Set By: EH

Teachers:

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KNOX GRAMMAR SCHOOL
MATHEMATICS FACULTY

2005
TRIAL HSC EXAMINATION

Mathematics Extension 2 (Year 12)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks (120)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question
- Write your **Board of Studies Student Number and Class Teacher's Initials** on the front cover of each of your writing booklets

Board of Studies Student Number: ________________

Class Teacher's Initials: ________________

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find $\int xe^{-2x}\ dx$. **2**

(b) Find $\int \sin^3 x\ dx$. **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{d\theta}{4+2\sin 2\theta}$, using the substitution, $t=\tan\theta$. **4**

(d) (i) Find real numbers A, B and C such that: **3**

$$\frac{A}{x-2}+\frac{Bx+C}{x^2+4}\equiv\frac{1}{(x-2)(x^2+4)}$$

(ii) Evaluate $\int_{-2}^{0}\frac{8}{(x-2)(x^2+4)}dx$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z_1=2+3i$ and $z_2=4-5i$, find $z_1\times\overline{z}_2$, in the form $x+yi$. **2**

(b) A unit circle has its centre at the origin O. The point z_1 moves on this circle and $z_2=\frac{\sqrt{2}-3i}{z_1}$.

(i) Calculate $|z_1z_2|$. **1**

(ii) Hence find the Cartesian equation of the locus of z_2. **2**

(c) Sketch the locus of z in the Argand plane such that:

(i) $|z+2|=|z-3i|$ **1**

(ii) $\arg(z-i)=\frac{3\pi}{4}$ **2**

(d) (i) Determine the Cartesian equation of the locus of z, such that: **2**

$$\text{Re}\left(z-\frac{1}{z}\right)=0$$

(ii) Hence or otherwise, sketch the locus of z, in the Argand plane, if $\text{Re}\left(z-\frac{1}{z}\right)=0$. **2**

Question 2 continues on the next page

Question 2 continued:

(e)

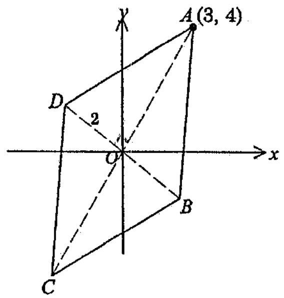


Diagram is not to scale

In the diagram, $ABCD$ is a rhombus whose diagonals meet at O, the origin. A represents the complex number $3 + 4i$ and $OD = 2$ units.

Find the complex number represented by:

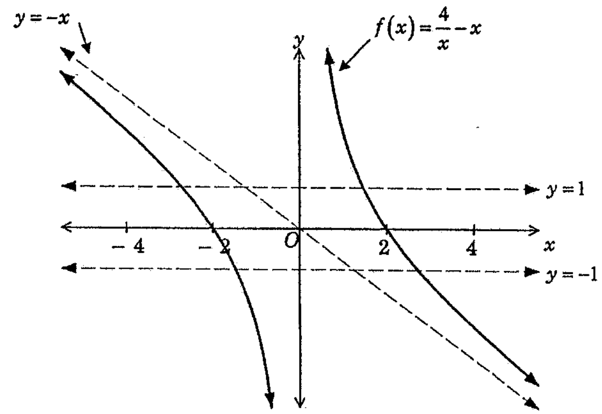
(i) C **1**

(ii) D **2**

Question 3 commences on the next page

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



For this part of Question 3, your answers are to be superimposed on the appropriate sketches on the separate answer sheet and then handed in with your writing booklet for this question.

The diagram above shows the graph of $y = f(x)$, where $f(x) = \frac{4}{x} - x$.

Sketch on separate number planes the graphs of:

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y = f(|x|)$ **2**

(v) $y = e^{f(x)}$ **2**

(b) Evaluate $\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2 - 4}}{x}\,dx$, using the substitution $x = \frac{2}{5}\sec\theta$. **5**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The equation $x^3-3x+2=0$ has roots α, β and γ.

(i) Form the cubic polynomial equation with roots: **1**

$$\frac{1}{\alpha}, \frac{1}{\beta} \text{ and } \frac{1}{\gamma}$$

(ii) Form the cubic polynomial equation with roots: **3**

$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma} \text{ and } \frac{\gamma}{\alpha\beta}$$

(b) (i) Prove that if a polynomial $P(x)$ has a root α of multiplicity r then $P'(x)$ has a root α of multiplicity $(r-1)$. **2**

(Hint: Start with $P(x)=(x-\alpha)^r Q(x)$)

(ii) Given $x=1$ is a double root of the equation $x^4-5x^3+16x^2-21x+9=0$, and using the result of b(i), or otherwise, find the other roots. **3**

(c) Given $z=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$ and $w=\sqrt{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

(i) Express $\frac{w}{z}$ in modulus argument form. **1**

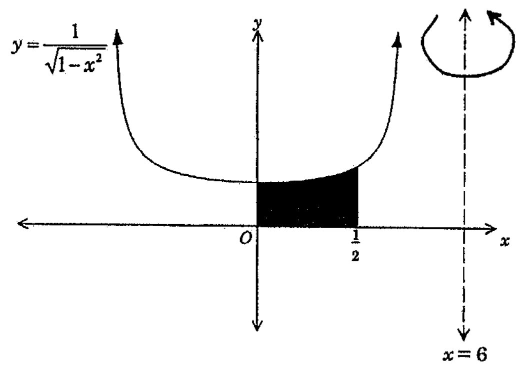
(ii) Use Mathematical Induction to prove for positive integers n that: **3**

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

(iii) Hence, using the result in c(ii), find the value of $\left(\frac{w}{z}\right)^{12}$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The shaded region bounded by the curve $y=\frac{1}{\sqrt{1-x^2}}$, the coordinate axes and the line $x=\frac{1}{2}$ is rotated through one complete revolution about the line $x=6$. **4**

Use the method of cylindrical shells to find the volume of the solid of revolution formed in cubic units.

