THE KING'S SCHOOL

2005

Higher School Certificate

Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\dfrac{2}{1-x^2}$ in partial fractions. **2**

(ii) Show that $\displaystyle\int_0^{\frac{1}{4}} \frac{2}{1-x^2}\,dx = \ln\left(\frac{5}{3}\right)$ **2**

(iii) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{2x}{1-x^4}\,dx$ **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2}{1+\sin 2x+\cos 2x}\,dx$ **3**

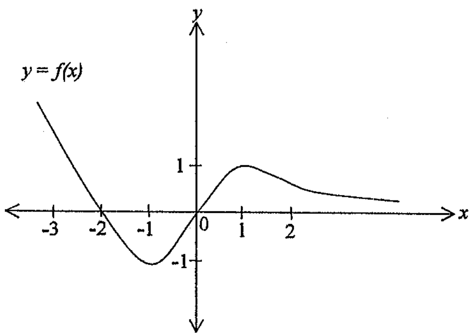
(c) Use completion of square to prove that

$\displaystyle\int_0^{1} \frac{4}{4x^2+4x+5}\,dx = \tan^{-1}\left(\frac{4}{7}\right)$ **3**

Question 1 is continued on the next page

Question 1 (continued) **Marks**

(d)



On separate diagrams, sketch the graphs of:

(i) $y = \ln f(x)$ **2**

(ii) $y = e^{\ln f(x)}$ **1**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Use integration by parts to show that

$$\int_0^1 (x-1)\, f'(x)\, dx = f(0) - \int_0^1 f(x)\, dx$$ **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{x-1}{(x+1)^2}\, dx$ **2**

(b) Let $z = x + iy$, x, y real, where $\arg z = \dfrac{3\pi}{5}$

(i) Sketch the locus of z **1**

(ii) Find $\arg(-z)$ **1**

(c) Sketch the region in the complex plane where $|z - i| \le |z + 1|$ **2**

(d) $z = x + iy$, x, y real, is a complex number such that $(z + \bar{z})^2 + (z - \bar{z})^2 = 4$

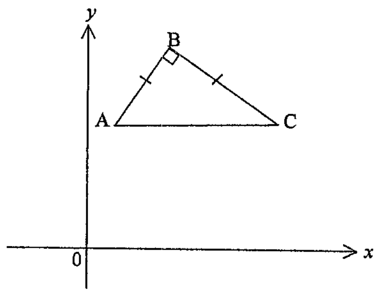
(i) Find the cartesian locus of z **2**

(ii) Sketch the locus of z in the complex plane showing any features necessary to indicate your diagram clearly. **2**

Question 2 is continued on the next page

Question 2 (continued) **Marks**

(e)



In the Argand diagram, ΔABC is right-angled at B and isosceles.

A, B, C represent the complex numbers a, b, c respectively.

(i) Find the complex number $\overrightarrow{BA}$ in terms of a and b. **1**

(ii) Prove that $c = ai + b(1-i)$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the parabola $y = \dfrac{1+x^2}{2}$ and use it to sketch the curve $y = \dfrac{2}{1+x^2}$ on the same diagram. **2**

(ii) Hence, or otherwise, find the range of the function $y = \dfrac{2}{1+x^2} - 1$ **1**

(b) Consider the function $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(i) By using (a), or otherwise, find the range of the function. **2**

(ii) Show that $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \dfrac{2x}{(1+x^2)\sqrt{x^2}}$ and

give the simplest expressions for the derivative if

(α) $x > 0$ and (β) $x < 0$ **3**

(iii) Sketch the curve $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **2**

(iv) The region bounded by $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ and the line $y = \dfrac{\pi}{2}$ is revolved about the y axis.

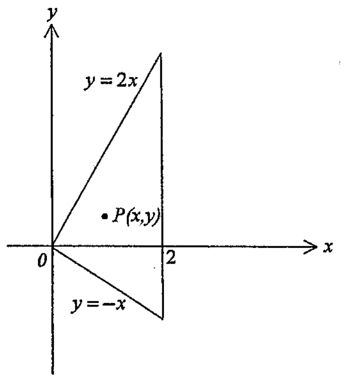
Show that the volume of the solid of revolution is given by

$$V = \pi \int_0^{\frac{\pi}{2}} \frac{1-\cos y}{1+\cos y}\, dy$$ **2**

(v) Find the volume V. **3**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The base of a solid is the triangular region bounded by the lines $y = 2x$, $y = -x$ and $x = 2$.

At each point $P(x, y)$ in the base the height of the solid is $4x^2 + x$

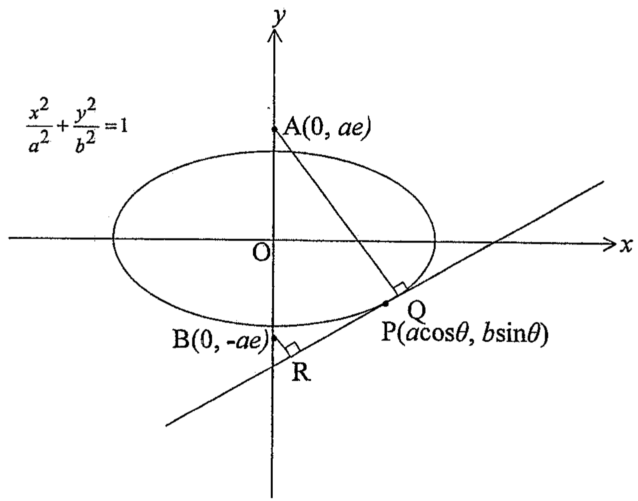
Find the volume of the solid. **4**

(b) If $xy^2 + 1 = x^2$, $y \neq 0$, show that $\dfrac{dy}{dx} = \dfrac{1}{y} - \dfrac{y}{2x}$ **2**

Question 4 is continued on the next page

Question 4 (continued) **Marks**

(c)



P($a\cos\theta$, $b\sin\theta$) is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b > 0$, where e is the eccentricity of the ellipse.

