KINCOPPAL-ROSE BAY

Cor Unum

2003

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown on every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value
- Start each question in new writing booklet

Examiners KH,MN June 2003

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (12 marks) (Start a new booklet)

(a) Evaluate $\lim_{x\to 0} \frac{\tan 4x}{x}$. **1**

(b) Find $\frac{d}{dx}\left(2x^3e^{3x}\right)$. **2**

(c) Solve $\frac{1}{2-x} \geq 3$. **3**

(d) State the domain and range of the function **2**

$$f(x) = 2\cos^{-1}\left(\frac{x}{3}\right) .$$

(e) Find the acute angle between the lines **2**

$$y = 3x - 5$$
$$2x + y - 7 = 0$$

(f) Evaluate $\int \frac{\cos x}{1+2\sin x}\, dx$ **2**

Question 2 (12 marks) (Start a new booklet)

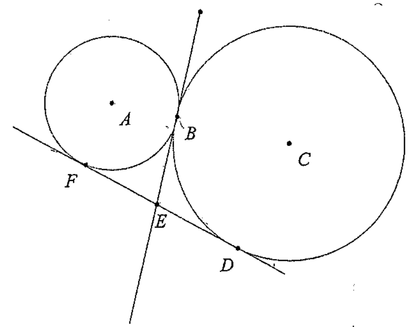
(a) Evaluate $\int_0^2 \frac{8x}{\sqrt{1+2x^2}}\, dx$; using the substitution $u = 1 + 2x^2$. **3**

(b) Find the general solution to $\sqrt{3}\tan x - 1 = 0$. **2**
Express your answer in terms of π .

(c) Prove that (x-2) is a factor of $2x^4 - 4x^3 + 4x^2 - 15x + 14$ **1**

(d) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 2x \; dx$ **3**

(e) Tangents *BE* and *FD* are common to the circles with centres *A* and *C*.



(i) Explain why $BE = EF = DE$ **1**

(ii) Let $\angle BFE = \alpha$ and $\angle BDE = \beta$. **2**
Prove that $\angle FBD = 90°$

Question 3 (12 marks) (Start a new booklet)

(a) Six people are seated in a straight line.

(i) How many seating arrangements are possible? **1**

(ii) How many arrangements are possible if Tarzan and Jane occupy the seats at either end? **2**

(b) (i) Show that $f(x) = x^3 + 2x - 17$ has a root between $x=2$ and $x=3$. **1**

(ii) Using an approximation of $x = 2\cdot 4$, use one application of Newton's method to find a better approximation for this root. Give your answer to two decimal places. **3**

(c) Use a table of standard integrals to evaluate **2**

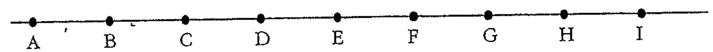
$$\int \frac{1}{\sqrt{x^2+9}}\, dx.$$

(d) Evaluate **3**

$$\int_0^{\frac{3}{4}} \frac{1}{9+16x^2}\, dx.$$

Question 4 (12 marks) (Start a new booklet)

(a) (i) In what ratio does I divide AG? **1**



(ii) W(2,4) divides XY internally in the ratio $k:l$ where X(-1,1) and Y(7,9). Find the ratio $k:l$. **2**

(b) The polynomial $P(x) = x^3 - 3x^2 + kx - 2$ has roots α, β, γ.

(i) Find the value of $\alpha + \beta + \gamma$. **1**

(ii) Find the value of $\alpha\beta\gamma$. **1**

(iii) It is known that two roots are the reciprocal of each other. Find the value of the third root and hence find the value of k. **2**

(c) Marvin the Martian has a body temperature of $100\,^\circ$C at the instant he falls asleep. When Marvin sleeps his body temperature obeys Newton's Law of Cooling according tho the law $\frac{dT}{dt} = k(T - A)$, where T is Marvin's body temperature and A is the temperature of the surrounding air.