(b) Let $I_n=\int_0^{\frac{\pi}{2}} x^n \cos x \, dx$.

(i) Show that if n is a positive integer greater than one, then: **2**

$$I_n=\left(\frac{\pi}{2}\right)^n-n(n-1)I_{n-2}$$

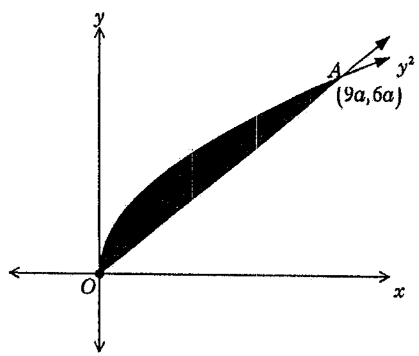
(ii) Evaluate I_4. **2**

Question 5 continues on the next page

Question 5 continued:

(c)

Diagram is not to scale



The base of a certain solid is the area bounded by the parabola $y^2 = 4ax$ (for $y \geq 0, a \geq 0$) and the chord joining $(0,0)$ and $A(9a, 6a)$. Cross-sections of this solid, determined by planes taken perpendicular to the x-axis, are semicircles with the diameter completely in the base of the solid. **4**

By using the method of slicing, find the total volume of the solid formed.

(d) (i) Sketch on the same number plane the graphs of $y = |x| - 2$ and $y = 4 + 3x - x^2$. **1**

(ii) Hence or otherwise solve $\dfrac{|x| - 2}{4 + 3x - x^2} > 0$. **2**

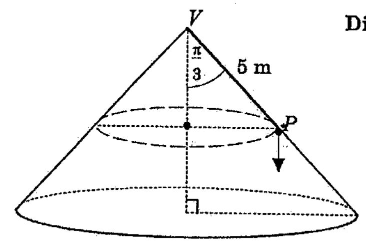
Question 6 commences on the next page

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Solve for x if $\cos x = \sin\left(\dfrac{x}{2}\right)$. **3**

(b)

Diagram is not to scale



A circular cone of semi-vertical angle $\dfrac{\pi}{3}$ is fixed with its vertex upwards as shown. A particle P of mass m kg is attached to the vertex at V by a light inextensible string of length 5 metres. The particle P rotates with uniform angular velocity ω rad/sec in a horizontal circle whose centre is vertically below V, on the outside surface of the cone and in contact with it. Let T be the tension in the string and N the normal reaction force at P.

(i) Draw a diagram showing all the forces acting on the particle. **1**

(ii) Find the tension T in the string and the normal force N on P in Newtons. Leave your answers in terms of m, g and ω. **3**

(iii) Show that for the particle to remain in uniform circular motion on the surface of the cone, then $\omega^2 < \dfrac{2g}{5}$, where g is the acceleration due to gravity. **2**

Question 6 continues on the next page

Question 6 continued:

(c)

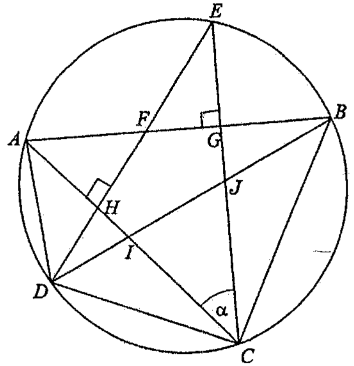


Diagram is not to scale

For this part of Question 6, your answers are to be placed on the answer sheet supplied and then handed in with your writing booklet for this question.

In the diagram, EC and ED are perpendicular to BA and AC at G and H respectively. The chords AC and BD meet at I. Let $\angle ECA = \alpha$.

(i) Prove that ΔBCD is isosceles. **2**

(ii) Prove that $\Delta CID \,|||\, \Delta CDA$. **2**

(iii) Given that $\Delta CIB \,|||\, \Delta CBA$ and $AB + AD = 2BC$. **2**

Prove that $CI = \frac{BD}{2}$.

Question 7 commences on the next page

Question 7 (15 marks) Use a SEPARATE writing booklet

Marks

(a) A projectile of unit mass is moving through air and experiences a resistance force R proportional to the square of its speed v. That is, $R = kv^2$, where k is a positive constant. *In this question, regard the direction of motion as positive.*

(i) Suppose the projectile is fired vertically upwards from the ground with an initial speed of u metres per second. Prove that the maximum height H reached by the projectile, where g is the acceleration due to gravity, is given by: **3**

$$H = \frac{1}{2k}\log_e\left(1 + \frac{ku^2}{g}\right)$$

(ii) Prove that the time T taken to reach this maximum height is given by: **2**

$$T = \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)$$

(b) (i) Differentiate with respect to x the function $h(x) = \frac{\log_{10} x}{x}$, **2**

(ii) Given that the only stationary point of $h(x)$ is a maximum, deduce, without calculating any numerical values, $e^{\pi} > \pi^{e}$. **3**

Question 7 continues on the next page

Question 7 continued:

(c) (i) A vehicle of mass m (in kg) is moving with speed v (in m/s) around a curve of radius r (in metres) banked at angle α with the horizontal. The normal reaction between the road and the vehicle is N, the friction (taken to be up the slope) is F_r, and the acceleration due to gravity is g (in m/s²). **2**

Draw a diagram that represents the forces on the vehicle.

By resolving forces parallel and perpendicular to the road, show that:

$$F_r = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$$

(ii) A train is travelling around a curve of radius 3000 metres at a speed of 180 km/h. The width of the rails is 1.5 metres. **3**

Taking the acceleration due to gravity to be 9.8 m/s², find how much higher than the inner rail must the outer rail be, in order for lateral thrust (F_r) on the rails to be avoided? Give your answer to the nearest centimetre.

