THE KING'S SCHOOL

2004
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

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Attempt Questions 1-10
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find the value of $\sin 2x$ if $x = 0.006$, correct to 2 significant figures. **2**

(b) State the domain and range of the function $y = \log_e x$ **2**

(c) Find $\lim\limits_{x \to 0} \dfrac{x^3 - 3x}{6x}$ **2**

(d) Solve the equation $(2x+3)^2 = 4$ **2**

(e) Find a primitive function of $(2x+3)^4$ **2**

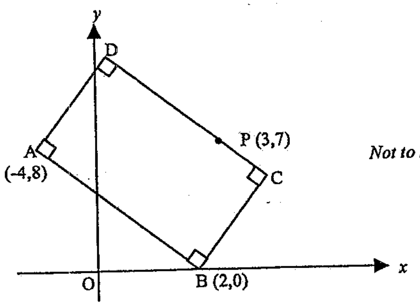
(f) For what value of x do $y = 1 - 2x$ and $2y = 7 + 6x$ hold simultaneously? **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the gradient of the normal to the curve $y=\dfrac{x}{x^2+1}$ at the point $(0, 0)$ **3**

(b)



Not to scale.

In the diagram, ABCD is a rectangle where A = (-4,8) and B = (2,0). P(3,7) is a point on the side CD.

(i) Find the gradient of line AB. **1**

(ii) Deduce that the equation of line AB is $4x+3y-8=0$ **2**

(iii) Hence, or otherwise, show that the length of side BC is 5 units. **2**

(iv) Write down the mid-point of side AB and deduce that P(3,7) is the mid-point of side CD. **2**

(v) Write down the point of intersection of the diagonals AC and BD. **1**

(vi) Find the coordinates of point D. **1**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \sin x \; dx$ **2**

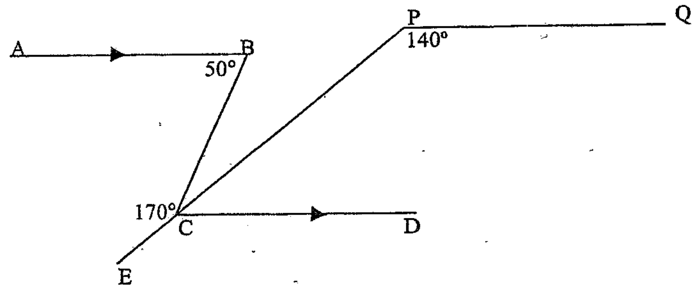
(ii) $\displaystyle\int_0^1 \frac{6x^2}{x^3+1} \; dx$ **2**

(b) For what values of k is x^2+2x+k

(i) concave upward? **1**

(ii) positive for all values of x? **2**

(c)



In the diagram, $AB \parallel CD$ and ECP is a straight line.

$\angle ABC = 50°$, $\angle ECB = 170°$, $\angle QPC = 140°$

(i) Find $\angle BCP$, giving reasons. **1**

(ii) Find $\angle ECD$, giving reasons. **2**

(iii) Prove that $PQ \parallel AB$ **2**

End of Question 3

Question 4 **(12 marks)** Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the curve $y = \tan\frac{x}{2}$ for $-2\pi \le x \le 2\pi$ **2**

(ii) State the period of $y = \tan\frac{x}{2}$ **1**

(iii) Solve the equation $\tan\frac{x}{2} = 1$ for $-2\pi \le x \le 2\pi$ **2**

(b) Consider the two arithmetic series

$A = 11 + 13 + 15 + \ldots$

and $B = -14 - 11 - 8 - \ldots$

(i) Find the sum of the first 40 terms of series A **1**

(ii) The two series have the same number of terms and the same sum. How many terms are in the series? **3**

(c) Find the values of a, b, c if **3**

$a(x+1)^2 + b(x+1)(x-1) + c(x-1) \equiv x^2 + 7x$

End of Question 4

Question 5 **(12 marks)** Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x) = x^4 - 6x^2 + 8x$

(i) Show that $f'(x) = 4(x+2)(x-1)^2$ **2**

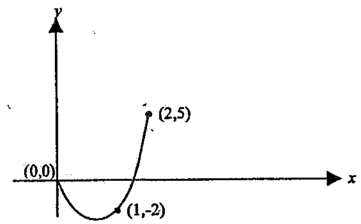
(ii) For what values of x is the function increasing? **2**

