KESSER TORAH COLLEGE

MATHEMATICS DEPARTMENT

Mathematics
Year 12

Advanced 2012

General Instructions

- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided.
- All necessary working should be shown in every question

Total marks (70)

- 10 Objective Response (10 marks)
- 4 Extended Response (60 marks)
- Use a **SEPARATE** answer sheet for each question

STUDENT NUMBER:_

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x>0$

Objective Response Questions (10 Marks)

Circle the correct answer for each question.

1. The solution to the inequality $\frac{7-5x}{4} \le 8$ is:

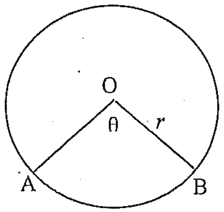
   A. $x \ge \frac{-1}{5}$ B. $x \le -5$ C. $x \ge -5$ D.

2. If $(3+\sqrt{2})(2+\sqrt{8}) = a + b\sqrt{2}$, then the value of

   A. $a = 10,\ b = 8$ B. $a = 22;\ b = 8$

   C. $a = 10,\ b = 6$ D. $a = 10,\ b = 2$

3. Which formula gives the length of the arc AB?



   A. $L = \pi r\theta$ B. $L = \frac{r\theta}{2}$ C. $L = \frac{r^2\theta}{2}$ D. $L = r\theta$

4. The value of $\log_2 16$ is

   A. 8   B. 4   C. 3   D. $\frac{1}{4}$

5. Find $\int \cos 2x\, dx$

   A. $2\cos 2x$ B. $2\sin 2x$

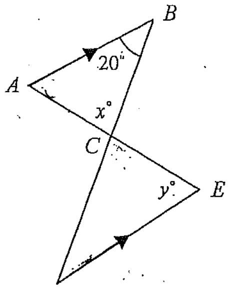
   C. $\frac{1}{2}\cos 2x$ D. $\frac{1}{2}\sin 2x$

6. The exact value of $\sec\frac{\pi}{4}$ is:

   A. $\frac{-1}{4}$ B. $\frac{1}{\sqrt{2}}$

   C. -4 D. $\sqrt{2}$

7. The value of $x + y$ in the diagram is



NOT TO SCALE

   A. 160°   B. 100°   C. 140°   D. cannot be determined

8. The derivative of $y = 3e^{-2x}$ is

   A. $\frac{dy}{dx} = -6e^{2x-1}$ B. $\frac{dy}{dx} = \frac{-1}{6e^{2x}}$

   C. $\frac{dy}{dx} = \frac{-6}{e^{2x}}$ D. $\frac{dy}{dx} = \frac{3e^{-2x}}{-2}$

9. The second derivative of $\sin x$ is

   A. $\sin x$ B. $-\sin x$ C. $\cos x$ D. $-\cos x$

10. The period of the curve $y = 2\sin 3x$ is

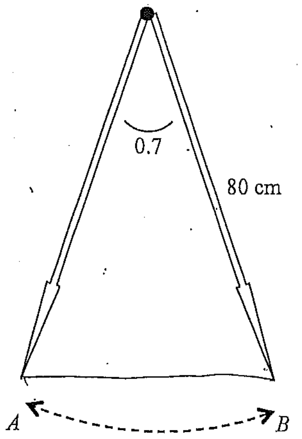
   A. 2π B. 6π C. π D. $\frac{2\pi}{3}$

Extended Response Questions (60 Marks)

Question 1 (15 marks)

a) If $\frac{\pi}{k} = 60°$, then find the value of k. 1

b)



NOT TO SCALE

A pendulum is 80cm long and swings through an angle of 0.7 radians.
The extreme positions of the pendulum are indicated by the points A and B.

i) Find the area of the sector swept out by the pendulum. 2

ii) Find the straight line distance between the extreme positions of the pendulum (correct to 2 decimal places) 2

c) Differentiate the following functions:

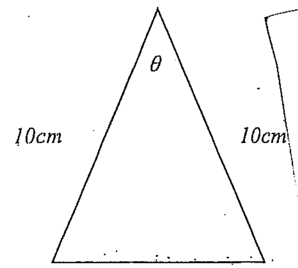
i) $y = 2e^{3x}$ 1

ii) $y = \ln(3x - 1)$ 1

iii) $y = (\sin x)^2$ 2

iv) $y = x^3 \cos 2x$ 3

d) The triangle drawn below has an area of $25cm^2$.
Find the two different values of θ. 3



Question 2 (15 marks) Start a new page

a) i) Find the y-intercept of the curve $y = 4\cos 2x$ 1

ii) Find the x-intercepts of the curve $y = 4\cos 2x$ for $0 \le x \le \pi$ 2

iii) Sketch the curve $y = 4\cos 2x$ for $0 \le x \le \pi$ 1

b) Find:

i) $\int 1 - e^{-x} dx$ 2

ii) $\int \frac{1}{x-1} dx$ 1

c) Find the equation of the tangent to the curve $y = e^{\tan x}$ at the point where $x = \frac{\pi}{4}$. 3

d) Consider the function $y = 2^x$

i) Copy and complete the table below 1

x	-1	0	1	2	3
2^x					

ii) Using the trapezoidal rule with five function values, find an approximation for $\int_{-1}^{3} 2^x\,dx$ 3

iii) Explain why the trapezoidal rule gives an overestimate for the value of $\int_{-1}^{3} 2^x\,dx$ 1

Question 3 (15 marks) Start a new page

a) i) Evaluate $\int_0^2 \sec^2 \frac{x}{2}\,dx$ correct to 1 decimal place. 2

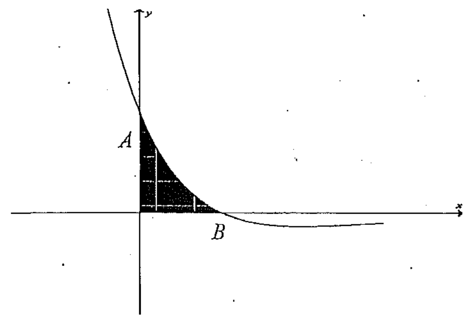
ii) Show that $\int_{\ln 2}^{\ln 4} e^{2x}\,dx = 6$ 2

b) i) Solve the equation $2y^2 - 3y - 2 = 0$ 2

ii) Hence, solve the equation $2\sin^2 x - 3\sin x - 2 = 0$ by using the substitution $M = \sin x$.
Give your answers such that $0 \le x \le 2\pi$ 2

c) i) Show that if $y = \frac{x}{e^x}$, then: $\frac{dy}{dx} = \frac{1-x}{e^x}$ 2

The following graph is of the function: $y = \frac{1-x}{e^x}$.
The points A and B are the intercepts with the x and y axes, respectively.



ii) Find the coordinates of the points A and B 2

iii) Use your result from part i) to calculate the shaded area bounded by the curve $y = \frac{1-x}{e^x}$ and the coordinate axes.
Give your answer in terms of e. 3

Question 4 (15 marks) Start a new page

a) Find the exact volume of the solid of revolution formed when the section of the curve $y = \sqrt{1+\sin 2x}$ between $x = 0$ to $x = \frac{\pi}{2}$ is rotated about the x-axis. 4

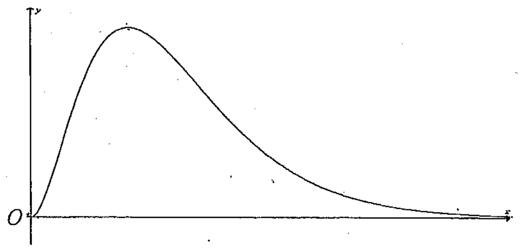
b) Given that $\log_a b = 2.5$ and $\log_a c = 0.5$, find:

i) the value of $\log_a\left(\frac{b}{c}\right)$. 1

ii) the value of $\log_a(c^2)$. 1

iii) b in terms of c. 2

c) The cross section of a hill can be modelled by the graph of the function $y = x^2 e^{-x}$ where x and y are measured in kilometres. Dan starts climbing up the hill from point O.