Question 8 commences on the next page

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find all the roots of $z^5 - 1 = 0$ and then show that these roots can be represented as $1, \omega, \omega^2, \omega^3$ and ω^4 where $0 < \arg\omega < \frac{\pi}{2}$. **4**

(b) Prove that $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$. **2**

(c) Show that $(1-\omega)(1-\omega^4) = 2 - 2\cos\frac{2\pi}{5}$. **2**

(d) Hence or otherwise, show that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$. **3**

(e) Suppose P_0, P_1, P_2, P_3, P_4 are the corresponding points of $1, \omega, \omega^2, \omega^3$ and ω^4 in the Argand plane.

(i) Show that $\left|\overline{P_0P_1}\right| = 2\sin\frac{\pi}{5}$. **2**

(ii) Hence, or otherwise deduce that $\left|\overline{P_0P_1}\right| \times \left|\overline{P_0P_2}\right| \times \left|\overline{P_0P_3}\right| \times \left|\overline{P_0P_4}\right| = 5$. **2**

End of Paper

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1.(a) $\int x e^{-2x}\, dx$

$u = x \quad v = -\frac{1}{2}e^{-2x}$

$u' = 1 \quad v' = e^{-2x}$

$I = \frac{-xe^{-2x}}{2} + \left[\frac{e^{-2x}}{2}\right]$ ✓

$= \frac{e^{-2x}}{2}[1 - x]$ ✓

(b) $\int \sin^3 x \, dx$

$= \int \sin x(1 - \cos^2 x)\, dx$

Let $u = \cos x$

$du = -\sin x \, dx$

$= -\int (1 - u^2)\, du$ ✓

$= -\left[u - \frac{u^3}{3}\right] + c$

$= \frac{\cos^3 x}{3} - \cos x + c$ ✓

(c). $\int_0^{\pi/4} \frac{d\theta}{4 + 2\sin 2\theta}$

Let $t = \tan\theta$

$dt = \sec^2\theta \, d\theta$

$dt = (1 + t^2)\, d\theta$ ✓

$\frac{dt}{1+t^2} = d\theta$ At $\theta = \pi/4$, $t = 1$

$\sin 2\theta = \frac{2t}{1+t^2}$ $\theta = 0$, $t = 0$.

$I = \int_0^1 \frac{1}{\frac{4 + 4t^2 + 4t}{1 + t^2}} \cdot \frac{dt}{1 + t^2}$

$= \int_0^1 \frac{dt}{4t^2 + 4t + 4}$

$= \frac{1}{4}\int_0^1 \frac{dt}{t^2 + t + 1}$

$= \frac{1}{4}\int_0^1 \frac{dt}{(t + 1/2)^2 + 3/4}$

$= \frac{2}{4\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)$

$= \left[\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$ ✓

$= \frac{1}{2\sqrt{3}}\left[\tan^{-1}(3/\sqrt{3}) - \tan^{-1}(1/\sqrt{3})\right]$

$= \frac{1}{2\sqrt{3}}\left[\pi/3 - \pi/6\right]$

$= \frac{1}{2\sqrt{3}} \times \frac{\pi}{6}$

$= \frac{\pi}{12\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ ✓

$= \frac{\pi\sqrt{3}}{36}$

(d) (i) $\frac{A}{x-2} + \frac{Bx + C}{(x^2+4)} = \frac{1}{(x-2)(x^2+4)}$

$A(x^2 + 4) + (Bx + C)(x - 2) = 1$

Let $x = 2$, $8A = 1$

$A = 1/8$ ✓

Compare x^2: $A + B = 0$. Compare constants: $4A - 2C = 1$

$1/8 + B = 0$. $1/2 - 2C = 1$

$B = -1/8$ ✓ $-2C = 1/2$

$C = \frac{-1}{4}$ ✓

(ii) $\int_{-2}^{0} \frac{8}{(x-2)(x^2+4)}\, dx$

$= 8\int \frac{1/8}{(x-2)} + \frac{(-1/8 x - 1/4)}{(x^2+4)}$

$= \int_{-2}^{0} \frac{dx}{(x-2)} - \int_{-2}^{0} \frac{x + 2}{x^2 + 4}\, dx$

$= \left[\ln(x-2)\right]_{-2}^{0} - \frac{1}{2}\left[\ln(x^2+4)\right]_{-2}^{0} - \left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{0}$

$= \ln(-2) - \ln(-4) - \frac{1}{2}\left[\ln(4) - \ln(8)\right] - \left[\tan^{-1} 0 - \tan^{-1}(1)\right]$

$= \ln(-2) - \ln(-4) + \ln 2 - \ln(2\sqrt{2}) - \pi/4$

$= \ln\left(\frac{-2 \times 2}{-4 \times 2\sqrt{2}}\right) - \frac{\pi}{4}$ ✓

$= \ln\left(\frac{1}{2\sqrt{2}}\right) - \pi/4$

$= -\ln(2\sqrt{2}) - \pi/4$

2.(a). $z_1 = 2 + 3i$

$z_2 = 4 - 5i$

$z_1 \times \overline{z_2} = (2 + 3i)(4 + 5i)$

$= -7 + 22i$ ✓

(b). $|z_1| = 1$.

$z_2 = \sqrt{2} - 3i$

$1 \quad z_1$

$|z_2| = \frac{|\sqrt{2} - 3i|}{|z_1|}$

$= \frac{\sqrt{11}}{|z_1|}$

$\therefore |z_1 z_2| = |z_1||z_2|$

$= \frac{\sqrt{11}}{|z_1|} \times |z_1|$

$= \sqrt{11}$

(ii). $z_1 z_2 = \sqrt{2} - 3i$

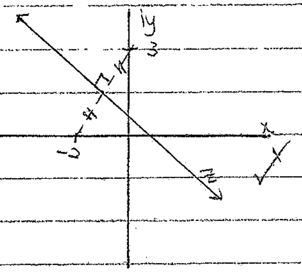
$|z_1| = 1$

$|z_2| = \sqrt{11}$ ⟹ $x^2 + y^2 = 11$ circle!