From A(0, ae) and B(0, $-ae$) perpendiculars are drawn to meet the tangent at P($a\cos\theta$, $b\sin\theta$) at Q and R, respectively.

(i) Prove that the equation of the tangent at P is $\dfrac{\cos\theta}{a}x + \dfrac{\sin\theta}{b}y = 1$ **3**

(ii) Hence, or otherwise, show that the line $x\cos\alpha + y\sin\alpha = k$ is a tangent to the ellipse if $a^2\cos^2\alpha + b^2\sin^2\alpha = k^2$ **2**

(iii) Hence, or otherwise, prove that $AQ^2 + BR^2 = 2a^2$ **4**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m moving with speed v experiences air resistance mkv^2, where k is a positive constant. g is the constant acceleration due to gravity.

(i) The particle of mass m falls from rest from a point O.

Taking the positive x axis as vertically downward, show that $\ddot{x} = k(V^2 - v^2)$, where V is the terminal speed. **2**

(ii) Another particle of mass m is projected vertically upward from ground level with a speed V^2, where V is the terminal speed as in (i).

Prove that the particle will reach a maximum height of $\frac{1}{2k}\ln(1 + V^2)$ **3**

(iii) Prove that the particle in (ii) will return to the ground with speed U where $U^{-2} = V^{-2} + V^{-4}$ **4**

(b) The ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is revolved about the line $x = 4$.

(i) Use the method of cylindrical shells to show that the volume of the solid of revolution is given by

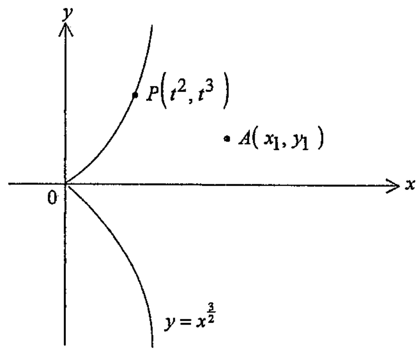
$$V = 8\sqrt{3}\,\pi \int_{-2}^{2} \sqrt{4 - x^2}\, dx - 2\sqrt{3}\,\pi \int_{-2}^{2} x\sqrt{4 - x^2}\, dx$$ **4**

(ii) Prove that the volume $V = 16\sqrt{3}\,\pi^2$ **2**

End of Question 5

Question 6 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a)



$P(t^2, t^3)$ is any point in the curve $y = x^{\frac{3}{2}}$

(i) Show that the equation of the tangent at $P(t^2, t^3)$ is $3tx - 2y - t^3 = 0$ **2**

(ii) $A(x_1, y_1)$ is a point not on the curve $y = x^{\frac{3}{2}}$

Deduce that at most three tangents to the curve pass through A. **1**

(iii) If the tangents with parameters t_1, t_2, t_3 do pass through $A(x_1, y_1)$, show that

(α) $t_1^3 + t_2^3 + t_3^3 = -6y_1$ **2**

(β) $(t_1t_2)^2 + (t_2t_3)^2 + (t_3t_1)^2 = 9x_1^2$ **2**

(iv) Find a cubic equation with roots $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ **2**

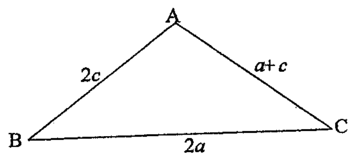
Question 6 is continued on the next page

Question 6 (continued) **Marks**

(b) (i) Given that $\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$, show that

$$\sin A+\sin C=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$ **1**

(ii) Consider ΔABC where



(α) Use the sine rule to show that $\sin A+\sin C=2\sin B$ **2**

(β) Deduce that $\sin\frac{B}{2}=\frac{1}{2}\cos\frac{A-C}{2}$ **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(n)=(n+1)^3+(n+2)^3+\ldots+(2n-1)^3+(2n)^3$, $n=1,2,3,\ldots$

(i) Show that $f(n+1)-f(n)=(2n+1)^3+7(n+1)^3$ **2**

(ii) Show that

$$(2n+1)^3-\frac{2n+1}{4}(3n+1)(5n+3)=\frac{2n+1}{4}(n+1)^2$$ **1**

(iii) Use mathematical induction for integers $n=1,2,3,\ldots$ to prove that

$$f(n)=(n+1)^3+(n+2)^3+\ldots+(2n)^3=\frac{n^2}{4}(3n+1)(5n+3)$$ **4**

(iv) Given that $1^3+2^3+\ldots+n^3=\left[\frac{n}{2}(n+1)\right]^2$, prove that

$(n+1)^3+(n+2)^3+\ldots+(2n)^3=\frac{n^2}{4}(3n+1)(5n+3)$ without induction. **2**

(b) (i) Show that $\dfrac{\binom{n}{k}}{n^k}=\dfrac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!}$, $2\le k\le n$ **2**

(ii) Deduce that $\dfrac{\binom{n+1}{k}}{(n+1)^k}>\dfrac{\binom{n}{k}}{n^k}$, $2\le k\le n$ **2**

(iii) Deduce that, if n is a positive integer, $\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$ **2**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the equation
$z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