(iii) Show that there is a minimum turning point at $x = -2$ **2**

(iv) Show that there is a horizontal point of inflection at $x = 1$ **2**

(b) (i) Briefly explain why Simpson's Rule gives the exact value of $\int_a^b f(x)\,dx$ if $f(x)$ is a quadratic function. **1**

(ii)

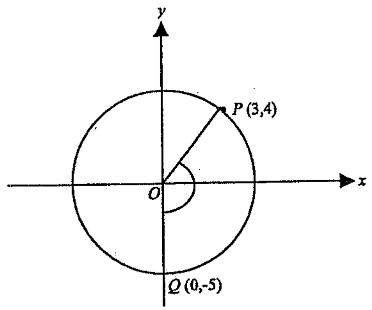


A parabola $y = f(x)$ passes through the points (0,0), (1,-2) and (2,5). Find the value of $\int_0^2 f(x)\,dx$ **3**

End of Question 5

Question 6 **(12 marks)** Use a SEPARATE writing booklet. **Marks**

(a)



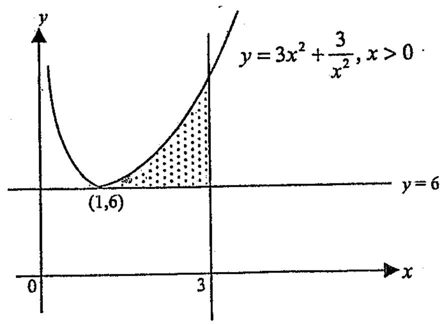
In the diagram, $P\ (3,4)$ and $Q\ (0,-5)$ are points on the circle $x^2 + y^2 = 25$.
Let $\angle POQ = \theta$, as marked on the diagram.

(i) Find the length of chord PQ. **1**

(ii) Show that $\cos\theta = -\frac{4}{5}$ **2**

(iii) Find the area of minor sector POQ correct to 1 decimal place. **2**

(b)



In the diagram, the shaded region is bounded by the curve $y = 3x^2 + \frac{3}{x^2}, x > 0$, and the two lines $y = 6$ and $x = 3$. Find the area of this shaded region. **4**

Question 6 continues next page

Question 6 **(continued)**

(c) The directrix of a parabola is the x axis and the focus is the point $(0,4)$.

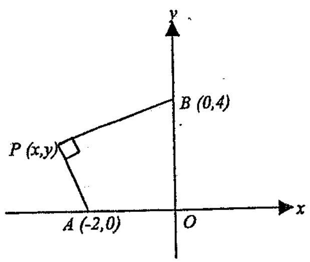
(i) Write down the focal length of the parabola. **1**

(ii) Find the equation of the parabola. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

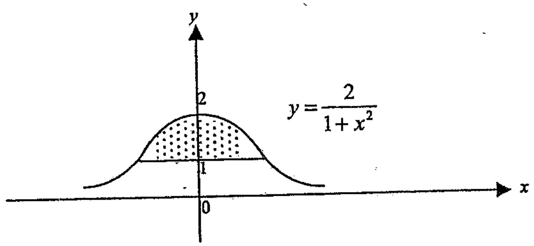
(a) $A(-2,0)$ and $B(0,4)$ are two points in the number plane. $P(x,y)$ is any point such that AP is perpendicular to BP.



(i) Prove that the equation of the locus of $P(x,y)$ is $x(x+2)+y(y-4)=0$ **3**

(ii) Deduce that the equation in (i) represents a circle and find its centre and radius. **3**

(b)



Consider the region bounded by the curve $y=\frac{2}{1+x^2}$ and the line $y=1$ as shown in the diagram.

The region is revolved about the y axis. Find the volume of the solid of revolution generated. **4**

Question 7 continues next page

Question 7 (continued) **Marks**

(c) The population, P, of a rural town is growing exponentially according to the equation $P=5000e^{0.1t}$, t measured in years. Currently, i.e. $t=0$, the population is increasing at a rate of 500 people/year.

