



2008

HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 60

- Attempt Questions 1 – 5
- All questions are of equal value

Kincoppal-Rose Bay, School of the Sacred Heart
Mathematics Extension 1/Extension 2 Common Half Yearly Examination, 2008

Total Marks – 60

Attempt Questions 1-5

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet.

Marks

- (a) Solve $\frac{x^2 - 5x}{x - 4} \leq 3$ 3
- (b) Twelve people are going to the Easter Show.
Five of them are going by car and the rest are going by train.
- (i) How many different groups of five people can be found to fill the car? 1
- (ii) In one of these groups of five, it is found that only 2 of the people can drive.
In how many ways can the seats be filled in this group under this condition? 1
- (c) When $P(x) = x^3 + bx + c$ is divided by $x - 1$, the remainder is -4
When $P(x) = x^3 + bx + c$ is divided by $x + 2$, the remainder is 11 .
Find the values of b and c . 2
- (d) If α, β and γ are the roots of $x^3 - 5x^2 - 3x + 2 = 0$, find the values of:
- (i) $\alpha + \beta + \gamma$ 1
- (ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ 1
- (e) Find the size of the acute angle between the curves $y = \log_e x$ and $y = x^2 - 4x + 3$
at the point $(1, 0)$, to the nearest degree. 3

Question 2 (12 marks) Use a SEPARATE writing booklet.

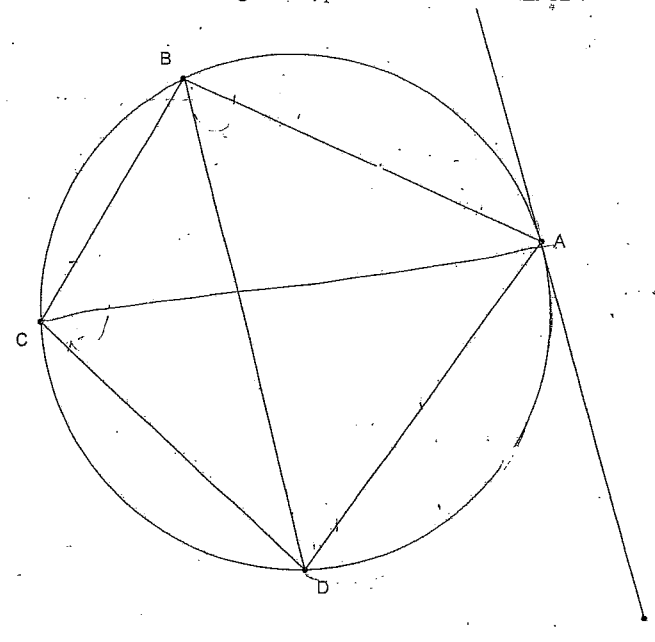
Marks

- (a) (i) Use the Factor Theorem to show that $x+1$ is a factor of the polynomial $x^3 - x^2 - 3x - 1 = 0$ 1
- (ii) Hence, or otherwise, factorise the polynomial $P(x) = x^3 - x^2 - 3x - 1$ into its two factors. 2
- (b) The point $Q(a, b)$ divides the interval joining $A(-1, 5)$ and $B(6, -4)$, externally, in the ratio 3:2. Find the values of a and b . 2
- (c) Find the definite integral $\int_0^1 xe^{x^2+1} dx$, using the substitution $u = x^2 + 1$. Leave your answer in terms of e . 3
- (d) (i) Find $\frac{d}{dx}(\sin x + \cos x)$ 1
- (ii) Hence, prove that $\int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\sin x + \cos x} dx = \frac{1}{2} \ln 2$ 3

Question 3 (12 marks) Use a SEPARATE writing booklet.

Marks

- (a) Find $\int \sin^2 3x dx$ 3
- (b) $ABCD$ is a cyclic quadrilateral. If $AB = AD$ and AE is a tangent at A , prove that AC bisects $\angle BCD$. 4



- (c) (i) Using the expansions of $\cos 2\theta$, show that $\sqrt{\frac{1+\cos x}{1-\cos x}} = \cot \frac{x}{2}$, where $0 < x < \pi$. 3
- (ii) Hence prove that $\cot 67\frac{1}{2}^\circ = \sqrt{2} - 1$ 2

Question 4 (12 marks) Use a SEPARATE writing booklet.