(i) Show that $v = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a complex root of $z^7 - 1 = 0$ **1**

(ii) Show that the other five complex roots of $z^7 - 1 = 0$ are

$$v^k \text{ for } k = 2, 3, 4, 5, 6$$ **2**

(iii) Show that $\overline{\left(v^{7-k}\right)} = v^k$ for $k = 1, 2, \ldots, 6$

i.e. show that the conjugate of v^{7-k} is v^k **2**

(iv) Deduce that $v + v^2 + v^4$ and $v^3 + v^5 + v^6$ are conjugate complex numbers. **1**

(v) Deduce that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ **3**

Question 8 is continued on the next page

Question 8 (continued) **Marks**

(b) (i) Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \cos x \sin^{n-1} x \, dx = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$ **1**

(ii) Show by integration that

$$\int x \sin x \, dx = -x\cos x + \sin x$$ **1**

(iii) Let $t_n = \int_0^{\frac{\pi}{2}} x \sin^n x \, dx, \quad n = 0, 1, 2, \ldots$

Use integration by parts to prove that

$$t_n = \frac{1}{n^2} + \frac{n-1}{n} t_{n-2}, \quad n = 2, 3, 4, \ldots$$ **4**

End of Examination

Kings 2005 4u Trial — P. Ong

1.(a)(i). $\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$

$2 = A(1+x) + B(1-x)$

Let $x=1$ — Let $x=-1$

$A = 1$ — $B = 1$.

$\therefore \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$.

(ii) $\int_0^{1/4} \frac{2}{1-x^2}\,dx$

$= -\int \frac{-1}{1-x} + \int \frac{1}{1+x}$

$= \left[-\ln|1-x| + \ln|1+x|\right]_0^{1/4}$

$= -\ln 3/4 + \ln 5/4$

$= \ln(5/3)$

(iii) $\int_0^{1/2} \frac{2x}{1-x^4}\,dx$

$= \int_0^{1/2} \frac{x}{(1+x^2)} + \frac{x}{(1-x^2)}\,dx$

$= \frac{1}{2}\int \frac{2x}{(1+x^2)}\,dx - \frac{1}{2}\int \frac{-2x}{(1-x^2)}\,dx$

$= \left[\frac{1}{2}\ln|1+x^2| - \frac{1}{2}\ln|1-x^2|\right]_0^{1/2}$

$= \frac{1}{2}\ln\left|\frac{1+x^2}{1-x^2}\right|_0^{1/2}$

$= \frac{1}{2}\ln|5/3|$

or let $u = x^2$

$du = 2x\,dx$

$x = 0,\ u = 0$

$x = \frac{1}{2},\ u = \frac{1}{4}$

$\frac{1}{2}\int_0^{1/4} \frac{du}{1-u^2} = \frac{1}{2}\ln\left|\frac{5}{3}\right| + c$ from (ii)

(b). $\int_0^{\pi/4} \frac{2}{1+\sin 2x + \cos 2x}\,dx$

Let $\tan x = t$

$\therefore \frac{dt}{dx} = \sec^2 x$

$= 1 + \tan^2 x$

$= 1 + t^2$

$\frac{dt}{1+t^2} = dx$

$\sin 2x = 2\sin x\cos x = \frac{2t}{1+t^2}$ — $\cos 2x = \cos^2 x - \sin^2 x = \frac{1-t^2}{1+t^2}$

At $x = \pi/4,\ t = 1$

$x = 0,\ t = 0$.

$I = \int_0^1 \frac{2}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{dt}{1+t^2}$

$= \int_0^1 \frac{2}{1+t^2+2t+1-t^2}\,dt$

$= \int_0^1 \frac{dt}{1+t}$

$= \left[\ln|1+t|\right]_0^1$

$= \ln 2$.

(c). $\int_0^1 \frac{4}{4x^2+4x+5}\,dx$

$= \int_0^1 \frac{dx}{x^2+x+5/4}$

$= \int_0^1 \frac{dx}{x^2+x+1/4+1}$

$= \int_0^1 \frac{dx}{(x+1/2)^2+1}$

$= \left[\tan^{-1}\left(\frac{x+1/2}{1}\right)\right]_0^1$

$= \tan^{-1}(3/2) - \tan^{-1}(1/2)$

Let $\alpha = \tan^{-1}(3/2)$ — $\beta = \tan^{-1}(1/2)$

$\tan\alpha = 3/2$ — $\tan\beta = 1/2$

$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{3/2 - 1/2}{1 + 3/4}$

$= \frac{1}{7/4}$

$= 4/7$

$\therefore I = \tan^{-1}(4/7)$

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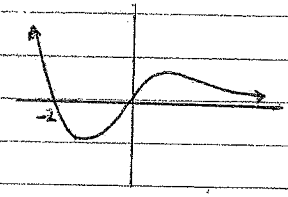
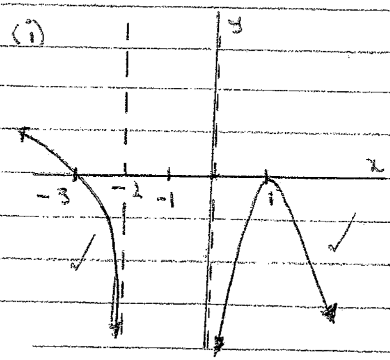
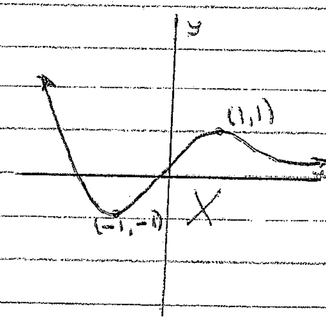
(d)