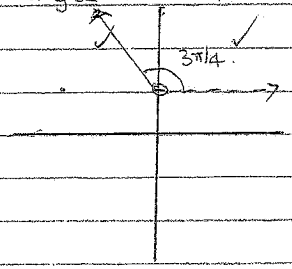
part of the locus. in cartesian form.

arg(z_2) ~~$\sqrt{2}-3i$~~

(c)(i) $|z+2| = |z-3i|$



(ii) $\arg(z-i) = \frac{3\pi}{4}$



(a)(i) $Re\left(z - \frac{1}{z}\right) = 0.$

$Re\left(z - \frac{1}{z}\right) = 0.$

$Re\left(\frac{z^2-1}{z}\right) = 0.$

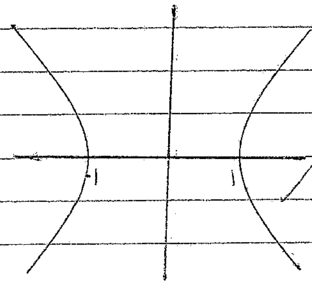
$x + iy = z$

$\therefore Re\frac{(x^2+2ixy-y^2-1)}{(x+iy)} = 0.$

$\therefore \frac{x^2-y^2-1}{(x+iy)} = 0.$

$\therefore \underline{x^2-y^2 = 1.}$ $x \neq 0.$

(ii).



2(a)(i) $A(3+4i)$

(i) $\therefore \underline{c = -3-4i}$

$D = \cancel{5}(\text{cis}\,\pi/2 \times (\text{cis}\,53°7')) \times \frac{2}{5}$

$= 2i(\cos 53°7' + i\sin 53°7')$

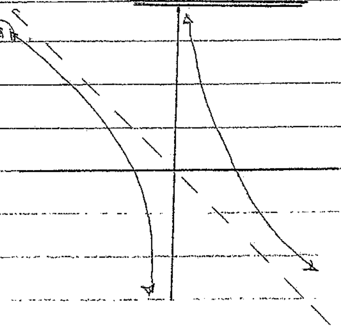
$= 2i\cos 53°7' - 2\sin 53°7'$

$= \frac{2i \times 3}{5} - \frac{8}{5}.$

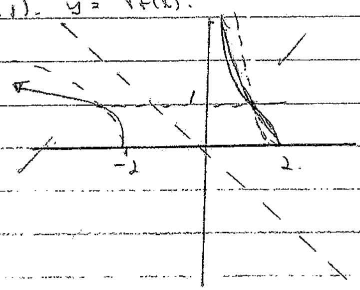
$= -\frac{8}{5} + i\frac{6}{5}$

$= \underline{\frac{1}{5}(-8+i6)}$

3(a)

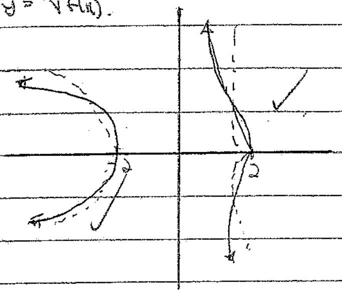


(i). $y = \sqrt{f(x)}$.

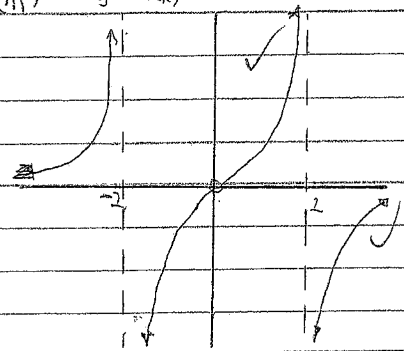


(ii) $y^2 = f(x)$.

$y = \pm\sqrt{f(x)}$.



(iii) $y = \frac{1}{f(x)}$

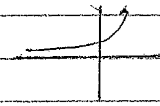


$\frac{4}{x} - x$

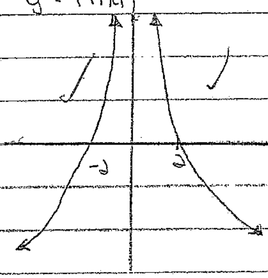
$\frac{4-x^2}{x}$.

$\frac{x}{4-x^2}$

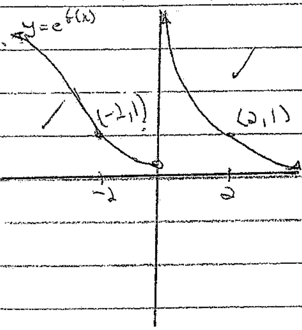
$= \frac{x}{4} - x$



(iv) $y = f(|x|)$



(v). $y = e^{f(x)}$



(b). $\int_{-4/5}^{-2/5} \frac{\sqrt{25x^2-4}}{x}\,dx.$

Let $x = 2/5\sec\theta$ At $x = -2/5$, $\theta = \pi$

$dx = 2/5\sec\theta\tan\theta\,d\theta$ $x = -4/5$, $\theta = \frac{2\pi}{3}$.

$I = \int_{2\pi/3}^{\pi} \frac{\sqrt{4\sec^2\theta - 4}}{\cancel{2/5\sec\theta}} \cdot \cancel{2/5\sec\theta}\tan\theta\,.d\theta$

$= 2\int_{2\pi/3}^{\pi} \tan^2\theta\,.d\theta$

$= 2\int_{2\pi/3}^{\pi} \sec^2\theta - 1\,.d\theta$

$= 2\left[\tan\theta - \theta\right]_{2\pi/3}^{\pi}$

$= 2\left[\sqrt{3} - \pi/3\right]$

$= \underline{2\sqrt{3} - \frac{2\pi}{3}}$

4.(a). $x^3 - 3x + 2 = 0.$

$\frac{1}{\alpha}$ ie $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}.$

$\frac{1}{y^3} - \frac{3}{y} + 2 = 0.$

$1 - 3y^2 + 2y^3 = 0.$

$\therefore \underline{x^3 - 3x^2 + 1 = 0.}$

i) $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$.

