KAMBALA

Student Number: _______________

Trial Examination

August 2014

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- Answer Section 1 on the multiple-choice answer sheet attached on page 15
- Answer Section II in the writing booklets provided
  Start each question in a new booklet
- A table of standard integrals is provided on a separate page
- Show all necessary working in Questions 11 – 16

Total marks – 100

Section I
10 marks

- Attempt Questions 1 - 10
  Allow about 15 minutes for this section

Section II
90 marks

- Attempt Questions 11 - 16
  Allow about 2 hours 45 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\ \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Section I

10 Marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet provided on page 15 to answer Questions 1–10.

1 Find the exact value of $sin\frac{\pi}{3}$.

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{2}}$

2 What is the solution to the inequality $|2-5x|<7$.

(A) $x<-1$ or $x<\frac{9}{5}$ (B) $x>-1$ or $x<\frac{9}{5}$

(C) $x>-1$ or $x>\frac{9}{5}$ (D) $x<-1$ or $x>\frac{9}{5}$

3 What is the solution of the equation $4\sqrt{m}-9=0$?

(A) $m=\frac{18}{8}$

(B) $m=\frac{3}{2}$

(C) $m=\frac{81}{16}$

(D) $m=6$

4 Which of the following equations represent the locus of a point, $P(x,y)$, which is always 6 units from the point $(3,-4)$.

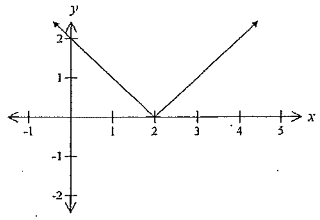
(A) $(x-3)^2+(y+4)^2=6$ (B) $(x+3)^2+(y-4)^2=6$

(C) $(x-3)^2+(y+4)^2=36$ (D) $(x+3)^2+(y-4)^2=36$

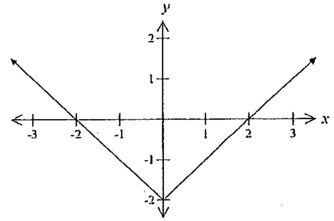
Section I Continued

5 Which of the following is the graph of $y=|x|-2$.

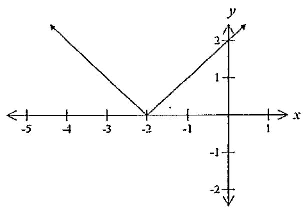
(A)



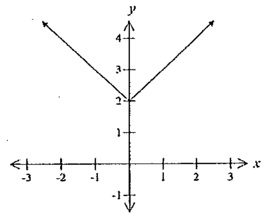
(B)



(C)



(D)



6 Which of the following is true for the equation $2x^2-5x-3=0$?

(A) No real roots.

(B) One real roots.

(C) Two real and rational roots.

(D) Two real and irrational roots.

7 Consider $f(x)=\frac{6}{x}$ and $g(x)=2x+4$. Find the values of x for which $f(x)=g(x)$?

(A) $x=-1$ or $x=3$ (B) $x=-1$ or $x=-3$

(C) $x=1$ or $x=3$ (D) $x=1$ or $x=-3$

Section I Continued

8 Which of the following is true for the function $f(x) = 3x^2 - x$?

(A) $f(x)$ is an even function.

(B) $f(x)$ is an odd function.

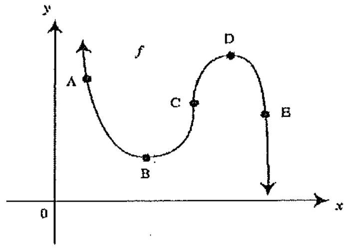
(C) $f(x)$ is neither odd or even.

(D) $f(x)$ is not a function.

9 What is the value of $\lim_{x \to c} \frac{x^2 - c^2}{x - c}$?

(A) Undefined (B) c^2 (C) 0 (D) $2c$

10 The graph below represent the gradient function $y = f'(x)$.



At point A, which of the following statements is true

(A) $f'(x) < 0$ and $f''(x) < 0$

(B) $f'(x) < 0$ and $f''(x) > 0$

(C) $f'(x) > 0$ and $f''(x) < 0$

(D) $f'(x) > 0$ and $f''(x) > 0$

End of Section I

Section II

90 Marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Start each question in a new writing booklet. Extra booklets are available.
All necessary working should be shown in every question.