(i)



ii



(a). $\int_0^1 (x-1) f'(x)\,dx$ * (e)

$u = (x-1)$ $v = f(x)$

$u' = 1$ $v' = f'(x)$

$I = [f(x)(x-1)]_0^1 - \int_0^1 f(x)\,dx$

$= +f(0) - \int_0^1 f(x)\,dx$

(ii). $\int_0^1 (x-1)(x+1)^{-2}\,dx$

$\therefore f'(x) = (x+1)^{-2}$

$f(x) = -(x+1)^{-1}$

$I = f(0) - \int_0^1 f(x)\,dx$

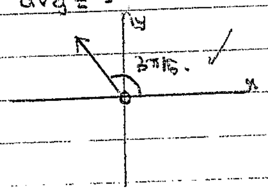
$= -1 + \int_0^1 \frac{1}{(x+1)}\,dx$

$= -1 + [\ln(x+1)]_0^1$

$= -1 + \ln 2.$

(b) (i) $z = x + iy$

$\arg z = 3\pi/5.$



(ii). $\arg(z) = \frac{3\pi}{5} + \pi$

$= 8\pi/5.$

$= -2\pi/5.$ (principle).

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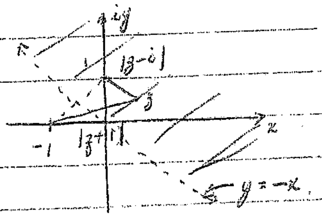
(c). $|z - i| \le |z + 1|$



(0,1) (-1,0) Midpt $= (-\frac{1}{2}, \frac{1}{2})$

$y - \frac{1}{2} = -1(x + \frac{1}{2})$

$2y - 1 = -2x - 1$

$2y + 2x = 0$

$y + x = 0.$ $\Rightarrow y = -x$

(d) (i) $z = x + iy$

$(z + \bar{z})^2 + (z - \bar{z})^2 = 4.$

$(2x)^2 + (2iy)^2 = 4.$

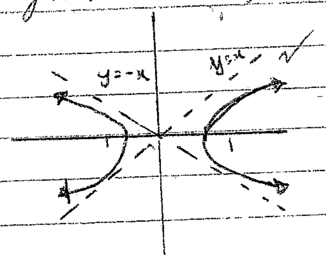
$4x^2 - 4y^2 = 4$

$x^2 - y^2 = 1.$

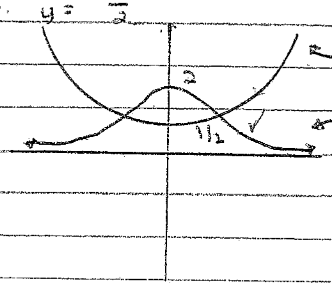
$y^2 = x^2 - 1$

$y = \pm\sqrt{x^2 - 1}$

(ii).



3.(a). $y = \frac{1+x^2}{2}$



(ii). Range: $-1 < y \le 1$

(b) (i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

As $x \to \infty$

$\therefore \to \cos^{-1}(-1)$

$y \to \pi$

As $x \to -\infty$

$\cos^{-1}(-1)$

$y \to \pi$

$\therefore$ Range $= 0 < y < \pi$

(ii) $\frac{d}{dx} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$f(x) = \frac{1-x^2}{1+x^2}$

$= 1 - \frac{2x^2}{1+x^2}$

$u = 2x^2$ $v = 1+x^2$

$f'(x) =$ $u' = 4x$ $v' = 2x$

$= \frac{-4x(1+x^2) + 4x^3}{(1+x^2)^2}$

$= \frac{-4x - 4x^3 + 4x^3}{(1+x^2)^2}$

$= \frac{-4x}{(1+x^2)^2}$

$\therefore \frac{d}{dx} \cos^{-1}(f(x)) = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$

$= \frac{\frac{4x}{(1+x^2)^2}}{\sqrt{1 - \frac{(1-x^2)^2}{(1+x^2)^2}}}$ $= \frac{4x}{(1+x^2)^2 \frac{\sqrt{(1+x^2)^2 - (1-x^2)^2}}{(1+x^2)}}$

$= \frac{4x}{(1+x^2)\sqrt{4x^2}}$

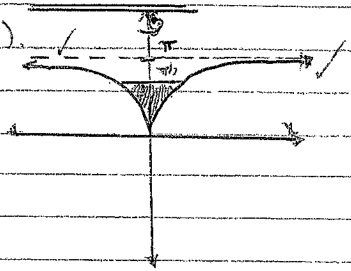
$= \frac{2x}{(1+x^2)\sqrt{x^2}}$

(α). If $x > 0$. (β) If $x < 0$

$\frac{dx}{dx} = \frac{2x}{(1+x^2)x}$ $\quad$ $\frac{-2}{(1+x^2)}$ ✓

$= \frac{2}{1+x^2}$ ✓

(iii). ✓



(iv). $V = \pi \int^{\pi/2} x^2 \, dy$

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\cos y = \frac{(1-x^2)}{(1+x^2)}$

$\cos y + x^2 \cos y = 1 - x^2$ ✓

$x^2(1+\cos y) = 1 - \cos y$

$x^2 = \frac{1-\cos y}{1+\cos y}$ ✓

$\therefore V = \pi \int_0^{\pi/2} \frac{1-\cos y}{1+\cos y} \, dy$

(v). Let $t = \tan y/2$

$\frac{dt}{dy} = \frac{1}{2}\sec^2 y/2$ $\quad \cos y = \frac{1-t^2}{1+t^2}$

$= \frac{1}{2}(1+t^2)$

$\frac{2dt}{1+t^2} = dy$ ✓

At $x = \pi/2$, $t = 1$

$x = 0$, $t = 0$.