What rate of increase, correct to the nearest hundred, is expected after 20 more years? **2**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Simplify the expression $x^{-1}y^{2}\left(x^{\frac{1}{2}}-y^{-1}\right)\left(x^{\frac{1}{2}}+y^{-1}\right)$, giving your answer with positive indices. **2**

(b) A particle moves on a straight line so that its velocity, $v\ m/s$, at any time t seconds is given by $v=(t-1)^{4}+\frac{t}{2},\ t\geq 0$

(i) Find the initial velocity and show that the particle never stops. **2**

(ii) Find the initial acceleration of the particle. **2**

(iii) Find the least value of the velocity. **2**

(iv) Sketch the velocity-time graph. **2**

(v) Find the distance travelled by the particle in the first 2 seconds. **2**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the land of Welf the interest on investments is paid continuously. If an interest rate of $100\,k\,\%$ p.a. is given then it can be shown that the amount in the fund after t years, $t\geq 0$, is $A(t)$ where

$A(t)=Pe^{kt}$, P is the initial investment.

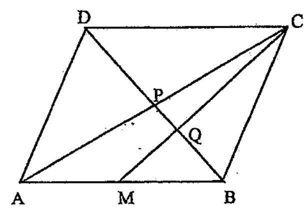
(i) A particular fund, F, in Welf pays 10% p.a. interest. Show that $k=0.1$ **1**

(ii) Sally invests \$5000 into fund F for 20 years. Find, correct to the nearest dollar, the amount Sally would have after the 20 years. **1**

(iii) Sally wants to be more wealthy in Welf and decides not to terminate her investment of \$5000 until it has grown to at least \$100 000. For how many years will she need to wait? **2**

(iv) Lucy also invests in fund F. She decides to deposit \$1500 into the fund at the start of each year for 20 years. Find, correct to the nearest dollar, the amount Lucy would have after the 20 years. **3**

(b)



In the diagram, $ABCD$ is a parallelogram. The diagonals AC and BD meet at P. M is the mid-point of AB and MC meets BD at Q.

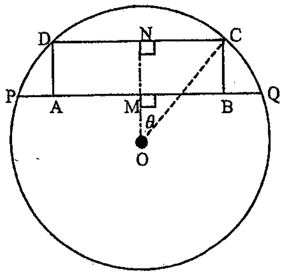
(i) Show that ΔCDQ is similar to ΔMBQ **1**

(ii) Deduce that $DQ=2BQ$ **2**

(iii) Prove that $BQ=2QP$ **2**

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**



In the diagram, O is the centre of a circle of radius $\sqrt{6}$ cm and PQ is a chord of length 4 cm. ABCD is a rectangle constructed in the minor segment cut off by chord PQ. OM is drawn perpendicular to PQ so that M is the mid-point of both chord PQ and side AB. N is the mid-point of side CD.

Let $\angle CON = \theta$, θ in radians.

(i) Show that $OM = \sqrt{2}$ cm **1**

(ii) Show that $0 < \theta < 1$ **2**

(iii) Show that the area, a, of rectangle ABCD is given by $a = 4\sqrt{3}\sin\theta\left(\sqrt{3}\cos\theta - 1\right)$. **3**

(iv) Show that $\dfrac{da}{d\theta} = 4\sqrt{3}\left(2\sqrt{3}\cos^2\theta - \cos\theta - \sqrt{3}\right)$ **3**

(v) Find the maximum area of the rectangle. **3**

End of Examination

TKS MATHEMATICS TRIAL 2004 SOLUTIONS

Qu 1

(a) $\sin 0.012 = 0.012$, 2 sig. figs

(b) Domain $x > 0$, Range all real values for y

(c) $\lim_{x\to 0} \dfrac{x(x^2-3)}{6x} = \lim_{x\to 0} \dfrac{x^2-3}{6} = -\dfrac{1}{2}$