Question 11 (15 marks) **Start a New Writing Booklet** — **Marks**

(a) Simplify $\frac{x}{2} - \frac{2x-3}{5}$. **2**

(b) Find integers a and b such that $(3-\sqrt{5})^2 = a + b\sqrt{5}$ **2**

(c) Simplify $\frac{\cot^2\theta}{2\operatorname{cosec}^2\theta - 2}$ **2**

(d) Show that the line $y = 2x - 8$ is a tangent to the curve $y = x^2 - 4x + 1$ at the point where $x = 3$. **2**

(e) Given the quadratic equation $x^2 - (2 + k)x + 3k = 0$, find the value of k if the two roots of the equation are reciprocals of each other. **2**

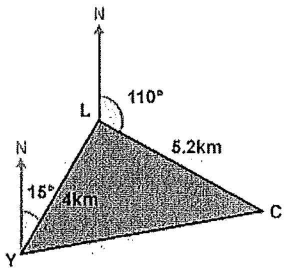
(f) Find $\sum_{n=2}^{4} \frac{6}{n}$ **1**

Question 11 continues on page 6

Section 11 Continued

(g) A hiker begins her journey at a youth hostel, Y, and walks 4km on a bearing of $15°$ to her lunch stop, L. She then walks on a bearing of $110°$ for 5.2 km until she reaches a waterfall, C. The hiker then returns to the youth hostel for the night.

The diagram below represents the hikers journey.



(i) Find the size of the angle $\angle YLC$ **1**

(ii) Find the total distanced walked by the hiker. Give your answer to two decimal places. **2**

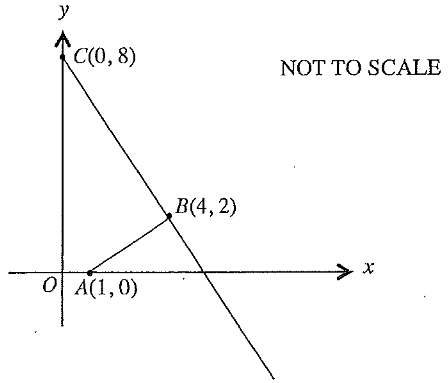
(iii) Find the bearing of the lunch stop from the waterfall. **1**

End of Question 11

Question 12 (15 marks) Start a New Writing Booklet

Marks

(a)



The diagram shows the points $A(1, 0)$, $B(4, 2)$ and $C(0, 8)$ on the Cartesian plane.

(i) Show that the equation of BC is $3x+2y-16=0$. **2**

(ii) Find the length of AB. **1**

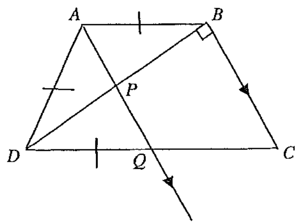
(iii) Find the equation of the circle with centre A that passes through B. **1**

Use the answer sheet provided on page 15 to answer Question 12 (a) part (iv).

(iv) On the answer sheet provided shade the region that satisfies both the inequalities: $3x+2y-16 \geq 0$ and $y \leq 0$ **1**

Section 12 Continued

(b) In the diagram, $AB = AD = DQ$ and $\angle DBC = 90°$. Line AQ is parallel to line BC and meets DB at P and DC at Q.



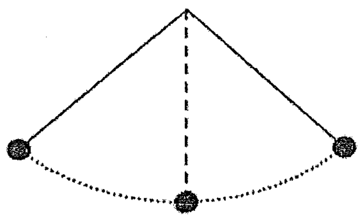
(i) Give a reason why $\angle APB = 90°$. **1**

(ii) Prove that ΔAPD is congruent to ΔDPQ. **2**

(iii) What type of quadrilateral is ABCD, Justify your answer with appropriate reasoning. **2**

(c) The sum of the first 7 terms of an arithmetic progression is 77. If the 1st term is 2, find the 8th term of the series. **2**

(d) A arm of a pendulum swings through an arc on its first swing



Each successive swing is 90% of the length of the previous swing. Let the length of the arc $= x$.