$I = \pi \int_0^1 \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \, dy$

$= \pi \int_0 \frac{\cancel{2}t^2}{\cancel{2}} \cdot \frac{2dt}{1+t^2}$

$= 2\pi \int_0^1 \frac{t^2}{1+t^2} \, dt$. ✓

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$V = 2\pi \int_0^1 \frac{t^2}{1+t^2} \, dt$.

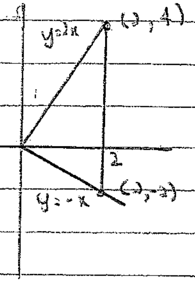
$= 2\pi \int_0^1 1 - \frac{1}{1+t^2} \, dt$. ✓

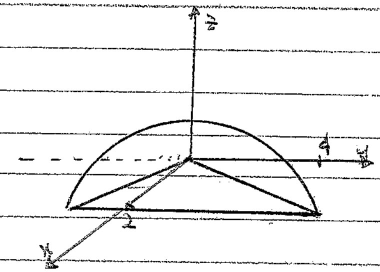
$= 2\pi[\cancel{t} - \tan^{-1}(t)]_0^1$

$= 2\pi[1 - \pi/4]$ ✓

$= 2\pi - \pi^2/2$.

4(a).





Use simpson's rule of trap. ~~rule~~

Area of Triangle

$= 1/2 \, bh$

$= 1/2 \times 6 \times 2$.

$= 6$.

$\therefore V = \int_0^{1/8} A(x) \, dx$

$=$

$4x^2 + x$

T.P at $8x + 1 = 0$.

$8x = -1$

$x = -1/8$.

$\therefore$ Assuming height from 0 to $-1/8$

$\therefore$ Height is $1/8$.

(b). $xy^2 + 1 = x^2$

Differentiate wrt x.

$x 2y \cdot \frac{dy}{dx} + y^2 = 2x$ ✓

$2xy \frac{dy}{dx} = 2x - y^2$

$\frac{dy}{dx} = \frac{2x}{2xy} - \frac{y^2}{2xy}$ ✓

$= \frac{1}{y} - \frac{y}{2x}$

5

(c)(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

D. wrt x

$\frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0.$

$\frac{b^2 2x}{a^2} = -2y\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$ ✓

At P(a cosθ, b sinθ)

$m = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$ ✓

Eqn: $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ya\sin\theta - ab\sin^2\theta = ab\cos^2\theta - xb\cos\theta$

$ya\sin\theta + xb\cos\theta = ab(\sin^2\theta + \cos^2\theta)$

$ya\sin\theta + xb\cos\theta = ab$ ✓

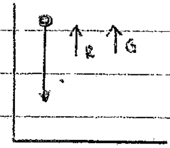
$\frac{y\sin\theta}{b} + \frac{x\cos\theta}{a} = 1$

(ii) For a line to be a tangent to E, solve simultaneously & use Δ = 0

try again

(iii) Use perp distance formula to find AQ & BR ... try again

5.(a). $r = mkv^2$



$\therefore m\ddot{x} = -mg - R$

$m\ddot{x} = -mg - mkv^2$

$\ddot{x} = -(g + kv^2)$

At Terminal Speed

$\ddot{x} = 0,\ v = V$ ✓

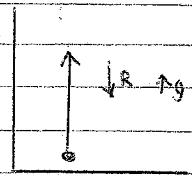
$0 = -g - kV^2$

$-g = kV^2$

$\therefore \ddot{x} = kV^2 - kv^2$ ✓

$= k(V^2 - v^2)$

(ii)



$m\ddot{x} = +mg - mkv^2$

$\ddot{x} = -kv^2 + g$

$v\frac{dv}{dx} = -kv^2 + g$

$\frac{dv}{dx} = kv + \frac{g}{v}$

$= \frac{-kv^2 + g}{v}$

$\frac{dx}{dv} = \frac{v}{-kv^2 + g}$ ✓

$$\frac{dx}{dv} = \frac{v}{-kv^2+g}$$

$$\int \frac{dx}{dv}\,dv = \int \frac{v}{-kv^2+g}\,dv$$

$$x = \frac{-1}{2k}\int \frac{-2kv}{-kv^2+g}\,dv$$

$$x = -\frac{1}{2k}\ln|kv^2-g| + c$$

At $x=0,\ v=V^2$

$$0 = -\frac{1}{2k}\ln|kV^4-g| + c.$$

$$\therefore x = \frac{1}{2k}\ln|kV^4-g| - \frac{1}{2k}\ln|kv^2-g|$$

At $x=H,\ v=0$.

