KRB
Maths Dept.

2002
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved scientific calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 38

- Attempt all questions 1-5
- Start a new page for each question

Total marks – 38
Attempt Questions 1-5

Answer each question on a new page.

Question 1 (9 marks)

a) The polynomial $P(x)$ is given by $P(x) = x^3 - 4x^2 + x + 6$.

i) Show that $x = 2$ is a zero of $P(x)$. 1

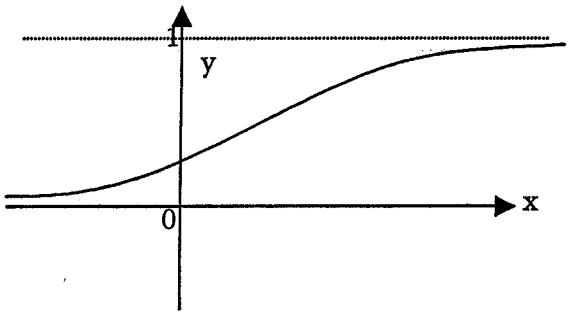
ii) Express $P(x)$ in factorised form and hence graph $P(x)$, indicating all intercepts on your graph 4

iii) Hence or otherwise solve $x^3 - 4x^2 \leq -x - 6$. 1

b) Use one application of Newton's method to estimate the root of $\tan^{-1}(x-1)+x=0$ near $x = 0.5$ (answer in radians correct to 2 decimal places). 3

Question 2 (6 marks) Start a new page.

The graph of $f(x) = \dfrac{e^x}{1+e^x}$ is sketched below.



The following information may be useful :
$as\ x \to \infty, f(x) \to 1$
$as\ x \to -\infty, f(x) \to 0$

i) Explain why $f(x)$ has an inverse function. 1
ii) State the domain and range of $f^{-1}(x)$. 2
iii) Find an expression for $f^{-1}(x)$ 3

Question 3 (8 marks) (Start a new page)

i) State the domain and range of $f(x) = 2\sin^{-1}\left(\frac{x}{2}\right)$. 2

ii) Sketch the graph of $f(x) = 2\sin^{-1}\left(\frac{x}{2}\right)$ indicating all important features on your graph. 2

iii) Show that the exact area bounded by the curve $f(x) = 2\sin^{-1}\left(\frac{x}{2}\right)$ and the y axis, from $x = 0$ to $x = 1$ is given by $-4\left(\frac{\sqrt{3}}{2} - 1\right)$ units^2. 4

Question 4 (7 marks) (Start a new page)

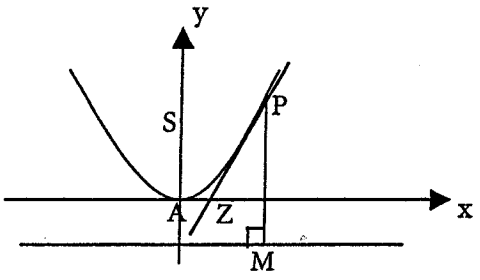
a) Find the exact value of $\cot\left(\sin^{-1}\left(\frac{5}{7}\right)\right)$. Show all working. 2

b) Differentiate $y = \left(\sin^{-1}\frac{x}{2}\right)^3$ 3

c) Find $\int \frac{dx}{\sqrt{9-4x^2}}$ 2

Question 5 (8 marks) (Start a new page)

Let $P(2at, at^2)$ be a variable point on the parabola $x^2 = 4ay$. Let the vertex of this parabola be $A(0, 0)$ and the focus be $S(0, a)$. The point M is the point on the directrix such that PM is perpendicular to the directrix. The tangents at P and A meet at Z.



i) Explain why $PS=PM$ 1

ii)Use calculus to show that the equation of the tangent at P is given by $y = tx - at^2$. 3

iii) Prove that $SZ^2 = AS.PS$ 4

KRB
Maths Dept.