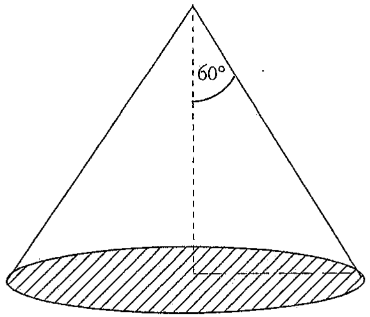
(i) Show that $T = A + Ce^{kt}$, where C and k are constants, satisfies Newton's Law of Cooling. **2**

(ii) Marvin goes to sleep at 10 pm. His temperature at midnight is $95\,^\circ$C. Marvin's bedroom is air conditioned with the temperature set at $20\,^\circ$C. Assuming Marvin continues to sleep what will be his body temperature to the nearest degree at 8am? **3**

Question 5 (12 marks) (Start a new booklet)

(a) Use the principle of Mathematical Induction to show that $7^n + 13^n$ is divisible by 10 for n positive odd integers. **3**

(b) Sand pours onto the ground and forms a cone where the semi-vertical angle is $60°$. The height of the cone at time t seconds is h *cm* and the radius of the base is r *cm*. Sand is being poured onto the pile at a rate of $12 cm^3/s$.



(i) Show that $r = \sqrt{3}h$ **1**

(ii) Find the rate at which the height is increasing at the instant when the height is 12 *cm*. **3**

[Volume of a cone $= \frac{1}{3}\pi r^2 h$]

(c) Consider the function

$$f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$$

(i) State any values of x for which $f(x)$ is undefined. **1**

(ii) Show that $f(1) = \frac{\pi}{2}$ **1**

(iii) Show that $f'(x) = 0$ **2**

(iv) Sketch the graph of $y = f(x)$ **1**

Question 6 (12 marks) (Start a new booklet)

A particle moves in Simple Harmonic Motion with amplitude a, in the form $\ddot{x} = -4x$ where x is the displacement, in metres, from the origin O and t is the time in seconds.

(i) Prove that $v^2 = 4(a^2 - x^2)$ **3**

(ii) The particle moves so that $x = 2, v = 4$ find the value of a. **1**

(iii) Find an expression for v in terms of displacement. **2**

(iv) By setting $v = \frac{dx}{dt}$ and taking the reciprocal, prove that $x = 2\sqrt{2}\sin 2t$ if when $t = \frac{\pi}{4}, x = 2\sqrt{2}$. **3**

(v) Where would you expect the maximum speed to occur? **1**

(vi) Hence, or otherwise, find the maximum speed of the particle. **2**

Question 7 (12 marks) (Start a new booklet)

A particle moves according to the equation $x = 2e^{-t}(\cos t + \sin t)$.
It moves in the interval $0 \le t \le 2\pi$.

(i) Show that $\dot{x} = -4e^{-t}\sin t$ and find the acceleration function $\ddot{x}$. **2**

(ii) Discuss the displacement as $t \to \infty$. **1**

(iii) Find the times when the particle is at the origin. **2**

(iv) When is the particle moving in the positive direction. **1**

(v) Find the times when the particle will be stationary. **2**

(vi) Find the displacement at the times when the particle is stationary. (Give your answers correct to three decimal places). **1**

(vii) Draw a neat, **full-page** sketch of $x = 2e^{-t}(\cos t + \sin t)$, giving endpoints, stationary points and intercepts . (In the interval $0 \le t \le 2\pi$). **3**

Trial Extension 1 solutions 2003

Question 1:

(a) $4\lim_{x\to 0}\frac{\tan 4x}{4x}=4$ ✓

(b) $2x^3 3e^{3x}+6x^2e^{3x}=6x^2e^{3x}(x+1)$ ✓ ✓

(c) $(2-x)^2\times\frac{1}{2-x}\ge 3(2-x)^2$ $\quad x\ne 2$ ✓

$2-x\ge 3(2-x)^2$

$2-x-3(2-x)^2\ge 0$

$(2-x)(1-3(2-x))\ge 0$

$(2-x)(3x-5)\ge 0$

$\frac{5}{3}\le x<2$

✓ ✓

(d) $-1\le\frac{x}{3}\le 1$ ✓ $\quad 0\le f(x)\le 2\pi$

$-3\le x\le 3$

✓

(e)