or $\lim_{x\to 0}\left(\dfrac{x^2}{6} - \dfrac{3}{6}\right) = -\dfrac{1}{2}$

(d) $2x+3 = 2$ or $2x+3 = -2$

$\therefore x = -\dfrac{1}{2}$ or $-\dfrac{5}{2}$

(e) $\dfrac{(2x+3)^5}{5\times 2} = \dfrac{(2x+3)^5}{10}$

(f) $2(1-2x) = 7+6x$

$2 - 4x = 7 + 6x$

$\therefore \quad 10x = -5$

$x = -\dfrac{1}{2}$

Qu 2

(a) $\frac{dy}{dx} = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

At $(0,0)$, $\frac{dy}{dx} = 1$

$\therefore$ gradient of normal at $(0,0)$ is -1

(b) (i) gradient $AB = \frac{0-8}{2--4} = -\frac{8}{6} = -\frac{4}{3}$

(ii) line AB is $y = -\frac{4}{3}(x-2)$

or $3y = -4x + 8$

ie $4x + 3y - 8 = 0$

(iii) BC = perpendicular distance from P to line AB

$\therefore BC = \frac{12 + 21 - 8}{\sqrt{4^2+3^2}} = \frac{25}{5} = 5$

(iv) $M_{AB} = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4)$

$\therefore$ From (iii), if $PM_{AB} = 5$ then P is mid-point of CD

$PM_{AB} = \sqrt{4^2 + 3^2} = 5$ $\therefore$ result.

(v) Diagonals meet at mid-point of PM_{AB}, from (iv)

ie at $\left(\frac{2}{2}, \frac{11}{2}\right) = \left(1, \frac{11}{2}\right) = Q$, say

(vi) [LOTS OF WAYS]

e.g. $B \to Q = Q \to D$

$\Rightarrow D = (0, 11)$

Qu 3

(a) (i) $\left[-\cos x\right]_0^{\frac{\pi}{6}} = -\frac{\sqrt{3}}{2} - -1 = 1 - \frac{\sqrt{3}}{2}$

(ii) $= 2\int_0^1 \frac{3x^2}{x^3+1}\,dx = 2\left[\ln(x^3+1)\right]_0^1$

$= 2\ln 2$

(b) (i) ∪ $\therefore$ all values of k

(ii) Need $\Delta < 0 \Rightarrow 4 - 4k < 0$

ie $4k > 4$ or $k > 1$

(c) (i) $\angle ECP = 180°$, ECP a straight line

$\therefore \angle BCP = 10°$

(ii) $\angle BCD = \angle ABC = 50°$, alternate $\angle s$ in // lines AB, CD

$\therefore \angle ECD = 360° - (170° + 50°)$, angles at point C

$= 140°$

(iii) From (ii), $\angle QPC = \angle ECD = 140°$ and these

- are corresponding angles

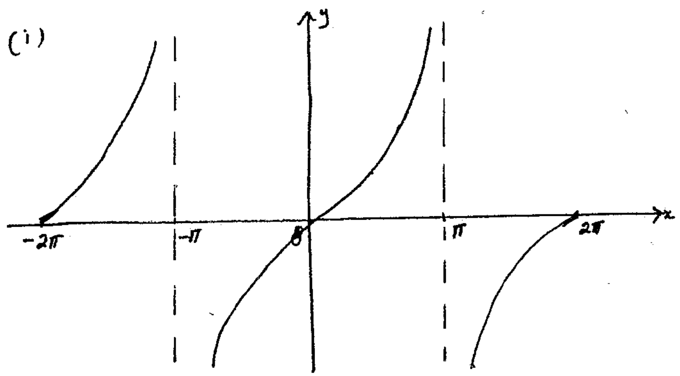
$\therefore$ $PQ \parallel CD$

But $AB \parallel CD$

$\therefore$ $PQ \parallel AB$

Qu 4

(a) (i)



(ii) period $= 2\pi$

(iii) $\therefore \frac{x}{2} = \frac{\pi}{4}$ or $-\frac{3\pi}{4}$ for $-\pi \le \frac{x}{2} \le \pi$

[Sketch helps]

$\Rightarrow x = \frac{\pi}{2}$ or $-\frac{3\pi}{2}$

(b) (i) $S_{40} = \frac{40}{2}\left[22 + 39 \times 2\right] = 20 \times 100 = 2000$

(ii) $\therefore \frac{n}{2}\left(22 + (n-1)\,2\right) = \frac{n}{2}\left(-28 + (n-1)\,3\right)$

$\Rightarrow 22 + 2n - 2 = -28 + 3n - 3$

$\therefore n = 51$ i.e. there's 51 terms.

