comments
Q3 (b)	Alternative (ii) From part (i) $-C = \frac{p}{2}$ $P + Q = q - C^2$ (from ii) $r = -C(P+Q)$ $\therefore r = \frac{p}{2}(q - C^2)$ $r = \frac{p}{2}\left(q - \left(\frac{-p}{2}\right)^2\right)$ $r = \frac{p}{2}\left(q - \frac{p^2}{4}\right)$ $2r = pq - \frac{p^3}{4}$ $8r = 4pq - p^3$ $\therefore p^3 + 8r = 4pq$ as required.		