$$H = \frac{1}{2k}\ln|kV^4-g| - \frac{1}{2k}\ln|-g|$$

But $-g = kV^{-2}$

$$H = \frac{1}{2k}\ln|kV^4+kV^2| - \frac{1}{2k}\ln|kV^2|$$

$$= \frac{1}{2k}\ln\left|\cancel{\frac{kV^4+kV^2}{kV^2}}\right|$$

$$= \frac{1}{2k}\ln|V^2+1|$$

(iii)

$$\ddot{x} = k(V^2-v^2) \quad (\text{part (i)})$$

$$v\frac{dv}{dx} = k(V^2-v^2)$$

$$\frac{dv}{dx} = \cancel{\frac{kV^2-v}{v}}$$

$$= \frac{k(V^2-v^2)}{v}$$

$$\frac{dx}{dV} = \frac{v}{k(V^2-v^2)}$$

$$\int\frac{dx}{dv} = \int\frac{v}{kV^2-v^2}\,dv \qquad x = -\frac{1}{2k}\int\frac{-2v}{V^2-v^2}\,dv$$

$$= \frac{-1}{2}\int\frac{-2v}{kV^2-v^2}\,dv \quad X \qquad = -\frac{1}{2k}\ln|V^2-v^2| + c$$

$$x = \frac{-1}{2}\ln|kV^2-v^2| + c.$$

At $x=0,\ v=0$. ... continue

$$0 = -\frac{1}{2}\ln|kV^2| + c.$$

$$\therefore c = \tfrac{1}{2}\ln|kV^2|$$

$$\therefore x = +\frac{1}{2}\cancel{\ln\left|\frac{kV^2-v^2}{kV^2}\right|} \qquad x = \frac{1}{2}\ln\left|\frac{kV^2}{kV^2-v^2}\right|$$

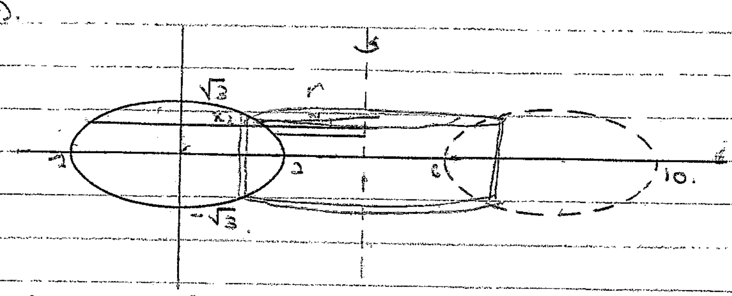
$$= \frac{1}{2}\ln\left|1-\frac{v^2}{kV^2}\right|$$

At $x=H,\ v=U$

$$\therefore H = \frac{1}{2}\ln\left|\frac{kV^2}{kV^2-U^2}\right|$$

now equate H + solve .... continue

(b).



(i). $V = 2\pi\int rh\,dx$

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \Rightarrow \frac{x^2}{4} = \frac{3-y^2}{3}$$

$r = x_2 - x_1 \Longrightarrow 4 -$

$h = 2y$

$$x = \pm\frac{2}{\sqrt{3}}\sqrt{3-y^2}$$

$$y^2 = \frac{3}{4}(4-x^2)$$

$$y = \pm\frac{\sqrt{3}}{2}\sqrt{4-x^2}$$

$$H = 2y = \sqrt{3}\sqrt{4-x^2}$$

$$R = 4-x$$

$$V = 2\pi\int_{-2}^{2} R\,H\,dx = 2\pi\int_{-2}^{2}(4-x).\sqrt{3}\sqrt{4-x^2}\,dx$$

$$= 2\pi\int_{-2}^{2}4\sqrt{3}\sqrt{4-x^2}\,dx - 2\pi\int_{-2}^{2}x\sqrt{3}\sqrt{4-x^2}\,dx$$

$$= 8\pi\sqrt{3}\int_{-2}^{2}\sqrt{4-x^2}\,dx - 2\sqrt{3}\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$$

(ii) $V = 8\sqrt{3}\pi\int_{-2}^{2}\sqrt{4-x^2} - 2\sqrt{3}\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$

$$= 8\sqrt{3}\pi\times\frac{\pi\times4}{2} + \sqrt{3}\int_{-2}^{2}-2x\sqrt{4-x^2}\,dx$$

$$= 16\sqrt{3}\pi + \sqrt{3}\left[(4-x^2)^{3/2}\right]_{-2}^{2}$$

$$= 16\sqrt{3}\pi + \sqrt{3}[0-0]$$

$$= 16\sqrt{3}\pi + 0.$$

$$= 16\sqrt{3}\pi$$

6(a)(i) $y = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

At $x = t^2$

$m = \frac{3\sqrt{t^2}}{2}$

$y - t^3 = \frac{3t}{2}(x - t^2)$

$2y - 2t^3 = x3t - 3t^3$

$\therefore 3xt - 2y - t^3 = 0.$

(ii). Since eqn of tangent is cubic

$\therefore$ 3 roots

$\therefore$ 3 values of t, hence at most 3 tangents.

(iii)(α) $3x_1 t_1 - 2y_1 - t_1^3 = 0.$ — ①

$3x_1 t_2 - 2y_1 - t_2^3 = 0.$ — ②

$3x_1 t_3 - 2y_1 - t_3^3 = 0.$ — ③

Add 1, 2, 3

$-t_1^3 - t_2^3 - t_3^3 = 2y_1 + 2y_1 + 2y_1$

$-(t_1^3 + t_2^3 + t_3^3) = 6y_1$

$t_1^3 + t_2^3 + t_3^3 = -6y_1$

(β) $(t_1 t_2)^2 + (t_2 t_3)^2 + (t_3 t_1)^2 =$ ~~[illegible]~~ Sum of prod of roots taken 2 at a time

i.e. $t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2 = \frac{c}{a}$ of

$\leq$ –

But $3tx - 2x^{3/2} - t^3 = 0$

$(2x^{3/2})^2 = (3tx - t^3)^2$

$4x^3 = 9t^2x^2 - 6t^4x + t^6$

$\therefore 4x^3 - 9t^2x^2 + 6t^4x - t^6 = 0$

$t^3 - 3tx + 2y = 0.$

Let $x = \frac{1}{y}$

$y = \frac{1}{x}$

$\left(\frac{1}{x}\right)^3 - 2t\left(\frac{1}{x}\right) + 2y = 0.$

$\frac{1}{x^3} - 2 + 2y = 0$

$1 - 2x^3 + 2x^3 y = 0.$

$2 + t^3(x+y) - 1 = 0.$

(b)(i) $\sin(X+Y) + \sin(X-Y) = 2\sin X \cos Y$

Let $X + Y = A$, $X - Y = C$.