(c) Put $x = 1$, $\therefore 4a = 1 + 7 \Rightarrow a = 2$

$\therefore$ Equating coefficients of x^2, $2 + b = 1$

$\Rightarrow b = -1$

Put $x = -1$, $\therefore -2c = 1 - 7$

$\therefore c = 3$

[LOTS OF WAYS, of course]

Qu 5

(a) (i) $f'(x) = 4x^3 - 12x + 8 = 4(x^3 - 3x + 2)$

Now, $(x+2)(x-1)^2 = (x+2)(x^2 - 2x + 1)$

$= x^3 - 2x^2 + x + 2x^2 - 4x + 2$

$= x^3 - 3x + 2$

$\therefore f'(x) = 4(x+2)(x-1)^2$

(ii) A test for increasing function is $f'(x) > 0$

$4(x+2)(x-1)^2 > 0 \Rightarrow x + 2 > 0$ since $4(x-1)^2 \ge 0$ for all x

$\therefore f(x)$ is increasing for $x > -2$

[Note If $x = 1$, $f'(x) = 0$ but the function is still increasing.
A curve is increasing if as x increases, $f(x)$ increases]

(iii) $f'(-2) = 0$ $\therefore$ at $x = -2$ there's a stationary point

But for $x < -2$, $f'(x) < 0$
& $x > -2$, $f'(x) > 0$ $\Rightarrow$

$\therefore$ at $x = -2$ there's a minimum turning point

(iv) $f'(x) = 4(x^3 - 3x + 2)$

$\therefore f''(x) = 4(3x^2 - 3) = 12(x^2 - 1)$

$\therefore f''(1) = 0$ and $f''(0) < 0$, $f''(2) > 0$
change in concavity

and, as well, $f'(1) = 0$

$\therefore$ at $x = 1$ there's a horiz. pt. of inflection

(b) (i) Simpson's rule uses the arc of a parabola (quadratic function) to approximate the arc of the curve, $y = f(x)$

(ii) From (i),

$$\int_0^2 f(x)\,dx = \frac{1}{6} \cdot 2\left[0 + 5 + 4 \times -2\right]$$

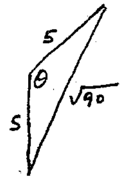
$$= \frac{1}{3}(-3)$$

$$= -1$$

Qu 6

(a) (i) $PQ = \sqrt{3^2 + 9^2} = \sqrt{90}$

(ii)



$\therefore \cos\theta = \dfrac{25 + 25 - 90}{2 \times 5 \times 5} = \dfrac{-40}{50} = -\dfrac{4}{5}$

(iii) Area $= \frac{1}{2} \times 5^2 \times \theta$ where $\cos\theta = -\frac{4}{5}$

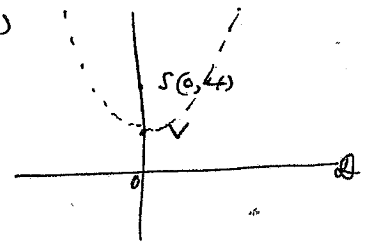
$= \frac{1}{2} \times 25 \times 2.498\ldots = 31.2\ u^2$, 1 d.p.

(b) $A = \displaystyle\int_1^3 3x^2 + \frac{3}{x^2} - 6\,dx$

$= \left[x^3 - \frac{3}{x} - 6x\right]_1^3$

$= 27 - 1 - 18 - (1 - 3 - 6) = 16\ u^2$

(c)