Find the total distance the arc travels before coming to rest. **3**

End of Question 12

Question 13 (15 marks) Start a New Writing Booklet

(a) Differentiate with respect to x:

(i) $x\log_e x$ **2**

(ii) $\dfrac{x}{\sin x}$ **2**

(b) A minute hand of a clock is 10cm long. Calculate the distance travelled by the minute hand between 1:20pm and 1:35 pm. Leave your answer in exact form **2**

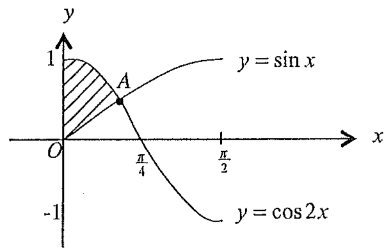
(c) (i) Sketch the parabola, P, which has a focus of $(2, 0)$ and a directrix $y = -2$. **1**

(ii) Find the equation of the parabola P. **2**

(d) Use Simpsons rule, with five function values, to estimate the area under the curve, $y = e^{2x}$, bounded by the curve, the $x-$axis and the lines $x = 1$ and $x = 2$. **3**

(e) The diagram shows the graphs of $y = \sin x$ and $y = \cos 2x$ for $0 \le x \le \frac{\pi}{2}$.

The graphs intersect at $A\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$. Find the exact area of the shaded region. **3**



End of Question 13

Question 14 (15 marks) Start a New Writing Booklet — Marks

(a) Consider the curve $y = xe^x$.

(i) Find the coordinates of the stationary point and determine its nature. **2**

(ii) Find the coordinates of the point of inflexion. **1**

(iii) Sketch the graph $y = xe^x$ showing all important features. **2**

(b) Show that $f(x) = \frac{1}{(2x-3)^3}$ is decreasing for all $x, x \neq \frac{3}{2}$ **2**

(c) The population P of a country town on January 1, 1995 was 1730 and on January 1, 2005 was 1160. The town's population is known to be changing according to the equation $\frac{dP}{dt} = kP$ where t is time in years measured from January 1, 1995 and k is a constant.

(i) Verify that $P = Ae^{kt}$, where A is a constant, satisfies the equation $\frac{dP}{dt} = kP$. **2**

(ii) State the value of A. **1**

(iii) Find the value of k, correct to 4 significant figures. **2**

(iv) Approximately how many people are expected to leave the town during the year 2005? **2**

(v) At what rate is the population changing at the end of 2005? **1**

End of Question 14

Question 15 (15 marks) Start a New Writing Booklet — Marks

(a) Find $\int \frac{4e^{2x}}{1+e^{2x}}\,dx$ **2**

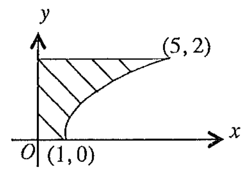
(b) The position of a particle is given by the equation: $x = t^3 - 12t^2 + 36t - 8$, where t is measured in seconds and x in metres.

(i) When does the particle come to rest? **2**

(ii) When does the particle first change direction? **1**

(iii) Find the total distance travelled by the particle during the first 10 seconds. **2**

(c)



The diagram shows the graph of $y = \sqrt{x-1}$ between $(1, 0)$ and $(5, 2)$. The shaded region is rotated about the y axis.

Find the volume of the solid formed. Leave your answer in exact form. **3**

(d) For the function $y = \frac{1}{2}\sin 2x$:

(i) Find the amplitude. **1**

(ii) Find the period. **1**

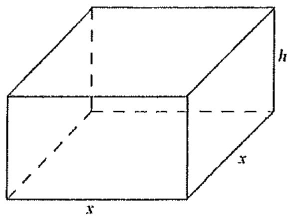
(iii) Sketch the graph $y = \frac{1}{2}\sin 2x$, where $-\pi \leq x \leq \pi$. **2**

(iv) Find the range of values for k that satisfies the equation $\sin 2x = 2k$ where $-\pi \leq x \leq \pi$. **1**

End of Question 15

Question 16 (15 marks) **Start a New Writing Booklet** **Marks**

(a)



The material for the square base of a rectangular box with an open top costs 27 cents per square cm and for the other faces costs 13.5 cents per square centimetre.