$= \frac{\alpha^2}{\alpha\beta\gamma}$

$y = \frac{x^2}{-2}$.

$-2y = x^2$

$\sqrt{-2y} = x$.

$P(\sqrt{-2y}) = \quad -2y\sqrt{-2y} \quad -3\sqrt{-2y} \quad +2 = 0$.

$-\sqrt{-2y}(-2y-3) \quad = -2$.

$-2y(-2y-3)^2 \quad = 4$.

$\cancel{-2y(4y^2+12y)}$

$\cancel{2y(} \quad -2y(4y^2+12y+9) = 4$.

$-8y^3 - 24y^2 - 18y = 4$

$8y^3 + 24y^2 + 18y - 4 = 0$.

$8x^3 + 24x^2 + 18x - 4 = 0$.

(b)(i) $P(x) = (x-\alpha)^r Q(x) \qquad \therefore P(\alpha) = 0$.

$u = (x-\alpha)^r \qquad V = Q(x)$

$u' = r(x-\alpha)^{r-1} \qquad V = Q'(x)$

$P'(x) = Q'(x)(x-\alpha)^r + rQ(x)(x-\alpha)^{r-1}$

$= (x-\alpha)^{r-1}[Q'(x)(x-\alpha) + rQ(x)]$

$P'(\alpha) = 0$.

$\therefore$ $P'(x)$ also a root.

(ii) $x^4 - 5x^3 + 16x^2 - 21x + 9 = 0$.

$4x^3 - 15x^2 + 32x - 21 = 0$.

$P'(1) = 0. \qquad \therefore 4 - 15 + 32 - 21 = 0$.

$$
\begin{array}{r}
4x^2 - 11x + 21 \\
(x-1)\overline{)\,4x^3 - 15x^2 + 32x - 21} \\
4x^3 - 4x^2 \\
-11x^2 + 32x \\
-11x^2 + 11x \\
21x - 21 \\
21x - 21 \\
0
\end{array}
$$

$(x-1)(4x^2 - 11x + 21)$

$\therefore \quad x = 1, \quad \frac{11 \pm \sqrt{121 - 336}}{8}$

$1, \quad \frac{11 \pm i\sqrt{215}}{8}$

(c). $z = \text{cis}\frac{2\pi}{3}$.

$w = \sqrt{3}\,\text{cis}\,\pi/3$

(i) $\frac{z}{w} = \frac{1}{\sqrt{3}}\,\text{cis}\,\pi/3$

(ii) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Let $n = 1$

LHS $= \cos\theta + i\sin\theta \qquad$ RHS $= \cos\theta + i\sin\theta$

$=$ LHS.

15

Assume true for $S(k)$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Prove for $S(k+1) \quad = \quad \cos(k+1)\theta + i\sin(k+1)\theta$

LHS $= (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta - \sin\theta\sin k\theta$

$= \cos(k+1)\theta + \sin(k+1)\theta$

$=$ RHS.

If true for $n = k$, then shown for $k+1$, therefore by the principle of Mathematical Induction, true for $n \geq 1$.

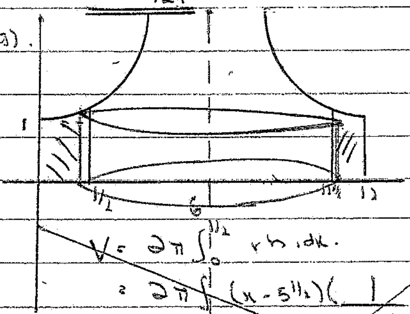
(iii) $\left(\frac{w}{z}\right)^{12}$

$= \left(\frac{1}{\sqrt{3}}\right)^{12} \text{cis}\,4\pi$

$= \frac{1}{729}\,\text{cis}\,0$

$= \frac{1}{729}$

5.(a).



$V = 2\pi \lim_{\delta x \to 0} \sum_{0}^{1/2} r\,h\,\delta x$.

$= 2\pi \int_0^{1/2} r\,h\,dx$.

$r = x_2 - x_1$

$x_2 - 5^{1/2}$.

$y = \frac{1}{\sqrt{1-x^2}}$

$y^2 \quad \frac{1}{y^2} = 1 - x^2$

$x^2 = 1 - \frac{1}{y^2}$

$= \frac{y^2 - 1}{y^2}$

$x = \frac{\sqrt{y^2-1}}{y}$

$V = 2\pi \int_0^{1/2} r\,h\,dx$.

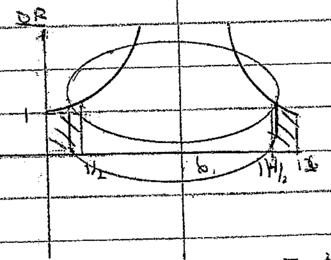
$= 2\pi \int (x - 5^{1/2})\left(\frac{1}{\sqrt{1-x^2}}\right) dx$.