Marks

(a) Show, by the process of mathematical induction that, for $n \geq 1$

3

$$\log_e 2 + \log_e \left(\frac{3}{2}\right) + \log_e \left(\frac{4}{3}\right) + \dots + \log_e \left(\frac{n+1}{n}\right) = \log_e (n+1)$$

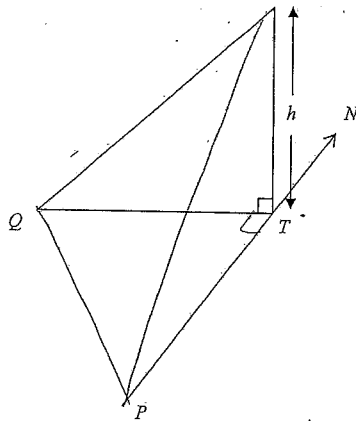
(b) At position T , a hill is h metres high. The angular elevation of the hill from a place P , due south of the hill, is 37° . The angular elevation of the hill from a place Q , due west of the hill, is 23° . The distance from P to Q is 3 km.

(i) Copy and complete the diagram showing all relevant information.

1

(ii) Find the height (h) of the hill in kilometres, to 1 decimal place.

3



(c) (i) Show that the equation of the tangent to the curve $y = e^{\frac{x}{2}}$ at the point $x = 2$, is $y = \frac{e}{2}x$.

2

(ii) Draw a sketch showing the curve and its tangent.

3

Evaluate in exact form, the area of the region bounded by the curve, its tangent and the y -axis.

5

Question 5 (12 marks) Use a SEPARATE writing booklet.

Marks

(a) The letters of the word CIRCLE are written at random on the circumference of a circle.

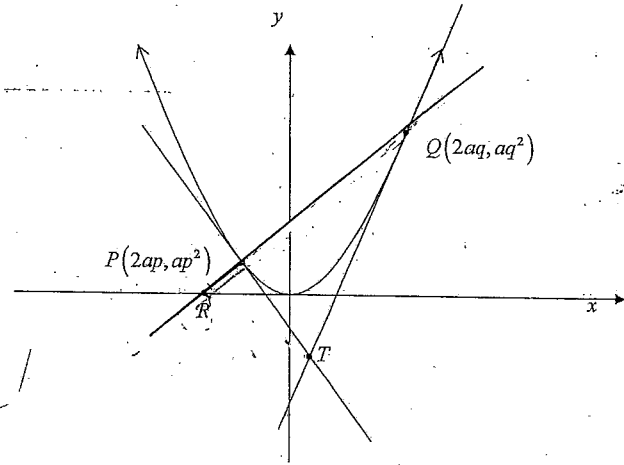
(i) How many different arrangements are possible?

2

(ii) In how many of these arrangements will the "C"s be separated?

2

(b)



$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The line PQ meets the x -axis at R . The tangents at P and Q meet at T .

(i) (Given that the equation of the secant PQ is $2y = x(p+q) - 2apq$, determine the coordinates of the point R in terms of p and q .)

1

(ii) Show that the equation of the tangent at $P(2ap, ap^2)$ is given by $y = px - ap^2$.

2

(iii) Hence show that the coordinates of the point T are given by $(a(p+q), apq)$.

2

Note - you do not have to derive the equation of the tangent at Q , you may quote it.

(iii) The secant PQ varies in position, but maintains a constant gradient of m . Prove that M , the midpoint of RT moves on a straight line and that this line is parallel to PQ .