(i) Show that the total cost of materials, C, for the box can be written as $C = 27x^2 + 54xh$ **1**

(ii) If the cost of making each box is \$65.61, find an expression for h in terms of x. **1**

(iii) Show that the formula for the volume of the box can be expressed as **2**

$$V = \frac{1}{2}(243x - x^3)$$

(iv) Find the maximum volume of a box than can be produced for \$65.61. **3**

(b) Ally borrowed \$20 000 from a bank to go on a vacation. Interest on the loan is calculated monthly at a rate of 0.5% per month. Her monthly repayments are \$$M$ and the loan is to be paid off after five years.

Let A_n be the amount of money owing on the loan after the n^{th} repayment has been made.

(i) Show that $A_3 = 20000(1.005)^3 - M(1 + 1.005 + 1.005^2)$ **2**

(ii) Show that $A_n = 20000 \times 1.005^n - 200M(1.005^n - 1)$ **1**

(iii) Calculate the value of the monthly repayments required by the bank to pay off the loan in the 5 years. **2**

(iv) After costing her trip, Ally found that she would actually need to borrow \$30 000 for her vacation. If the monthly repayments remained the same, how long would it take Ally to pay off her loan? **3**

End of Question 16

End of Examination

KAMBALA

Student Number: ______________

Trial Examination
Solutions
August 2014

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- Answer Section 1 on the multiple-choice answer sheet attached on page 15
- Answer Section II in the writing booklets provided
  Start each question in a new booklet
- A table of standard integrals is provided on a separate page
- Show all necessary working in Questions 11 – 16

Total marks – 100

Section I
10 marks

- Attempt Questions 1 - 10
  Allow about 15 minutes for this section

Section II
90 marks

- Attempt Questions 11 - 16
  Allow about 2 hours 45 minutes for this section

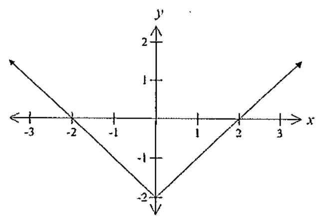
Section I

Qn	Solutions
1	**Question** Find the exact value of $=sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ **Solution** Answer = A
2	**Question** $\|2-5x\|<7$ **Solution** $2-5x<7$ or $2-5x>-7$ $-5x<5$ $-5x>-9$ $x>-1$ $x<\frac{9}{5}$ Answer = B
3	**Question** $4\sqrt{m}-9=0$ $4\sqrt{m}=9$ $\sqrt{m}=\frac{9}{4}$ $m=\frac{81}{16}$ Answer = C
4	**Question** Circle Centre=$(3,-4)$ Radius = 6 **Solution** $(x-3)^2+(y+4)^2=36$ Answer = C

5 **Question**

$y = |x| - 2$

Answer



Answer = B

6 **Question**

Find the roots of $2x^2 - 5x - 3 = 0$

Answer

$\Delta = 25 - 4 \times -3 \times 2 = 48$

Therefore there are two roots of the equation and they are rational as 49 is a square number.

Answer = C

7 **Question**

$f(x) = g(x)$

Solution

$\frac{6}{x} = 2x + 4$

$2x^2 + 4x - 6 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\therefore x = -3, 1$

Answer = D

8 **Question**

$f(x) = 3x^2 - x$

Solution

$f(x) = 3x^2 - x$

$f(-x) = 3x^2 - x$

$\therefore f(x)$ is neither odd nor even

Answer = C

9 **Question**

$\lim_{x \to c} \frac{x^2 - c^2}{x - c}$

Solution

$\lim_{x \to c} \frac{(x-c)(x+c)}{x-c}$

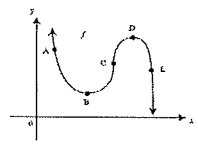
$= \lim_{x \to c}(x + c)$

$= 2c$

Answer =D

10 **Question**

$y = f'(x)$



Solution

$f'(x) > 0$ and $f''(x) < 0$

Answer =B

Question 11 (15 marks) **Marks**

(a) **Simplify** $\frac{x}{2}-\frac{2x-3}{5}$. **2**

$$\frac{5x}{10}-\frac{2(2x-3)}{10}$$

$$=\frac{5x-4x+6}{10}$$

$$=\frac{x+6}{10}$$

(b) **Find integers** a **and** b **such that** $(3-\sqrt{5})^2=a+b\sqrt{5}$ **2**