i) Show that: $y' = e^{-x}(2x - x^2)$ **2**

ii) What is the maximum height of the hill, to the nearest metre? **2**

iii) After reaching the top of the hill, Dan continues downward. Show that he reaches the steepest slope when he is approximately 3.41km to the right of his starting point O. **3**

Start here

Question One

a) $\frac{\pi}{k} = 60° = \frac{\pi}{3}$

$k = 3$ ✓

b)

i. $A = \frac{1}{2}r^2\theta$

✓ $= \frac{1}{2} \times 6400 \times 0.7$

$= 2240 \text{ cm}^2$

ii. $AB^2 = 80^2 + 80^2 - 2 \times 80^2 \times \cos 0.7$

✓ $= 12800 - 12800 \cos 0.7$

$= 3010.020$

✓ $AB = 54.86$ cm

c)

✓ i. $y = 2e^{3x}$

$\frac{dy}{dx} = 6e^{3x}$

ii. $y = \ln(3x-1)$

✓ $\frac{dy}{dx} = \frac{3}{3x-1}$

iii $y = (\sin x)^2$

✓ $\frac{dy}{dx} = 2 \times (\sin x)^1 \times \cos x$

$= 2\sin x \cos x$

iv. $y = x^3 \cos 2x$

$y' = uv' + vu'$

✓ $= x^3 \times -2\sin 2x + \cos 2x \times 3x^2$

$= x^2(3\cos 2x - 2x\sin 2x)$

Office Use Only – Do NOT write anything, or make any marks, below this line.

Start here

d) $A = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 10 \times 10 \times \sin\theta$

$25 = 50\sin\theta$

$\sin\theta = \frac{1}{2}$ ✓ ✓

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

15/15 ☺

Office Use Only – Do NOT write anything, or make any marks, below this line.

Start here

Question Two

a)

i. $y = 4\cos 2x$

Let $x = 0$

$y = 4 \times \cos 0$

$= 4$

y-intercept at $(0, 4)$

period : π

amplitude : 4

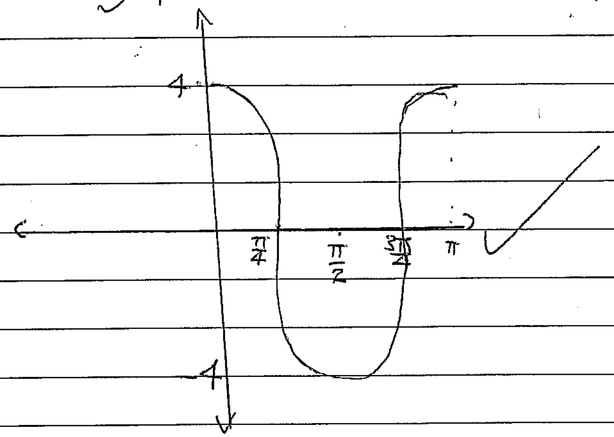
ii. $y = 4\cos 2x$

Let $y = 0$

$4\cos 2x = 0$

$\cos 2x = 0$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$



b)

i. $\int 1 - e^{-x}\, dx$

$= x + e^{-x} + c$

ii $\int \frac{1}{x-1}\, dx = \ln(x-1) + c$

(in next booklet)