(i) focal length $a = 2$

(ii) Vertex $= (0, 2)$

$\therefore$ equation is $(x - 0)^2 = 4 \times 2\,(y - 2)$

i.e. $x^2 = 8(y - 2)$

Qu 7

(a) (i) Since $AP \perp BP$ the product of their gradients is -1

$$\therefore \frac{y}{x+2} \times \frac{y-4}{x} = -1$$

or $y(y-4) = -x(x+2)$

or $x(x+2) + y(y-4) = 0$

(ii) $\therefore x^2 + 2x + y^2 - 4y = 0$

$\Rightarrow (x+1)^2 - 1 + (y-2)^2 - 4 = 0$

or $(x+1)^2 + (y-2)^2 = 5$

is a circle, centre $(-1, 2)$, radius $\sqrt{5}$

(b) $V = \pi \int_1^2 x^2 \, dy$ where $y = \frac{2}{1+x^2}$

$\therefore 1 + x^2 = \frac{2}{y}$ or $x^2 = \frac{2}{y} - 1$

$$\therefore V = \pi \int_1^2 \frac{2}{y} - 1 \, dy$$

$$= \pi \left[2 \ln y - y \right]_1^2$$

$$= \pi \left(2\ln 2 - 2 - (0 - 1) \right) = \pi (2\ln 2 - 1) \text{ u}^3$$

(c) $\frac{dP}{dt} = 5000 \times 0.1 \, e^{0.1t} = 500 \, e^{0.1t}$

$\therefore$ when $t = 20$, $\frac{dP}{dt} = 500 e^2 \approx 3700$ people/yr, nearest hundred.

Qu 8

(a) $x^{-1} y^2 (x - y^{-2}) = y^2 - x^{-1} = y^2 - \frac{1}{x}$

$\left[\text{or } \frac{y^2 x - 1}{x} \right]$

(b) (i) $t = 0$, $v = (-1)^4 + 0 = 1$ m/s

Now, $(t-1)^4 \geq 0$ for all $t \geq 0$

and $\frac{t}{2} > 0$ for all $t > 0$

$\therefore v > 0$ for all $t \geq 0$ ie particle never stops

(ii) $\ddot{x} = \frac{dv}{dt} = 4(t-1)^3 + \frac{1}{2}$

$\therefore$ when $t = 0$, $\ddot{x} = -4 + \frac{1}{2} = -3\frac{1}{2}$ m/s²

(iii) From (i) and (ii), least v occurs when $\ddot{x} = 0$

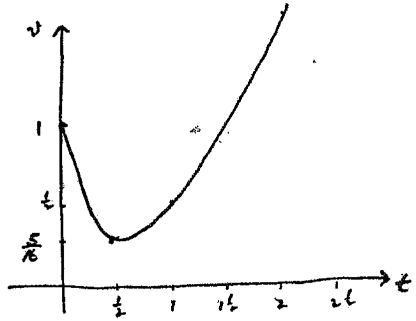
$\Rightarrow 4(t-1)^3 + \frac{1}{2} = 0$

or $(t-1)^3 = -\frac{1}{8}$

$\therefore t - 1 = -\frac{1}{2}$ or $t = \frac{1}{2}$

$\therefore$ least $v = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$ m/s

(iv)



(v) $x = \int_0^2 (t-1)^4 + \frac{t}{2} \, dt = \left[\frac{(t-1)^5}{5} + \frac{t^2}{4} \right]_0^2$

$= \frac{1}{5} + 1 - \left(-\frac{1}{5} + 0 \right) = 1\frac{2}{5}$ m

Qu 9

(a) (i) $\therefore\ 100k = 10 \Rightarrow k = 0.1$

(ii) $A(20) = 5000\, e^{0.1 \times 20} = 5000 e^{2}$

$= \$36945$, nearest dollar

(iii) Here, $A(t) = 5000\, e^{0.1t} \geq 100000$

$\therefore\ e^{0.1t} \geq 20$

or $0.1t \geq \ln 20$

$\therefore\ t \geq \frac{\ln 20}{0.1} = 29.957\ldots$

$\Rightarrow$ Sally needs to wait approx 30 years

(iv) Lucy would have

$1500\, e^{0.1 \times 20} + 1500 \times e^{0.1 \times 19} + \cdots + 1500\, e^{0.1 \times 1}$

$= 1500\, e^{.1} + 1500\, e^{.2} + 1500\, e^{.3} + \cdots + 1500\, e^{2}$,

is a geometric series where $a = 1500\, e^{.1}$, $r = e^{.1}$

$\therefore$ Lucy would have $\frac{1500\, e^{.1}\left((e^{.1})^{20} - 1\right)}{e^{.1} - 1}$

$= \frac{1500\, e^{.1}\left(e^{2} - 1\right)}{e^{.1} - 1}$

$= \$100707$, nearest dollar

Qu 9 (b)

(i) In Δs CDQ, MBQ

$\angle CDQ = \angle MBQ$, alternate $\angle$s in $\parallel$ lines CD, BA

$\angle DCQ = \angle BMQ$, "