$m_1=3$ ✓

$m_2=-2$

For an acute angle

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\tan\theta=\left|\frac{3--2}{1-6}\right|$

$\tan\theta=\left|\frac{5}{-5}\right|=1$

$\theta=45°$ ✓

(f) $\frac{1}{2}\log(1+2\sin x)+C$

✓ ✓

Question 2

(a) $\frac{du}{dx}=4x$ $\quad x=2\ \ u=9$ ✓ $\quad x=0\ \ u=1$

$dx=\frac{du}{4x}$

$\int_0^2\frac{8x}{\sqrt{1+2x^2}}\,dx=\int_1^9\frac{8x}{\sqrt{u}}\frac{du}{4x}=\int_1^9 2u^{-\frac{1}{2}}du=$ ✓

$\left[4u^{\frac{1}{2}}\right]_1^9=12-4=8$ ✓

(b) $\tan x=\frac{1}{\sqrt{3}}$ ✓

$x=n\pi+\tan^{-1}\frac{1}{\sqrt{3}}$

$x=n\pi+\frac{\pi}{6}$ ✓

(c) $P(2)=2(2)^4-4(2)^3+4(2)^2-15(2)+14$ ✓

$=0$

$\therefore (x-2)$ is a factor via the factor theorem

(d)

$\cos 2x=1-2\sin^2 x$

so $\cos 4x=1-2\sin^2 2x$

$\sin^2 2x=\frac{1}{2}(1-\cos 4x)$ ✓

$\int_0^{\frac{\pi}{4}}\sin^2 2x\ dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 4x)dx$

$=\frac{1}{2}\left[x-\frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$ ✓

$=\frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{\sin\pi}{4}\right)-\left(0-\frac{\sin 0}{4}\right)\right]$

$=\frac{\pi}{8}$ ✓

(e) (i) Two tangents from an exterior point to a circle are equal. ✓

(ii)

Since $\angle BFE=\alpha$, then $\angle FBE=\alpha$ (isos $\triangle$) ✓

Since $\angle BDE=\beta$, then $\angle DBE=\beta$ (isos $\triangle$)

In $\triangle BFD$ $\quad\alpha+\alpha+\beta+\beta=180$ ($\angle$ sum of $\triangle$)

$2\alpha+2\beta=180$ ✓

$\alpha+\beta=90$

$\therefore\angle FBD=90°$

Question 3.

(a) (i) $6!=720$ ✓

(ii) $4!\times 2!=48$ ✓ ✓

(b) (i)

$f(2)=(2)^3+2(2)-17$

$=8+4-17$

$=-5$ ✓

$f(3)=(3)^3+2(3)-17$

$=16$

Since $f(x)$ changes sign between $x=2$ and $x=3$ and since $f(x)$ is continuous for all x, $f(x)$ must be zero somewhere between $x=2$ and $x=3$.

(ii)

$f'(x)=3x^2+2$

$f'(2\cdot4)=3(2\cdot4)^2+2=19\cdot28$ ✓

$f(2\cdot4)=(2\cdot4)^3+2(2\cdot4)-17=1\cdot624$

$x_2=x_1-\frac{f(x)}{f'(x)}$

$=2\cdot4-\frac{1\cdot624}{19\cdot28}$ ✓

$=2\cdot315767635.....$

$=2\cdot32\ \ (2d.p)$ ✓

(c) $\ln(x+\sqrt{x^2+9})+C$

✓ ✓

(d) $\left[\frac{1}{16}\times\frac{4}{3}\tan^{-1}\frac{4x}{3}\right]_0^{\frac{3}{4}}$ ✓ ✓

$=\frac{1}{16}\times\frac{4}{3}\tan^{-1}1-\frac{1}{16}\times\frac{4}{3}\tan^{-1}0$

$=\frac{1}{16}\times\frac{4}{3}\times\frac{\pi}{4}$

$=\frac{\pi}{48}$ ✓

Question 4.