$9-6\sqrt{5}+5$

$14-6\sqrt{5}$

$\therefore a=14$ and $b=-6$

(c) **Simplify** $\frac{\cot^2\theta}{2\operatorname{cosec}^2\theta-2}$ **2**

$$\frac{cot^2\theta}{2(cosec^2\theta-1)}$$

$$=\frac{cot^2\theta}{2(cot^2\theta)}$$

$$=\frac{1}{2}$$

(d) **Show that the line** $y=2x-8$ **is a tangent to the curve** $y=x^2-4x+1$ **at the point where** $x=3$. **2**

Kambala –Mathematics – HSC Task #4 – Trial Examination – August 2014

Method 1:
sub (1)into (2)
$2x-8=x^2-4x+1$
$x^2-6x+9=0$
$\Delta=6^2-4\times9\times1$
$\Delta=0$

Method 2:
Sub $x=3$into both equations:
Sub $x=3$ into (1)
$y=2\times3-8$
$y=-2$
Sub $x=3$ into (2)
$y=3^2-4(3)+1$
$y=-2$

Method 3:
$(x-3)^2=0$
$x=3$
Sub $x=3$ into (1)
$y=2\times3-8$
$y=-2$

Therefore $y=2x-8$ is a tangent to the curve $y=x^2-4x+1$ as there is only one point of intersection at (3, -2).

(e) **Given the quadratic equation** $x^2-(2+k)x+3k=0$, **find the value of** k **if the two roots of the equation are reciprocals of each other.** **2**

$\alpha+\beta=\frac{-b}{a}$ and $\alpha\beta=\frac{c}{a}$

$\alpha\times\frac{1}{\alpha}=\frac{3k}{1}$

$1=3k$

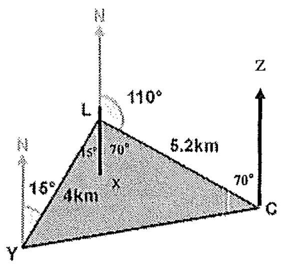
$k=\frac{1}{3}$

(f) **Find** $\sum_{n=2}^{4}\frac{6}{n}$ **1**

$$\sum_{n=2}^{4}\frac{6}{n}=\frac{6}{2}+\frac{6}{3}+\frac{6}{4}=\frac{13}{2}$$

(g) **A hiker begins her journey at a youth hostel, Y, and walks 4km on a bearing of 15° to her lunch stop, L. She then walks on a bearing of 110° for 5.2 km until she reaches a waterfall, C. The hiker then returns to the youth hostel for the night.**

The diagram below represents the hikers journey.



(i) **Find the size of the angle $\angle YLC$** — 1

$\angle YLX = 15°$ (Alternate angles in parallel lines are equal)
$\angle YLX = 70°$ (Straight angle)
$\therefore \angle YLC = 85°$

(ii) **Find the total distanced walked by the hiker. Give your answer to two decimal places.** — 2

$YC^2 = 4^2 + 5.2^2 - 2\times4\times5.2\times cos85$
$YC^2 = 39.4143211$
$YC = 6.28$

Total distance walked $= 4 + 5.3 + 6.28 = 15.48km$

(iii) **Find the bearing of the lighthouse from the waterfall.** — 1

$\angle LCZ = 70°$ (Cointerior angles in parallel lines are supplementary)
The bearing of the lunch stop from the waterfall is $290°T$

Question 12 (15 marks) — Marks

(a) **The points $A(1,0)$, $B(4,2)$ and $C(0,8)$ are on the Cartesian plane.**

(i) **Show that the equation of BC is $3x+2y-16=0$.** — 2

$$m_{bc} = \frac{8-2}{0-4}$$
$$m_{bc} = -\frac{3}{2}$$
$$y - 8 = -\frac{3}{2}(x-0)$$
$$2y - 16 = -3x$$
$$\therefore 3x + 2y - 16 = 0$$