$\therefore\ \Delta CDQ \ ||| \ \Delta MBQ$, 2 angles equal

(ii) From (i), $\frac{DQ}{BQ} = \frac{CD}{BM}$, ratios of corresponding sides

$= 2$, since M is the mid-point of AB, AB = CD

$\therefore\ DQ = 2BQ$

(iii) Now, $BQ + QP = PD$, diagonals of $\parallel$ogram bisect each other

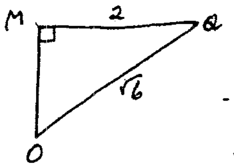
$= DQ - QP$

$= 2BQ - QP$, (ii)

$\therefore\ BQ = 2QP$

Qu 10

(i) Consider



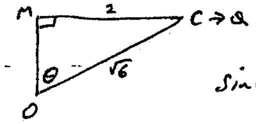
$\therefore OM^2 + 4 = 6$

$\Rightarrow OM = \sqrt{2}$ cm

(ii) as C approaches the vertical position above N the $\theta \to 0$, otherwise $\theta > 0$

max θ occurs as C approaches Q

$\Rightarrow$

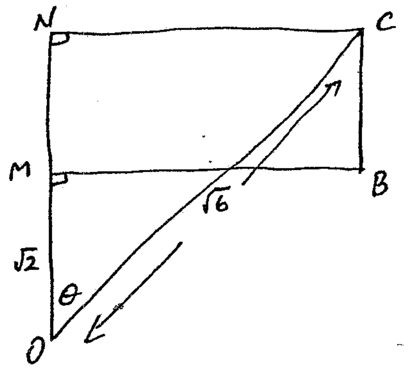


$\sin\theta = \frac{2}{\sqrt{6}}$

$\Rightarrow \theta = 0.955\ldots\ldots < 1$

$\therefore 0 < \theta < 1$

(iii) Consider



Then $\sin\theta = \frac{NC}{\sqrt{6}}$, $NC = \sqrt{6}\sin\theta$

$\therefore DC = 2\sqrt{6}\sin\theta$

and $\cos\theta = \frac{ON}{\sqrt{6}}$ $\therefore ON = \sqrt{6}\cos\theta$

$\therefore MN = \sqrt{6}\cos\theta - \sqrt{2} = CB$

$\therefore$ Area, $a = 2\sqrt{6}\sin\theta\,(\sqrt{6}\cos\theta - \sqrt{2})$

$= 2\sqrt{6}\sqrt{2}\sin\theta\,(\sqrt{3}\cos\theta - 1)$

$= 4\sqrt{3}\sin\theta\,(\sqrt{3}\cos\theta - 1)$

(iv) $\frac{da}{d\theta} = 4\sqrt{3}\left(\sin\theta\,(-\sqrt{3}\sin\theta) + (\sqrt{3}\cos\theta - 1)\cos\theta\right)$

$= 4\sqrt{3}\left(-\sqrt{3}\sin^2\theta + \sqrt{3}\cos^2\theta - \cos\theta\right)$

$= 4\sqrt{3}\left(-\sqrt{3}(1-\cos^2\theta) + \sqrt{3}\cos^2\theta - \cos\theta\right)$

$= 4\sqrt{3}\left(2\sqrt{3}\cos^2\theta - \cos\theta - \sqrt{3}\right)$

(v) $\frac{da}{d\theta} = 4\sqrt{3}\left((\sqrt{3}\cos\theta + 1)(2\cos\theta - \sqrt{3})\right)$

$= 0$ only if $2\cos\theta - \sqrt{3} = 0$ since $0 < \theta < 1$

and since as $\theta \to 0$ or $\theta \to 1$, then $a \to 0$,

the value of θ from $2\cos\theta - \sqrt{3} = 0$ must produce a maximum turning point $\Rightarrow$ maximum area

$\therefore$ max area occurs when $\cos\theta = \frac{\sqrt{3}}{2}$ and $\therefore \sin\theta = \frac{1}{2}$

$\therefore$ max $a = 4\sqrt{3}.\frac{1}{2}\left(\sqrt{3}.\frac{\sqrt{3}}{2} - 1\right)$ cm^2

$= \sqrt{3}$ cm^2