$= 2\pi \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} - 2\cdot 11\pi \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

$= 2\pi\left[\sqrt{1-x^2}\right]_0^{1/2} - \left[11\pi\sin^{-1}x\right]_0^{1/2}$

$= \sqrt{3}\pi - 2\pi \quad \frac{11\pi^2}{6} \quad 0. \quad = \pi\left(\sqrt{3} - \frac{11\pi}{6}\right)$

OR



$$2\pi \int_0^{k} r\,h\;dx \quad \text{where } r = 6 - x$$
$$h = y$$
$$= 2\pi \int_0^{\frac{1}{2}} (6-x)\frac{1}{\sqrt{1-x^2}}\,dx \;\ldots\; \text{try again}$$

$\therefore \quad \pi r^2 h = \quad 36\pi \quad -25\pi \quad + 2\pi\int_0^{1/2} \left(x - 5\tfrac{1}{2}\right)\frac{1}{\sqrt{1-x^2}}\,dx$

$\cancel{\pi(\quad-\quad 6)^2}$ $\quad = 9\pi \quad + 2\pi\int_0^{1/2}\frac{x}{\sqrt{1-x^2}} - \frac{11}{2\sqrt{1-x^2}}\,dx$

$+\left[-2\pi\sqrt{1-x^2} - \frac{2\pi\cdot 11}{2}\sin^{-1}(x)\right]_0^{1/2}$

$= 9\pi + 2\pi\left(\frac{\sqrt{3}}{2} - 1\right) - 11\pi\left(\frac{\pi}{6}\right)$

$= 9\pi + \pi\sqrt{3} - 2\pi - \frac{11\pi^2}{6}$

$= 7\pi + \pi\sqrt{3} - \frac{11}{6}\pi^2$

(b)(i). $I_n = \int_0^{\pi/2} x^n \cos x\,dx$

$= \quad u = x^n \quad v = \sin x$

$u' = nx^{n-1} \quad v' = \cos x$

$I_n = \left[x^n \sin x\right]_0^{\pi/2} - n\int_0^{\pi/2} x^{n-1}\sin x\,dx$

$= [\pi/2]^n - \quad u = x^{n-1} \quad v = -\cos x$

$u' = (n-1)x^{n-2} \quad v' = \sin x$

$= [\pi/2]^n + n\left[x^{n-1}\cos x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2}\cos x \cdot x^{n-2}\,dx$

$= [\pi/2]^n - n(n-1)\,I_{n-2}$

$I_4 = [\pi/2]^4 - 12 I_2$

$I_2 = (\pi/2)^2 - 2I_0$

$I_0 = \int \cos x\,dx$

$= [\sin x]_0^{\pi/2}$

$= 1$

$I_2 = (\pi/2)^2 - 2.$

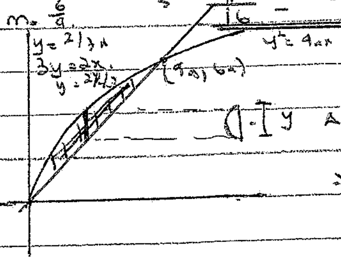
$I_4 = (\pi/2)^4 - 12(\pi/2)^2 + 24.$

$= \frac{\pi^4}{16} - 3\pi^2 + 24.$

$y = y_1 - y_2$

$y^2 = (y_1 - y_2)^2$

5 (a).



$V = \int_0^{9a} A(x)\,dx \quad = (4ax + \frac{4}{9}x^2 - 2(\frac{2}{3}x)(\sqrt{4ax})$

$A(x) = \pi(y)^2 = \pi y^2$

$= 4\pi\int_0^{9a} ax - \frac{x^2}{9}\,dx$

$y = (y_1)^2 - (y_2)^2$

$= 4\pi \left[\frac{ax^2}{2} - \frac{x^3}{27}\right]_0^{9a}$

$= 4\pi\left[\ldots\right]$

$(y_1^2 - y_2^2)\pi \quad = \left(4ax - \frac{4x^2}{9}\right)\pi$

$V = \int_0^{9a} A(x)\,dx$

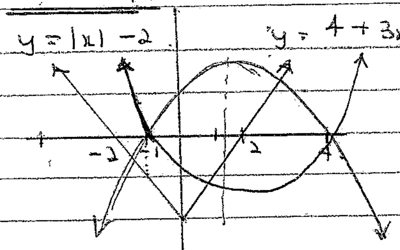
$= 4\pi\int_0^{9a} ax - \frac{x^2}{9}\,dx$

$= 4\pi\left[\frac{ax^2}{2} - \frac{x^3}{27}\right]_0^{9a}$

$= 4\pi\left[\frac{81a^3}{2} - 27a^3\right]$

$= 54a\pi$

(d). $y = |x| - 2 \qquad y = 4 + 3x - x^2$ — concave down



$x = \frac{3}{2}$

For $\frac{|x| - 2}{4 + 3x - x^2} > 0$

then $|x| - 2 > 0 \;\cap\; 4 + 3x - x^2 > 0$

$\boxed{2 < x < 4}$

(i). $x < -2 \cup -1 < x < 2 \cup x > 4.$ (from graph).

6. (a). $\cos x = \sin(x/2).$

$x = \pi/3$ (By Trial & error).

OR $\cos x = \sin(x/2)$

$\sin^{-1}(\cos x) = x/2$

Let $\cos x = \alpha$

$\sin^{-1}\alpha = x/2.$

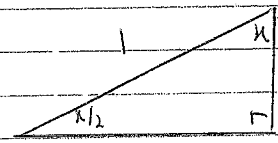
$\sin x/2 = \alpha.$

$\therefore x + x/2 = \pi/2$ (Angle sum)

$2x + x = \pi$

$3x = \pi$

$x = \pi/3.$



$1 - 2\sin^2 x/2 = \sin x/2$

$2\sin^2 x/2 + \sin x/2 - 1 = 0.$

$\sin x/2 = \frac{-1 \pm \sqrt{1+8}}{4}$

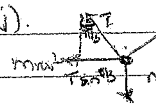
$= \frac{-1 \pm 3}{4}$

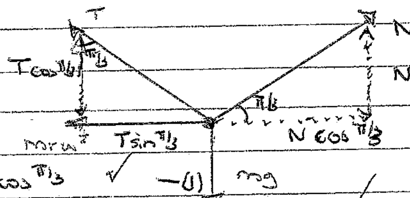
$\alpha = -1, \; 1/2.$

$\therefore x/2 = -\pi/2, \; \pi/6.$

$x = -\pi, \; \pi/3.$

(b)(i).