Start here

c) $y = e^{\tan x}$

$y' = \sec^2 x \, e^{\tan x}$

at $x = \frac{\pi}{4}$

$\sec^2 \frac{\pi}{4} \times e^{\tan \frac{\pi}{4}}$ 2·5

gradient $= 2e$

~~$e\sqrt{2}$ is the gradient~~

~~$y - y_1 = m(x - x_1)$~~

~~$y - e = e\sqrt{2}(x - \frac{\pi}{4})$~~

~~$y - e = e\sqrt{2}x - e\sqrt{2} \times \frac{\pi}{4}$~~

~~$y = e\sqrt{2}x - \frac{e\sqrt{2}\pi}{4} + e$~~

$y - y_1 = m(x - x_1)$

$y - e = 2e(x - \frac{\pi}{4})$

$y - e = 2ex - \frac{e\pi}{4}$

$\boxed{y = 2ex - (\frac{e\pi}{4} + e)}$

d) $y = 2^x$

x	-1	0	1	2	3
2^x	$\frac{1}{2}$	1	2	4	8

ii $A \approx \frac{h}{2}[(y_0 + y_n) + 2(\text{rest } y)]$

$\int_{-1}^{3} 2^x \, dx$

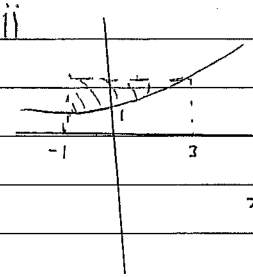
$A \approx \frac{1}{2}[(\frac{1}{2} + 8) + 2(1 + 2 + 4)]$

$\approx \frac{1}{2}[8.5 + 14]$

$\approx \frac{1}{2}[22.5]$

$\approx 11.25 \, u^2$

iii



$y = 2^x$

trapezoidal rule uses a rectangle to estimate, it covers more area than the function as shown

trapezium

13.5/15

Office Use Only – Do NOT write anything, or make any marks, below this line.

Start here

Question Three

a)

i. $\cancel{\int_0^2 \sec^2 \tfrac{1}{2} x}$

$\cancel{= [2\tan \tfrac{1}{2}x]_0^2}$

$\cancel{= [2\tan 1 - 2\tan 0]}$

$\int_0^2 \sec^2 \frac{x}{2}\, dx$

$= \left[2\tan \frac{x}{2}\right]_0^2$

$= 2\tan \frac{2}{2} = 2\tan 1 - 0$

$= 0.0349 = 0.03$

ii. $\int_{\ln 2}^{\ln 4} e^{2x}\, dx = 6$

To show

$LHS = \int_{\ln 2}^{\ln 4} e^{2x}$

$= \left[\frac{1}{2} e^{2x}\right]_{\ln 2}^{\ln 4}$

$= \left[\frac{1}{2} e^{2\ln 4} - \frac{1}{2} e^{2\ln 2}\right]$

$= \frac{1}{2} \times 16 - \frac{1}{2} \times 4$

$= 8 - 2$

$= 6$

$= RHS$

$\therefore LHS = RHS$

b)

i. $2y^2 - 3y - 2 = 0$

$(2y+1)(y-2) = 0$

$y = -\frac{1}{2}, 2$

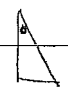
ii $2\sin^2 x - 3\sin x - 2 = 0$

Let $M = \sin x$

$2M^2 - 3M - 2 = 0$

$M = -\frac{1}{2}, 2$

$\sin x = -\frac{1}{2}$ $\quad$ $\sin x \neq 2 \rightarrow$ no soln.



$x =$ related angle $= \frac{\pi}{6}$

$= \frac{7\pi}{6}, \frac{11\pi}{6}$

Start here

c)

i. To show

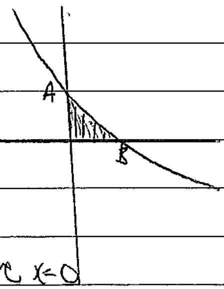
$y = \frac{x}{e^x}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{e^x \times 1 - e^x \times x}{e^{2x}}$

$= \frac{e^x(1 - x)}{e^{2x}}$

$\therefore \frac{dy}{dx} = \frac{1 - x}{e^x}$



ii. A is at the point where $x=0$

$y = \frac{1-0}{e^0}$

$= 1$

$A(0,1)$

B is at the point where $y=0$

$0 = \frac{1-x}{e^x}$

$1 - x = 0$

$x = 1$

$B(1,0)$

iii. $\int_0^1 \frac{1-x}{e^x}\, dx$

$= \left[\frac{x}{e^x}\right]_0^1$ (using part (i))