~~$\sin A + \sin C = 2\sin A$~~

$A - X = Y$, $Y = X - C$, $X = A - Y$, $X = C + Y$

$A - X = X - C$; $A - Y = C + Y$

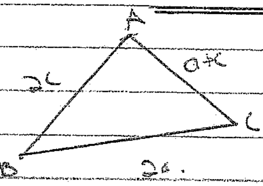
$A + C = 2X$; $A - C = 2Y$

$X = \frac{A+C}{2}$, $Y = \frac{A-C}{2}$

$\therefore \sin A + \sin C = 2\sin X \cos Y$

$= 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)$

(ii)



$\frac{\sin B}{a+c} = \frac{\sin A}{2a} = \frac{\sin C}{2c}$

$\frac{1}{2a+2c} = 2(\sin A + \sin C)$

$\frac{1}{a+c} = \frac{\sin A}{2} + \frac{\sin C}{2}$

$\therefore \sin B = \frac{\sin A}{2} + \frac{\sin C}{2}$

$2\sin B = \frac{\sin A}{1} + \sin C$

(iii) $2\sin B = \sin A + \sin C$

$4\sin\frac{B}{2}\cos\frac{B}{2} = \sin A + \sin C$

LHS $= \sin\frac{B}{2} = \frac{\sin A + \sin C}{4\cos B/2}.$

$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{4\cos B/2}$

$= \frac{\cancel{2}\cancel{\cos\frac{B}{2}}\cos\left(\frac{A-C}{2}\right)}{\cancel{4}\cancel{\cos B/2}}$ $_2$

$= \frac{1}{2}\cos\left(\frac{A-C}{2}\right) = R.H.S.$

But $A + B + C = \pi$

$\therefore A + C = \pi - B$

$\frac{A+C}{2} = \frac{\pi}{2} - \frac{B}{2}$

$\sin\left(\frac{A+C}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{B}{2}\right)$

$= \cos\frac{B}{2}$

7.(a)(i) $f(n) = (n+1)^3 + (n+2)^3 + \dots (2n-1)^3 + (2n)^3$

$f(n+1) - f(n)$

$= (n+2)^3 + (n+3)^3 + \dots (2n+1)^3 + (2n+2)^3 - [(n+1)^3 + (n+2)^3 + \dots (2n)^3]$

$= (2n+2)^3 - (n+1)^3 + (2n+1)^3$

$= (2(n+1))^3 - (n+1)^3 + (2n+1)^3$

$= 8(n+1)^3 - (n+1)^3 + (2n+1)^3$

$= 7(n+1)^3 + (2n+1)^3$

(ii) $(2n+1)^3 - \frac{(2n+1)(3n+1)(5n+3)}{4} = \frac{2n+1}{4}(n+1)^2$

LHS $= \frac{2n+1}{4}\left[4(2n+1)^2 - (3n+1)(5n+3)\right]$

$= \frac{2n+1}{4}\left[\cancel{8}\,16n^2 + 16n + 4 - 15n^2 - 14n - 3\right]$

$= \frac{2n+1}{4}\left[n^2 + 2n + 1\right]$

$= \frac{(2n+1)(n+1)^2}{4}$

= RHS.

(iii) $f(n) = (n+1)^3 + (n+2)^3 + \dots (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$

Let $n=1$

LHS $= (2)^3 = 8$ RHS $= \frac{1}{4}(4)(8) = 8$.

True

Assume true for $n=k$

$f(k) = \frac{k^2}{4}(3k+1)(5k+3)$

Prove for $f(k+1) = \frac{(k+1)^2}{4}(3k+4)(5k+8)$

$f(k+1) = (2k+1)^3 + 7(k+1)^3 + f(k)$ —— part (i)

$= (2k+1)^3 + 7(k+1)^3 + \frac{k^2(3k+1)(5k+3)}{4}$

part (ii) ← $= \frac{(2k+1)(k+1)^2}{4} + \frac{(2k+1)(3k+1)(5k+3)}{4} + \frac{k^2(3k+1)(5k+3)}{4}$

$= \frac{(3k+1)(5k+3)[2k+1+k^2]}{4} + \frac{(2k+1)(k+1)^2}{4}$

$\frac{(3k+1)(5k+3)(k+1)^2}{4} + \frac{(2k+1)(k+1)^2}{4}$

$= \frac{(k+1)^2[(3k+1)(5k+3) + 2k+1]}{4}$

$= \frac{(k+1)^2[15k^2 + 14k + 3 + 2k + 1]}{4}$

$= \frac{(k+1)^2[15k^2 + 16k + 4]}{4}$

$= \frac{(k+1)^2[(3k+4)(5k+8)]}{4}$

= RHS

if true for $n=k$, shown for $n=k+1$, therefore by the principles of Mathematical Induction true for $n \geq 1$.

(iv). $1^3 + 2^3 + \dots + n^3 = \left[\frac{n}{2}(n+1)\right]^2$

8.(a). $z^7 - 1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0$.

(i) $z^7 - 1 = 0$.