(ii)

$mrw^2 = T\sin\pi/3 = N\cos\pi/3 \quad -(1)$

$mg = T\cos\pi/3 + N\sin\pi/3. \quad -(2)$

$N\cos\pi/3 = T\sin\pi/3 - mrw^2 \quad -(1)$

$N\sin\pi/3 = mg - T\cos\pi/3. \quad -(2)$

$\cancel{\tan\pi/3} = \frac{T\sin\pi/3 - mrw^2}{mg - T\cos\pi/3}.$

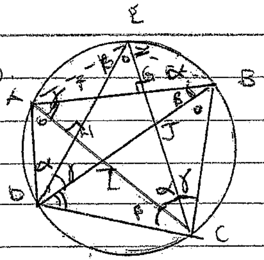
Sec Back

$\sqrt{3} = \dfrac{\frac{\sqrt{3}}{2}T - mr\omega^2}{mg - \frac{T}{2}}$ but $r = 5\sin\frac{\pi}{3}$

$= \frac{5\sqrt{3}}{2}$

$(mg - \frac{T}{2})\sqrt{3} = \frac{\sqrt{3}}{2}T - \frac{5\sqrt{3}}{2}m\omega^2$ ... Find T... try again

6(c).



(i). $E\hat{B}A = E\hat{D}A = \alpha$ (angles on the same arc EA)

$E\hat{D}B = E\hat{C}B = E\hat{A}B = \gamma$ (" " EB)

$D\hat{E}C = D\hat{A}C = D\hat{B}C = \circ$ (" " DC)

$A\hat{B}C = B\hat{D}C = C\hat{E}B =))$ (" " BC)

$\therefore$ In cyclic quad DACB, $\circ +)) = \alpha + \beta + \gamma$

In " " " , $\alpha + \gamma +)) = \beta + \circ$

$\therefore \circ =))$ $\therefore$ In Δ DBC

$D\hat{B}C = B\hat{D}C$ $\therefore \Delta DBC$ is isosceles.

(ii) In Δ's CID & CDA

$A\hat{B}D$ is common.

$D\hat{B}D = B\hat{D}A$ (proven above)

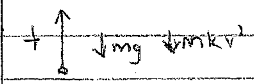
$\therefore$ and $D\hat{B}C = D\hat{A}C$

$\therefore B\hat{D}C = D\hat{A}C$

$\therefore \Delta CID \,|||\, \Delta CDA$ (equiangular).

(iii).

7.(a)



$\therefore m\ddot{x} = -mg - mkv^2$

$\ddot{x} = -g - kv^2$

$v\frac{dv}{dx} = -(g + kv^2)$

$\int \frac{dx}{dv} = \int \frac{-v}{g + kv^2} \cdot dv$

$x = \frac{-1}{2k}\int \frac{2kv}{g + kv^2} \cdot dv$

$x = \frac{-1}{2k}\ln|g + kv^2| + c.$

At $x = 0, v = u.$ $\therefore c = \frac{1}{2k}\ln|g + ku^2|$

$\therefore x = \frac{1}{2k}\ln\left|\frac{g + ku^2}{g + kv^2}\right|$

At $v = 0$, $x = H$

$\underline{\underline{\frac{1}{2k}\ln\left|1 + \frac{ku^2}{g}\right|}} \Longleftarrow \therefore H = \frac{1}{2k}\ln\left|\frac{g + ku^2}{g}\right|$

(ii) $\frac{dv}{dt} = -(g + kv^2)$

$\int \frac{dt}{dv} = \int \frac{-1}{g + kv^2} \cdot dv \Longrightarrow \frac{-1}{k}\int \frac{dv}{g/k + v^2}$

$t = \frac{-\sqrt{k}}{\sqrt{g}\,k}\tan^{-1}\left(\frac{v\sqrt{k}}{\sqrt{g}}\right) + c$

$= \frac{-1}{\sqrt{gk}}\tan^{-1}\left(\frac{v\sqrt{k}}{\sqrt{g}}\right) + c$

At $v = u, t = 0$

$\therefore c = \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)$

$\therefore t = \frac{-1}{\sqrt{gk}}\tan^{-1}\left(\frac{v\sqrt{k}}{\sqrt{g}}\right) + \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)$

$\therefore$ At $v = 0$, $t = T$

$\underline{\underline{T = \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)}}$

(b)(i). $h(x) = \frac{\log_{10} x}{x}.$

$= \frac{\ln x}{x\ln 10}$ (change of base)

$u = \ln x$ $\quad v = x\ln 10.$

$u' = \frac{1}{x}$ $\quad v' = \ln 10.$

$h'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{\ln 10 - \ln x \ln 10}{x^2(\ln 10)^2}$

$= \underline{\underline{\frac{1 - \ln x}{x^2(\ln 10)}}}$

(ii). $1 - \ln x = 0.$

$1 = \ln x.$

$x = e.$ $\therefore$ Max at (e, m)

$\therefore \cancel{h(e) < 1}$

$\cancel{\frac{\log_{10} e}{e} < 1}$

$h(e) = \frac{\log_{10} e}{e}$

$h(x) < h(e)$

$\therefore \frac{\log_{10} x}{x} < \frac{\log_{10} e}{e}$ since D: $x > 0$

$e\log_{10} x < x\log_{10} e$

$e\frac{\ln x}{\cancel{\ln 10}} < x\frac{\ln e}{\cancel{\ln 10}}$

$\frac{x}{e} > \ln x \Longleftarrow$

$e\ln x < x\ln e$

~~For $x = \pi$~~

$\frac{\pi}{e} > \ln\pi$

* $\pi > e\ln\pi$

$\pi > \ln\pi^e$

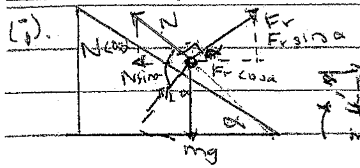
$e^\pi > \pi^e$ as req'd.