(a) (i) -4:1 ✓

(ii) $x=\frac{kx_2+lx_1}{k+l}$

$2=\frac{7k-l}{k+l}$ ✓

$2k+2l=7k-l$

$-5k=-3l$

$\frac{k}{l}=\frac{3}{5}$

$k:l=3:5$ ✓

(b) (i) $\alpha+\beta+\gamma=3$ ✓

(ii) $\alpha\beta\gamma=2$ ✓

(iii) $\alpha\times\frac{1}{\alpha}\times\gamma=2$

$\gamma=2$ ✓

$\alpha+\beta+2=3$

$\alpha+\beta=1$

$\alpha\beta+\beta\gamma+\alpha\gamma=k$

$1+2\beta+2\alpha=k$

$1+2(\beta+\alpha)=k$

$1+2\times1=k$

$k=3$ ✓

(c) (i)

$T = A + Ce^{kt}$

$\dfrac{dT}{dt} = kCe^{kt}$ ✓

$\dfrac{dT}{dt} = k(T - A)$ as $Ce^{kt} = T - A$ ✓

(ii)

When $t = 0$ $T = 100$

$100 = 20 + Ae^{0}$

$A = 80$ ✓

$T = 20 + 80e^{kt}$

when $t = 2$ $T = 95$

$95 = 20 + 80e^{k2}$

$e^{2k} = \dfrac{15}{16}$

$2k = \ln\dfrac{15}{16}$

$k = \dfrac{1}{2}\ln\dfrac{15}{16}$ ✓

when $t = 10$

$T = 20 + 80e^{\frac{1}{2}\ln\frac{15}{16}10}$

$T = 77.93571472$

$T = 78°$ ✓

Question 5:

(a) Let $7^n + 13^n = 10M$ where M is any integer.

For $n=1$ $7^1 + 13^1 = 20 = 10 \times 2$ which is divisible by 10. ✓

Assume $7^k + 13^k = 10M$ is true for $n=k$

Prove true for $n=k+2$

$13^k = 10M - 7^k$

$7^{k+2} + 13^{k+2} =$

$7^2 7^k + 13^2 13^k = 7^2 7^k + 13^2(10M - 7^k)$ ✓

$= 49.7^k + 1690M - 169.7^k$

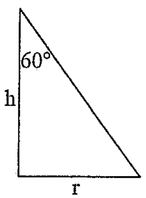
$= 1690M - 120.7^k$

$= 10(169 - 12.7^k)$ ✓

which is a multiple of 10, therefore true for $n=k+2$.

Since it is true for n=1, it is true for n=1+2 And so on, so it is true for all positive odd integers.

(b)



(i) $\tan 60° = \dfrac{r}{h}$

$h\tan 60° = r$ ✓

$r = \sqrt{3}h$

(ii) $\dfrac{dV}{dt} = 12$

$V = \dfrac{1}{3}\pi r^2 h$

$V = \dfrac{1}{3}\pi(3h^2)h$

$V = \pi h^3$ ✓

$\dfrac{dV}{dh} = 3\pi h^2$

$\dfrac{dV}{dh} = 432\pi$ when h=12 ✓

$\dfrac{dh}{dt} = \dfrac{dh}{dV}.\dfrac{dV}{dt}$

$= \dfrac{1}{432\pi}.12$

$= \dfrac{1}{36\pi} cm/s$ ✓

(c) (i) $x \neq 0$ ✓

(ii)

$f(1) = \tan^{-1}1 + \tan^{-1}\left(\dfrac{1}{1}\right)$

$= \dfrac{\pi}{4} + \dfrac{\pi}{4}$ ✓

$= \dfrac{\pi}{2}$

(iii)