$z^7 = 1$

$z = \text{cis}\frac{2\pi k}{7}$ $(k = 1, 2, 3, \dots 7)$

$\therefore z_1 = \text{cis}\frac{2\pi}{7}$

Sub $v = \text{cis}\frac{2\pi}{7}$ into LHS $= \left(\text{cis}\frac{2\pi}{7}\right)^7 - 1$

$= \text{cis}\, 2\pi - 1$

$= 1 - 1$

$= 0 = $ RHS.

(ii) $z = \text{cis}\frac{2\pi k}{7}$

$v = \text{cis}\frac{2\pi}{7} \Rightarrow v^2 = \text{cis}\frac{4\pi}{7}$, $v^3 = \text{cis}\frac{6\pi}{7}$, $v^4 = \text{cis}\frac{8\pi}{7}$

$= \text{cis}\left(-\frac{6\pi}{7}\right)\dots$

$\therefore v_2 = \text{cis}\frac{4\pi}{7}$ (DeMoivre's)

Similarly $v_3 = \text{cis}\frac{6\pi}{7}\dots$ ✓ $v_7 = \text{cis}\frac{14\pi}{7} = \text{cis}\,0$.

(iii) $(v^{7-k}) = \frac{\text{cis}(7-k)\pi}{7}$ $v = \text{cis}\frac{2\pi}{7}$ $\therefore v^{7-k} = \text{cis}\frac{2(7-k)\pi}{7}$

~~$= \text{cis}\left(2 - \frac{2k}{7}\right)\pi$~~

~~$= \text{cis}\left(2\pi - \frac{2k\pi}{7}\right) = \text{cis}\left(\frac{2k\pi}{7}\right)$~~

$\therefore v^{7-k} = \text{cis}\left(\frac{2k\pi}{7}\right)$

$= \frac{\cos(7-k)\pi - i\sin(7-k)\pi}{7}$

$= \frac{\cos 7\pi \cos k\pi + \sin 7\pi \sin k\pi - i\sin 7\pi \cos k\pi + i\cos 7\pi \sin k\pi}{7}$

$= \frac{-\cos k\pi - i\sin k\pi}{7} = -\frac{\text{cis}\, k\pi}{7}$

$= \text{cis}\frac{k\pi}{7}$

$= v^7$

$v^k = \left(\text{cis}\frac{2\pi}{7}\right)^k = \text{cis}\left(\frac{2\pi k}{7}\right) = \overline{v^{7-k}}$

(iv) $v + v^2 + v^4$

$= \text{cis}\frac{2\pi}{7} + \text{cis}\frac{4\pi}{7} + \text{cis}\frac{-6\pi}{7}$

$v^3 + v^5 + v^6$

$= \text{cis}\frac{6\pi}{7} + \text{cis}\frac{-4\pi}{7} + \text{cis}\frac{-2\pi}{7}$ ✓

$= \overline{(v + v^2 + v^4)}$ ✓

(v). Quad with $(v+v^2+v^4)$ & $(v^3+v^5+v^6)$ as roots.

$x^2 - (\Sigma\alpha)x + \alpha\beta$

$= x^2 - (v + v^2 + \dots v^6)x + (v+v^2+v^4)(v^3+v^5+v^6)$

$= x^2 + x + (3 + v + \dots v^6)$ ✓

$= x^2 + x + 2$.

Sum of Roots $\Sigma\alpha = v^6 + v^5 + \dots v = -b/a$

$\text{cis}\frac{2\pi}{7} + \text{cis}\frac{4\pi}{7} + \dots \text{cis}\frac{12\pi}{7} = -1$

$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$ ✓

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -1/2$

$\cos(\pi/2 - \theta) = \cos\theta$

$-\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{2\pi}{7} = -1/2$

$-\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{2\pi}{7} = 1/$.

8.(b) (i) $\int_0^{\pi/2} \cos x \sin^{n-1} x\, dx$

Let $u = \sin x$ at $x = \pi/2$, $u = 1$

$du = \cos x\, dx$ $x = 0$, $u = 0$.

$I = \int_0^{1} u^{n-1}\, du$

$= \left[\frac{u^n}{n}\right]_0^1$ ✓

$= \frac{1}{n}$.

(ii) $\int x\sin x\, dx$

$u = x$ $v = -\cos x$

$u' = 1$ $v' = \sin x$

$I = -x\cos x + \int \cos x\, dx$ ✓

$= -x\cos x + \sin x + c$

(iii). $t_n = \int_0^{\pi/2} x\sin^n x\, dx$

$= \int_0^{\pi/2} x\sin^{n-1}x \sin x\, dx$

$u = \sin^{n-1}x$ $v = -x\cos x + \sin x$

$u' = (n-1)\sin^{n-2}x.\cos x$ $v' = x\sin x$ ✓

$t_n = \left[\sin^{n-1}x(\sin x - x\cos x)\right]_0^{\pi/2} - (n-1)\int x\sin^{n-2}.(-\cos^2 x) - (n-1)\int \sin^{n-1}.\cos x\, dx$

$= 1 + (n-1)\int x\sin^{n-2}(1 - \sin^2 x)\, dx - (n-1)\left[\frac{1}{n}\right]$

$t_n = 1 - 1 + \frac{1}{n} + (n-1)\int x\sin^{n-2}\, dx - (n-1)\int x\sin^n\, dx$

$t_n = \frac{1}{n} + (n-1)t_{n-2} - (n-1)t_n$ ✓

$t_n(n+1-1) = \frac{1}{n} + (n-1)t_{n-2}$

$nt_n = \frac{1}{n} + (n-1)t_{n-2}$

$t_n = \frac{1}{n^2} + \frac{n-1}{n}t_{n-2}$ ✓