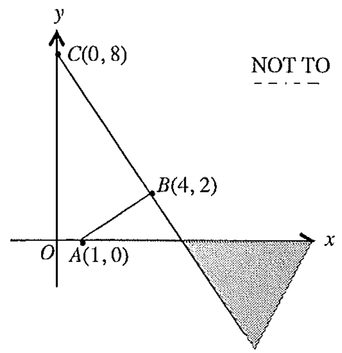
(ii) **Find the length of AB.** — 1

$$d_{AB} = \sqrt{(4-1)^2 + (2-0)^2}$$
$d_{AB} = \sqrt{13}$ units

(iii) **Find the equation of the circle with centre A that passes through B.** — 1

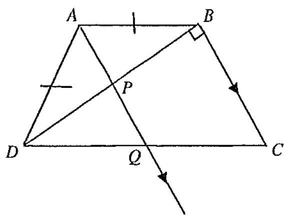
$$(x-1)^2 + y^2 = 13$$

(iv) On the answer sheet provided shade the region that satisfies both the inequalities: $3x+2y-16 \geq 0$ and $y \leq 0$ — 1



Section 12

(b) In the diagram, DQ= $AB = AD$ and $\angle DBC = 90°$. Line AQ is parallel to line BC and meets DB at P and DC at Q.



(i) **Give a reason why $\angle APB = 90°$.** 1

$\angle APB = \angle PBC = 90°$ (Alternate angles in parallel lines are equal)

(ii) **Prove that** Prove that ΔAPB is congruent to ΔDPQ. 2

In ΔABP and ΔDPQ
$DQ = AB$ (given)
$\angle APB = \angle DPQ$ (Vertically opposite angles are equal)
$\angle DAP = \angle PQD$ (Base angles are equal in isosceles triangle)
$\therefore \Delta ABP \equiv \Delta ADP$ (AAS)

(iii) **What type of quadrilateral is ABCD. Justify your answer with appropriate reasoning.** 2

$\angle ABD = \angle QBD$ (Corresponding angles in congruent triangles are equal)

$\therefore AB \parallel DC$ (Alternate angles are equal)
$\therefore ABCD$ is a trapezium (quadrilateral with one pair of parallel lines)

(c) **The sum of the first 7 terms of an arithmetic progression is 77. If the 1st term is 2, find the 8th term of the series.** 2

$a = 2$

$S_7 = 77$

$77 = \frac{7}{2}(4 + 6d)$

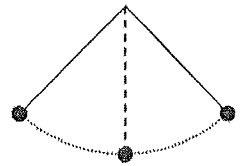
$154 = 28 + 42d$

$154 = 42d$

$d = 3$

$T_8 = 2 + 7\times 3 = 23$

(d) **A arm of a pendulum swings through an arc on its first swing**



Each successive swing is 90% of the length of the previous swing. Let the length of the arc = x. Find the total distance the arc travels before coming to rest. 3

$x + 0.9x + 0.9^2x + 0.9^3x \dots\dots$

$s_\infty = \frac{x}{1 - 0.9}$

$s_\infty = 10x$

Question 13 (15 marks)

(a) Differentiate with respect to x:

(i) $xlog_e x$ 2

$\frac{dy}{dx} = x\frac{1}{x} + log_e x$

$\frac{dy}{dx} = 1 + log_e x$

(ii) $\frac{x}{\sin x}$ 2

$\frac{dy}{dx} = \frac{sinx\times 1 - x\times cosx}{\sin^2 x}$

$\frac{dy}{dx} = \frac{sinx - xcosx}{\sin^2 x}$

(b) **A minute hand of a clock is 10cm long. Calculate the distance travelled by the minute hand between 1:20pm and 1:35 pm. Leave your answer in exact form** 2

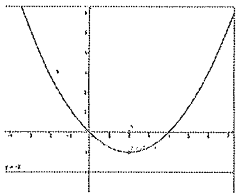
$r = 10 \text{ and } \theta = \frac{\pi}{2}$

$l = r\theta$

$l = \frac{10\pi}{2}$ cm

$l = 5\pi$ cm

(c) (i) **Sketch the parabola, P, which has a focus of $(2, 0)$ and a directrix $y = -2$.** 1