$= \left[\frac{1}{e} - 0\right]$

$= \frac{1}{e} u^2$

14.5/15

Start here

Question Four

13·5

a) $V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} 1 + \sin 2x \, dx$

$= \pi \left[x - \frac{1}{2}\cos 2x \right]_0^{\frac{\pi}{2}}$

$= \pi \left[\frac{\pi}{2} - \frac{1}{2}\cos\pi - \frac{1}{2}\cos 0 \right]$

$= \pi \left[\frac{\pi}{2} + \frac{1}{2} - \frac{1}{2} \right]$ 3·5

$= \frac{\pi^2}{2} \; u^3$

b) $\log_a b = 2.5$ $\log_a c = 0.5$

i. $\log_a \frac{b}{c}$

$= \log_a b - \log_a c$

$= 2.5 - 0.5$

$= 2$

ii $\log_a c^2$

$= 2 \log_a c$

$= 2 \times 0.5$

$= 1.$

iii. $\log_a b = 2.5$ $\log_a c = 0.5$

$\therefore b = a^{2.5}$ $c = a^{0.5}$

$= a^{0.5} \times a^{0.5} \times a^{0.5} \times a^{0.5} \times a^{0.5}$

$= c \times c \times c \times c \times c$

$= c^5$

$b = c^5$

Office Use Only – Do NOT write anything, or make any marks, below this line.

Start here

c)

i, $y = x^2 e^{-x}$

To show:

$y = x^2 e^{-x}$

$\frac{dy}{dx} = uv' + vu'$

$= x^2 \times -e^{-x} + e^{-x} \times 2x$

$= -x^2 e^{-x} + 2x e^{-x}$

$= e^{-x}(2x - x^2)$

ii. $y' = e^{-x}(2x - x^2)$

Let $y' = 0$

$e^{-x}(2x - x^2) = 0$

$e^{-x} \neq 0$

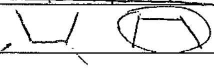
$\therefore 2x - x^2 = 0$

$x(2 - x) = 0$

$x = 0, 2$

$x = 2$ is a max

-1	0	1	2	3
-3e	0	$\frac{1}{e}$	0	$-3e^{-3}$



$y = x^2 \times e^{-x}$ 1·5

Let $x = 2$

$= 4 \times e^{-2}$

$= \frac{4}{e^2}$ $\frac{4}{e^2}$ is max ht $\approx .541$ km

$= 1$ km is the max $= 541$ m.

$\therefore$ 1000m is the max height of the hill.

Office Use Only – Do NOT write anything, or make any marks, below this line.

Start here

iii. the slope is the steepest the closest the ~~\$~~ gradient is to 0.

Is there a p.oi ?

$y' = e^{-x} \times 2x - e^{-x}x^2$

$y'' = uv' + vu' - (uv' + vu')$

$= (e^{-x} \times 2 + 2x \times -e^{-x}) - (e^{-x} \times 2x + x^2 \times -e^{-x})$

$= (2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2e^{-x})$

$= 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$

$= 2e^{-x} - 4xe^{-x} + x^2e^{-x}$

$= e^{-x}(2 - 4x + x^2)$

$y'' = 0$

$x^2 - 4x + 2 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{4 \pm \sqrt{16 - 8}}{2}$

$= \frac{4 \pm \sqrt{8}}{2}$

$= \frac{4 \pm 2\sqrt{2}}{2}$

$= 2 \pm \sqrt{2}$

2.64	$2+\sqrt{2}$	3.64
\	—	

$e^{-2.64}(2 - 10.56 + 6.9)$

$e^{-2.64}(-15.529)$

$2 + \sqrt{2} \approx 3.14$ 3.41

3.14 is a p.oi

$e^{-3.64}(2 - 14.56 + 13.2)$

$e^{-3.64}(-0.689)$

∴ the steepest slope on the hill is at approx. 3.14.

2.5