$\ln x^e < x$

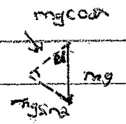
$\Leftarrow x^e < e^x$

$\underline{\underline{\pi^e < e^\pi}}$

(i).



$\cancel{mr\omega^2}$ ~~$m r\omega^2 \sin\alpha$~~ $m r\omega^2 \cos\alpha$



$\therefore$ ~~$E\cos\alpha$~~ $\quad \therefore$ ~~F~~ $\quad mr\omega^2 \cos\alpha = mg\sin\alpha + Fr.$

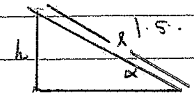
$\therefore Fr = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$

(ii). $r = 3000m = 3km$

$v = 180\ km/h$

$= 50\ m/s.$

$l = 1.5$



$Fr < mg\sin\alpha - \frac{mv^2}{r}\cos\alpha \Rightarrow$ for $Fr = 0 \Rightarrow \tan\alpha = \frac{v^2}{gr}$

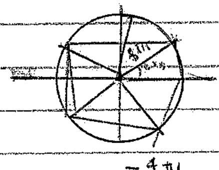
$= \frac{50^2}{9.8 \times 3000}$

$\sin\alpha = \frac{h}{1.5}$

$h = 1.5 \times \sin\alpha$

$= 0.13$ (to 2dp)

6.(a). $z^5 - 1 = 0.$



$z^5 = 1.$

$z = cis\left(\frac{2k\pi}{5}\right)$

$z_1 = cis\, 2\pi/5,\ z_2 = cis\, 4\pi/5,\ z_3 = cis\, -4\pi/5,\ z_4 = cis\, \frac{-2\pi}{5},\ z_5 = 1.$

$\therefore z_5 = 1,\ z_1 = w,\ z_2 = w_2,\ z_3 = \bar{w}_2,\ z_4 = \bar{w}_1.$

(b) $(1-w)(1-w^2)(1-w^3)(1-w^4).$

$= (1 - w^2 - w + w^3)(1 - w^4 - w^3 + w^7)$

$= 1 - w^4 - w^3 + w^7 - w^2 + w^6 + w^5 - w^9 - w + w^5 + w^4 - w^8$

$+ w^2 - w^7 - w^6 + w^{11}$

$= 1 - (w + w^2 + w^3 + w^4) + 1 + 1 - w^4.$

$= 1 + 1 + 1 - (-1) - w^4.$

$= 4 - w^4.$

$= 1 + w^2 - w^3 - w^4 - w^2 - w^4 + 1 + w - w - w^2 + w^3 + 1 + w^3 - w^2 - w + 1$

$= 4 - w^2 - w^3 - w^4 - w$

$= 4 - (-1)$

$= 5.$

(c). $(1-w)(1-w^4)$

$= 1 - w^4 - w + w^5$ $\qquad w = \bar{w}_4.$

$= 2 - w^4 - w$

$= 2 - (w + w^4)$

$= 2 - 2\cos\frac{2\pi}{5}.$

(d). $(1-w^2)(1-w^3)$

$= 1 - w^3 - w^2 + w^5.$

$= 2 - (w^2 + w^3)$

$= 2 - 2\cos\pi/5.$

$\therefore (1-w^2)(1-w^3)(1-w)(1-w^4) = (2 - 2\cos\pi/5)(2 - 2\cos 2\pi/5)$

$5 = \cancel{4} - (4\cos 2\pi/5 + 4\cos\pi/5) + 4\cos\pi/5\cos\frac{2\pi}{5}$

$5 = 4 - 4(\cos\frac{2\pi}{5} + \cos\pi/5) + 2(\cos\frac{3\pi}{5} + \cos\pi/5)$

$1 = 2\cos\frac{3\pi}{5} - 2\cos\frac{2\pi}{5} - 2\cos\pi/5.$

$1 = 2[2\sin\frac{2\pi}{5}\sin\pi/5] - 2\cos\pi/5.$

$*$?

(iii) $\sqrt{3}(mg - T\cos\pi/3) = T\sin\pi/3 - mr\omega^2$

$$\sqrt{3}\left(mg - \frac{T}{2}\right) = \frac{\sqrt{3}T}{2} - mr\omega^2$$

$$2\sqrt{3}mg - \sqrt{3}T = \sqrt{3}T - mr\omega^2$$

$$2\sqrt{3}mg + mr\omega^2 = \sqrt{3}T + T$$

$$= T(\sqrt{3}+1)$$

$$T = \frac{2\sqrt{3}mg + mr\omega^2}{\sqrt{3}+1}$$ but $r = 5\sin\frac{\pi}{3}$

$= \frac{5\sqrt{3}}{2}$

$$= \frac{2\sqrt{3}mg + \frac{5\sqrt{3}m\omega^2}{2}}{\sqrt{3}+1}$$

(?) $T/2 = mg - N\sin\pi/3.$

$$2\frac{\sqrt{3}mg + mr\omega^2}{\sqrt{3}+1} = 2mg - 2N\sin\pi/3.$$

$$N\sin\pi/3 = 2mg - \frac{2\sqrt{3}mg + mr\omega^2}{2(\sqrt{3}+1)}$$

$$N\frac{\sqrt{3}}{2} = 2mg - \frac{2\sqrt{3}mg + mr\omega^2}{2(\sqrt{3}+1)}$$

$$N = \frac{4mg}{\sqrt{3}} - \frac{mg}{(\sqrt{3}+1)} - \frac{mr\omega^2}{\sqrt{3}(\sqrt{3}+1)}$$

Then use either $T > 0$ or $N > 0$

to prove $\omega^2 < \frac{2g}{5}$