(ii) **Find the equation of the parabola P.** 2

$(x-2)^2 = 4(y+1)$

(d) **Use Simpsons rule, with five function values, to estimate the area under the curve, $y = e^{2x}$, bounded by the curve, the $x-$axis and the lines $x = 1$ and $x = 2$.** 3

x	1	1.25	1.5	1.75	2
$y = e^{2x}$	e^2	$e^{2.5}$	e^3	$e^{3.5}$	e^4

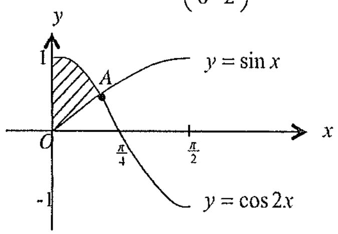
$A \approx \frac{1.25-1}{3}(e^2 + 4(e^{2.5}) + 2(e^3) + 4(e^{3.5}) + e^4)$

$A \approx \frac{1}{12}(e^2 + 4(e^{2.5}) + 2(e^3) + 4(e^{3.5}) + e^4)$

$A \approx 23.6$ units squared

(e) **The diagram shows the graphs of $y = \sin x$ and $y = \cos 2x$ for $0 \le x \le \frac{\pi}{2}$.**

The graphs intersect at $A\left(\frac{\pi}{6}, \frac{1}{2}\right)$. Find the exact area of the shaded region. 3



$A = \int_0^{\frac{\pi}{6}} \cos(2x)\,dx - \int_0^{\frac{\pi}{6}} \sin(x)\,dx$

$A = \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{6}} - \left[-\cos(x)\right]_0^{\frac{\pi}{6}}$

$A = \frac{1}{2}\sin\frac{\pi}{3} - \frac{1}{2}\sin 0 - (-\cos\frac{\pi}{6} + \cos 0)$

$A = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1$

$A = \frac{\sqrt{3} + 2\sqrt{3} - 4}{4} = \frac{3\sqrt{3} - 4}{4}$

Question 14 (15 marks) **Marks**

(a) **Consider the curve $y = xe^x$.**

(i) **Find the coordinates of the stationary point and determine its nature.** 2

$y' = xe^x + e^x$

$y' = e^x(x+1)$ and $y'' = xe^x + 2e^x$

Stationary points when $y' = 0$

$0 = e^x(x+1)$

$e^x = 0$ and $(x+1) = 0$

$x = -1$

$y = \frac{-1}{e}$

When $x = -1$

$y'' = -1e^{-1} + e^{-1} + 1$

$y'' > 0$ therefore point $\left(-1, \frac{-1}{e}\right)$ is a minimum

(ii) **Find the coordinates of the point of inflexion.** 1

Points of inflexion when $y'' = 0$

$y'' = xe^x + 2e^x$

$0 = e^x(x+2)$

$x = -2, y = \frac{2}{e^2}$

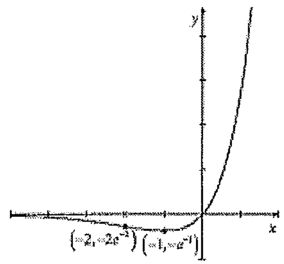
Test for change of concavity

x	-2.1	-2	-1.9
y''	-	0	+

Therefore change of concavity at $x = -2$

Therefore there is a point of inflexion at $(-2, \frac{2}{e^2})$

(iii) **Sketch the graph $y = xe^x$ showing all important features.** 2



Show that $f(x)=\dfrac{1}{(2x-3)^3}$ **is decreasing for all** x, $x \neq \dfrac{1}{2}$ **2**

$f'(x) = -3(2x-3)^{-2} \times 2$

$f'(x) = \dfrac{-6}{(2x-3)^2}$

$(2x-3)^2 > 0$

$\therefore \dfrac{-6}{(2x-3)^2} < 0$

$\therefore f'(x)$ is decreasing for all values of x $x \neq \dfrac{1}{2}$

(c) The population P of a country town on January 1, 1995 was 1730 and on January 1, 2005 was 1160. The town's population is known to be changing according to the equation $\dfrac{dP}{dt} = kP$ where t is time in years measured from January 1, 1995 and k is a constant.