3

$m =$

End of paper

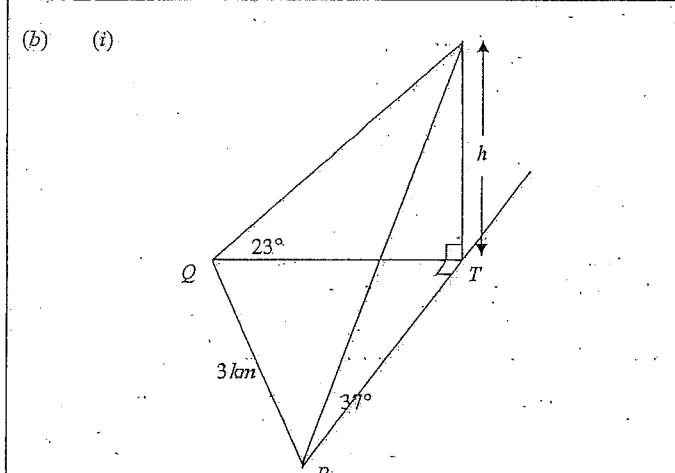
Question 1 (12 marks)	Marks
<p>(a) $\frac{x^2 - 5x}{x - 4} \notin 3$ $\frac{x^2 - 5x}{x - 4} = 3$ $x^2 - 5x = 3x - 12$ $x^2 - 8x + 12 = 0$ $(x - 6)(x - 2) = 0$ $x = 6, 2$</p> <p><i>Solution:</i> $x \notin 2, 4 < x \notin 6$</p>	<p>✓ $x \neq 4$ ✓ check ✓ solution</p>
<p>(b) (i) ${}^{12}C_5 = 792$ (ii) $2 \times {}^4C_4 = 48$</p>	<p>✓ (i) ✓ (ii)</p>
<p>(c) $P(1) = 1 + b + c = -4$ (1) $P(-2) = -8 - 2b + c = 11$ (2)</p> <p>from (2) $\Rightarrow 8 + 2b - c = -11$ (3)</p> <p>(1) + (3) $\Rightarrow 9 + 3b = -15$ $\underline{b = -8}$ $\underline{c = 3}$</p>	<p>✓ using remainder theorem ✓ solutions</p>
<p>(d) (i) $\alpha + \beta + \gamma = \frac{-b}{a} = 5$</p> <p>(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma)$ $= 2 \times 5$ $= 10$</p>	<p>✓ Part (i) ✓ Part (ii)</p>
<p>(e) $y = \log_e x$ $y = x^2 - 4x + 3$ $y' = \frac{1}{x}$ $y' = 2x - 4$ $m_1 = 1$ $m_2 = -2$</p> <p>$\tan \theta = \left \frac{1+2}{1-2} \right = -3$ $\theta \approx 72^\circ$</p>	<p>✓ derivatives ✓ gradients ✓ correct use of formula</p>

Question 2 (12 marks)	Marks
<p>(a) (i) $P(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1$ $= -1 - 1 + 3 - 1$ $= 0$</p> <p>(ii)</p> $\begin{array}{r} x^2 - 2x - 1 \\ x+1 \overline{) x^3 - x^2 - 3x - 1} \\ \underline{x^2 + x^2} \\ -2x^2 - 3x \\ \underline{-2x^2 - 2x} \\ -x - 1 \\ \underline{-x - 1} \\ 0 \end{array}$ <p>$P(x) = (x+1)(x^2 - 2x - 1)$</p>	<p>(i) ✓ (ii) ✓ ✓ correct use of division algorithm ✓ written as factors</p>
<p>(b)</p> <p>$(-1, 5)$ $(6, -4)$ $\swarrow \searrow$ $-3:2$</p> <p>$a = \frac{-2-18}{-3+2}$ $b = \frac{10+12}{-3+2}$ $\underline{a = 20}$ $\underline{b = -22}$</p>	<p>✓ correct use of formula ✓ solution</p>
<p>(c) $u = x^2 + 1$ $\frac{du}{dx} = 2x$ $xdx = \frac{1}{2} du$</p> <p>when $x = 0, u = 1$ when $x = 1, u = 2$</p> $\int_0^1 xe^{x^2+1} dx = \frac{1}{2} \int_1^2 e^u du = \frac{1}{2} [e^u]_1^2$ $= \frac{1}{2} [e^2 - e]$ $= \frac{e}{2} [e - 1]$	<p>✓ xdx ✓ correct substitution ✓ correct integral</p>

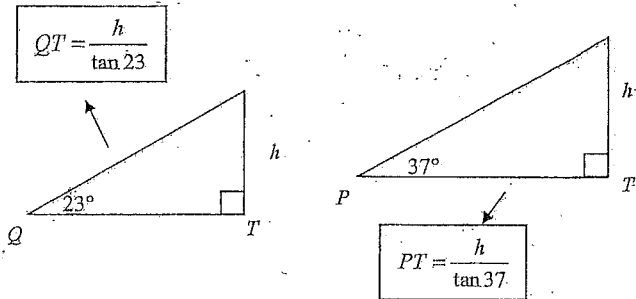
<p>(d) (i) $\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$</p> <p>(ii) $\int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\sin x + \cos x} dx = \ln \left[\sin x + \cos x \right]_0^{\frac{\pi}{4}}$</p> $= \ln \left[\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \right]$ $= \ln \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \ln 1$ $= \ln \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \right)$ $= \ln \sqrt{2}$ $= \frac{1}{2} \ln 2$	<p>(i) ✓</p> <p>(ii) ✓ integration ✓ substitution ✓ working to show</p>
<p>Question 3 (12 marks) Marks</p>	
<p>(a) $\cos 6x = 1 - 2 \sin^2 3x$</p> $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$ $\int \sin^2 3x dx = \frac{1}{2} \int (1 - \cos 6x) dx$ $= \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + c$ $= \frac{1}{6} (3x - \sin 6x) + c$	<p>✓ $\sin^2 3x$</p> <p>✓ integral</p> <p>✓ +c</p>
<p>(b) Let $\angle EAD = \theta$ then $\angle DBA = \theta$ (angle in alternate segment) now $\square ABC$ is isosceles (given, $AB = AD$) $\therefore \angle BDA = \theta$ (base angle isosceles $\square ABC$) then $\angle BAD = 180 - 2\theta$ (straight angle) $\therefore \angle BCD = 180 - (180 - 2\theta)$ (opp angles cyclic quadrilateral) $= 2\theta$ but $\angle ACD = \theta$ (angle in alternate segment) $\therefore \angle BCD = \theta$ $\therefore AC$ bisects $\angle BCD$</p>	<p>✓ angles in alt segment</p> <p>✓ cyclic quad</p> <p>✓ reasoning to get to do</p> <p>✓ conclusion</p>