Question 1

a) i) $P(x)=x^3-4x^2+x+6$

$P(2)=2^3-4(2^2)+2+6$

$=8-16+2+6$

$=0$

$\therefore P(2)=0, so\ x=2\ is\ a\ zero.$

ii)
$$x-2\overline{)x^3-4x^2+x+6}\quad x^2-2x-3$$

x^3-2x^2

$-2x^2+x$

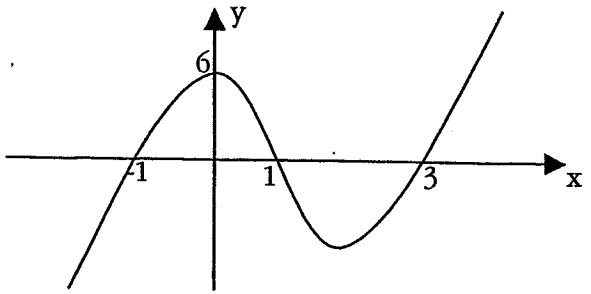
$-2x^2+4x$

$-3x+6$

$-3x+6$

$P(x)=(x-2)(x-3)(x+1)$

$P(0)=6$



iii) $2\le x\le 3\ or\ x\le -1$

b) $f'(x)=\dfrac{1}{(x-1)^2+1}+1$

$=\dfrac{1}{x^2-2x+2}+1$

$f'(0.5)=\dfrac{1}{0.5^2-1+2}+1$

$=1.8$

$f(0.5)=\tan^{-1}(-0.5)+0.5$

$=-\tan^{-1}(0.5)+0.5$

$=0.03635$

$x_1=0.5-\dfrac{0.03635}{1.8}$

$=0.4798$

$=0.48(2d.p)$

Question 2

a) f(x) has an inverse function since if a horizontal line is drawn, it cuts the curve at one point only. i.e. for each y value, only one x value exists.

b) domain : $0 < x < 1$
range : all real y

c) $x = \dfrac{e^y}{1+e^y}$

$x + xe^y = e^y$

$x = e^y(1-x)$

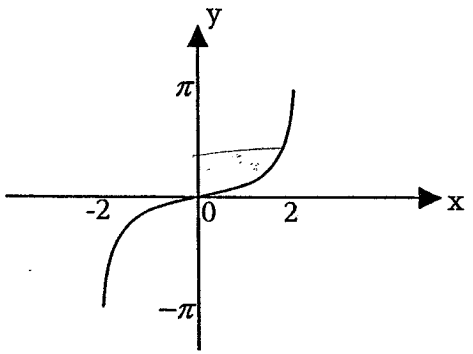
$e^y = \dfrac{x}{1-x}$

$y = \ln\left(\dfrac{x}{1-x}\right)$

Question 3

i) domain : all real x such that $-2 \le x \le 2$
range : all real y such that $-\pi \le y \le \pi$

ii)



iii) $x = 0, y = 0$

$x = 1, y = 2\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

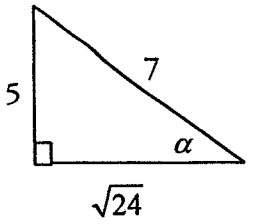
$$2\int_0^{\frac{\pi}{3}} \sin\left(\frac{y}{2}\right) dy = 2\left[2\times -\cos\left(\frac{y}{2}\right)\right]_0^{\frac{\pi}{3}}$$

$$= -4\left[\cos\frac{\pi}{6} - \cos 0\right]$$

$$= -4\left[\frac{\sqrt{3}}{2} - 1\right]$$

Question 4

a) let $\alpha = \sin^{-1}\frac{5}{7}$



so $\sin\alpha = \frac{5}{7}$

$\cot\alpha = \frac{\sqrt{24}}{5}$

$\therefore \cot\left(\sin^{-1}\frac{5}{7}\right) = \frac{2\sqrt{6}}{5}$

b) $\frac{dy}{dx} = 3\left(\sin^{-1}\frac{x}{2}\right)^2 \times \frac{1}{\sqrt{4-x^2}}$

$= \frac{3\left(\sin^{-1}\frac{x}{2}\right)^2}{\sqrt{4-x^2}}$

c) $\int \frac{dx}{\sqrt{4\left(\frac{9}{4}-x^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

Question 5

i) This is the locus of a parabola. i.e. the locus of a point that moves so that it is equidistant from a fixed point and a fixed line.

ii) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $x = 2at$, $\frac{dy}{sx} = t$

$y - at^2 = t(x - 2at)$

$y = tx - at^2$

iii) $AS = a$

$PS = PM = a + at^2$

$AS.PS = a^2 + a^2t^2$

$Z = (at, 0)$

$SZ^2 = (at-0)^2 + (0-a)^2$

$= a^2t^2 + a^2$

$\therefore SZ^2 = AS.PS$