$f'(x) = \dfrac{1}{1+x^2} + \dfrac{1}{1+\left(\dfrac{1}{x}\right)^2} . \dfrac{d}{dx}(x^{-1})$

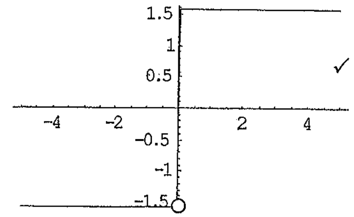
$= \dfrac{1}{1+x^2} + \dfrac{1}{\dfrac{x^2+1}{x^2}} . -x^{-2}$ ✓

$= \dfrac{1}{1+x^2} + \dfrac{x^2}{x^2+1} . -\dfrac{1}{x^2}$

$= \dfrac{1}{1+x^2} - \dfrac{1}{x^2+1}$

$= 0$ ✓

(iv)



✓

Question 6

(i) $\ddot{x} = -4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$ ☑

$\frac{1}{2}v^2 = -\frac{4x^2}{2} + C \qquad (v=0, x=a)$

$0 = -2a^2 + C$

$C = 2a^2$ ☑

$\frac{1}{2}v^2 = -2x^2 + 2a^2$

$v^2 = 4(a^2 - x^2)$ ☑

(ii) $v^2 = 4(a^2 - x^2) \qquad x=2, v=4$

$16 = 4(a^2 - 4)$

$a^2 - 4 = 4$

$a^2 = 8$

$a = 2\sqrt{2} \qquad (a>0)$ ☑

(iii) $v^2 = 4(8 - x^2)$

$v = \pm 2\sqrt{8 - x^2}$ ☑

but if $x=2, v=4>0$

$v = 2\sqrt{8-x^2}$ ☑

(iv) $v = 2\sqrt{8-x^2}$

$\frac{dx}{dt} = 2\sqrt{8-x^2}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{8-x^2}}$ ☑

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C, \left(t = \frac{\pi}{4}, x = 2\sqrt{2}\right)$ ☑

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) + C$

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}(1) + C$

$\frac{\pi}{4} = \frac{\pi}{4} + C$

$C = 0$

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)$ ☑

$\sin(2t) = \frac{x}{2\sqrt{2}}$

$x = 2\sqrt{2}\sin(2t)$

(v) at the origin (the centre of the motion. ☑

(vi) $v = 4\sqrt{2}\cos(2t)$ ☑

max speed $= 4\sqrt{2} \times 1$

$= 4\sqrt{2}\, m/s$ ☑

Question 7

(i) $x = 2e^{-t}(\cos t + \sin t)$

$\dot{x} = (\cos t + \sin t) \times -2e^{-t} + 2e^{-t}(-\sin t + \cos t)$

$= -2e^{-t} \times 2\sin t$ ☑

$= -4e^{-t}\sin t$

$\ddot{x} = \sin t \times 4e^{-t} + -4e^{-t}\cos t$

$= 4e^{-t}(\sin t - \cos t)$ ☑

(ii) As $t \to \infty$, $x \to 0$ since $e^{-t} \to 0$. ☑

(iii) $0 = 2e^{-t}(\cos t + \sin t)$ but $e^{-t} \neq 0$

$\cos t + \sin t = 0$

$\sin t = -\cos t$ ☑

$\tan t = -1$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$ ☑

(iv) $\dot{x} = -4e^{-t}\sin t$ moving in a positive direction $\dot{x} > 0$, ☑

$-4e^{-t}\sin t > 0$

$\sin t < 0$

$\pi < t < 2\pi$

(v) Stationary when $\dot{x} = 0 \to t = 0, \pi, 2\pi$ ☑☑

(vi) $x = 2e^{-t}(\cos t + \sin t) \quad t = 0, \pi, 2\pi$

$x = 2(\cos 0 + \sin 0) = 2$

$x = 2e^{-\pi}(\cos\pi + \sin\pi) = -0.086$

$x = 2e^{-2\pi}(\cos 2\pi + \sin 2\pi) = 0.004$ ☑☑

(vii)