<p>Question 3 cont (12 marks) Marks</p>	
<p>(c) (i) $\cos 2\theta = 2 \cos^2 \theta - 1 \Rightarrow 2 \cos^2 \theta = \cos 2\theta + 1$ $\cos 2\theta = 1 - 2 \sin^2 \theta \Rightarrow 2 \sin^2 \theta = 1 - \cos 2\theta$</p> $\frac{\sqrt{1 + \cos x}}{\sqrt{1 - \cos x}} = \frac{\sqrt{2 \cos^2 \frac{x}{2}}}{\sqrt{2 \sin^2 \frac{x}{2}}}$ $= \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}$ $= \cot \frac{x}{2}$	<p>(i) ✓ expressions for ✓ correct substitu</p>
<p>(c) (ii) $\cot 67 \frac{1}{2}^\circ = \sqrt{\frac{1 + \cos 135^\circ}{1 - \cos 135^\circ}}$</p> $= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}$ $= \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}$ $= \sqrt{(\sqrt{2} - 1)^2}$ $= \sqrt{2} - 1$	<p>(i) ✓ first line of sol ✓ exact values ✓ simplification</p> <p>Nb marks may be reorganised for this question</p>

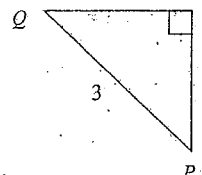
Question 4 (10 marks)	Marks
<p>(a) Step 1: Prove true for $n=1$.</p> <p>LHS = $\log_e 2$</p> <p>RHS = $\log_e (1+1) = \log_e 2 = \text{LHS}$</p> <p>Step 2: Assume true for $n=k$</p> <p>i.e. $\log_e 2 + \log_e \left(\frac{3}{2}\right) + \log_e \left(\frac{4}{3}\right) + \dots + \log_e \left(\frac{k+1}{k}\right) = \log_e (k+1)$</p> <p>Step 3: Required to prove true for $n=k+1$.</p> <p>i.e. $\log_e 2 + \log_e \left(\frac{3}{2}\right) + \log_e \left(\frac{4}{3}\right) + \dots + \log_e \left(\frac{k+1}{k}\right) + \log_e \left(\frac{k+2}{k+1}\right) = \log_e (k+2)$</p> <p>LHS = $\log_e 2 + \log_e \left(\frac{3}{2}\right) + \log_e \left(\frac{4}{3}\right) + \dots + \log_e \left(\frac{k+1}{k}\right) + \log_e \left(\frac{k+2}{k+1}\right)$</p> <p>$= \log_e (k+1) + \log_e \left(\frac{k+2}{k+1}\right)$</p> <p>$= \log_e \left[(k+1) \left(\frac{k+2}{k+1}\right) \right]$</p> <p>$= \log_e (k+2)$</p> <p>$= \text{RHS}$</p> <p>Step 4:</p> <p>Since the result is true for $n=1$ and $n=k+1$, then it is true for $n=2, 3$ etc. The result is true for all $n \geq 1$</p>	<p>✓ Steps 1 + 2</p> <p>✓ Step 3</p> <p>✓ Step 4</p>



completed diagram



✓ QT and PT
✓ tying together into $\square QPT$
✓ correct solution.



$$3^2 = \left(\frac{h}{\tan 23}\right)^2 + \left(\frac{h}{\tan 37}\right)^2$$

$$3^2 = h^2 \left(\frac{1}{\tan^2 23} + \frac{1}{\tan^2 37}\right)$$

$$h = \frac{3}{\sqrt{\left(\frac{1}{\tan^2 23} + \frac{1}{\tan^2 37}\right)}}$$

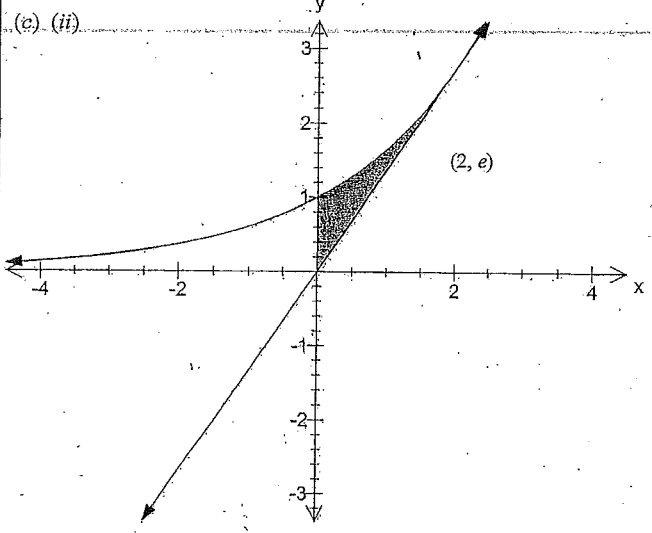
$$h \approx 1.1095 \text{ km}$$

$$= 1.1 \text{ km (1dp)}$$

(a) (i) $y = e^{\frac{x}{2}}$
 $y' = \frac{1}{2}e^{\frac{x}{2}}$

when $x = 2$,
 $m = \frac{e}{2}$ and $y = e$
 $y - e = \frac{e}{2}(x - 2)$
 $2y - 2e = ex - 2e$
 $2y = ex$
 $y = \frac{e}{2}x$

- derivative and gradient
- working to get eqn



- sketch showing region

$$A = \int_0^2 e^{\frac{x}{2}} dx - \frac{1}{2} \times 2 \times e$$

$$= 2 \left[e^{\frac{x}{2}} \right]_0^2 - e$$

$$= 2[e^1 - e^0] - e$$

$$= 2e - 2 - e$$

$$= (e - 2) \text{ units}^2$$

- integral
- answer

Question 5 (12 marks) Marks

(a) (i) $\frac{5!}{2!} = 60$

(ii) "c's" together = 4!
 \therefore apart $\Rightarrow \frac{5!}{2!} - 4! = 36$
 or
 "c's" can go in 3 ways apart
 rest in 4! ways
 $\therefore \frac{3 \times 4!}{2!} = 36$

- (i) 5!
- division by same elements
- (ii) "c's" together
- subtraction

(b) (i) When $y = 0$,
 $0 = x(p+q) - 2apq$
 $2apq = x(p+q)$
 $x = \frac{2apq}{p+q}$
 $R\left(\frac{2apq}{p+q}, 0\right)$

- (i)

(b) (ii) $y = \frac{x^2}{4a}$
 $y' = \frac{2x}{4a} = \frac{x}{2a}$
 at $x = 2ap$, $m = p$
 $y - ap^2 = p(x - 2ap)$
 $y = px - 2ap^2 + ap^2$
 $y = px - ap^2$

- derivative and gradient
- working to eqn

<p>(b) (iii) eqn at $Q: y = qx - aq^2$ solving sim:</p> $px - ap^2 = qx - aq^2$ $px - qx = ap^2 - aq^2$ $x(p - q) = a(p - q)(p + q) \quad \text{nb: } p \neq q$ $x = a(p + q)$ $y = p[a(p + q)] - qp^2$ $y = ap^2 + apq - ap^2$ $y = apq$	<input checked="" type="checkbox"/> eqn at Q and working for x value <input checked="" type="checkbox"/> working for y value
<p>(b) (iv) Since PQ has constant gradient m</p> $m = \frac{p+q}{2}$ <p>i.e. $p+q = 2m$</p>	<input checked="" type="checkbox"/> midpoint <input checked="" type="checkbox"/> $(p+q)$ and apq <input checked="" type="checkbox"/> substitution and resolution to (A) <input checked="" type="checkbox"/> conclusion
<p>let M be (X, Y)</p> $\text{then } X = \frac{a(p+q) + \frac{2apq}{p+q}}{2}$ $= \frac{a(p+q)^2 + 2apq}{2(p+q)}$ <p>and $Y = \frac{apq}{2} \Rightarrow \boxed{apq = 2Y}$</p> $\therefore X = \frac{a(2m)^2 + 2 \times 2Y}{4m}$ $4mX = 4am^2 + 4Y$ $4Y = 4mX - 4am^2$ $Y = mX - am^2 \quad (A)$ <p>Which is a straight line with gradient m.</p> <p>\therefore the locus of M is a straight line parallel to PQ</p>	