(i) **Verify that** $P = Ae^{kt}$, **where** A **is a constant, satisfies the equation** $\dfrac{dP}{dt} = kP$. **2**

$\dfrac{dP}{dt} = A \times ke^{kt}$

$\dfrac{dP}{dt} = kP$

(ii) **State the value of** A. **1**

$A = 1730$

(iii) **Find the value of** k**, correct to 3 significant figures.** **2**

$1160 = 1730 \times e^{10k}$

$\dfrac{1160}{1730} = e^{10k}$

$\dfrac{1160}{1730} = e^{10k}$

$log_e\left(\dfrac{116}{173}\right) = 10k$

$k = -0.03997$

(iv) **Approximately how many people are expected to leave the town during the year 2005?** **2**

$A_{10} = 1730 \times e^{-0.03997(11)} = 1114$

$1160 - 1114 = 45$

Therefore 45 people are expected to leave in 2005

(v) **At what rate is the population changing at the end of 2005?**

$\dfrac{dP}{dt} = -0.03997 \times 1114 = -44.53$

The towns population is decreasing at a rate of 44.53 people per year at the end of 2005.

Question 15 (15 marks)

(a) **Find** $\displaystyle\int \frac{4e^{2x}}{1+e^{2x}}\,dx$ **2**

$= 2\log_e(1 + e^{2x}) + c$

(b) **The position of a particle is given by the equation:** $x = t^3 - 12t^2 + 36t - 8$**, where t is measured in seconds and** x **in metres.**

(i) **When does the particle come to rest?** **2**

$\dot{x} = 3t^2 - 24t + 36$

Particle comes to rest when $\dot{x} = 0$

$3t^2 - 24t + 36 = 0$

$t^2 - 8t + 12 = 0$

$(t-6)(t-2) = 0$

$t = 2s$ and 6 seconds

(ii) **When does the particle first change direction?** **1**

when $t = 2$

(iii) **Find the total distance travelled by the particle during the first 10 seconds.** **2**

Total distance travelled

$T = 0\ at\ x = -8$

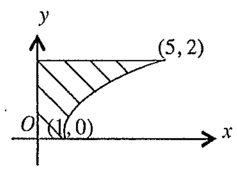
$T = 2\ x = 24$

$T = 6\ x = -8$

$T = 10\ \ x = 152$

Total Therefore total distance travelled is 224 metres.

(c)



The diagram shows the graph of $y = \sqrt{x-1}$ **between** $(1, 0)$ **and** $(5, 2)$.
The shaded region is rotated about the y **axis.**

Find the volume of the solid formed. **3**

$v = \pi \int_0^2 x^2\,dy$ and $x = y^2 + 1$

$v = \pi \displaystyle\int_0^2 x^2\,dy$

$v = \pi \displaystyle\int_0^2 (y^4 + 2y^2 + 1)\,dy$

$v = \pi \left[\dfrac{y^5}{5} + \dfrac{2y^2}{3} + y\right]_0^2$

$v = \frac{206}{15}\pi\ units^3$

(d) **For the function** $y=\frac{1}{2}\sin 2x$ **:**

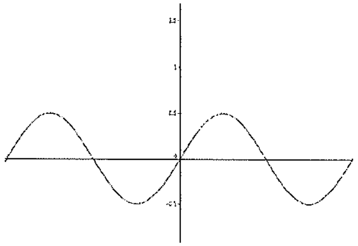
(i) **Find the amplitude.** **1**

$a=\frac{1}{2}$

(ii) **Find the period.** **1**

Period $=\pi$

(iii) **Sketch the graph** $y=\frac{1}{2}\sin 2x$ **, where** $-\pi \le x \le \pi$**.** **2**



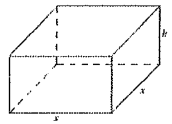
(

iv) **Find the range of values for** k **that satisfies the equation** **1**
$\sin 2x = 2k$ **where** $-\pi \le x \le \pi$**.**

$-0.5 \le k \le 0.5$

Question 16 (15 marks) Marks

(a)



The material for the square base of a rectangular box with an open top costs 27 cents per square cm and for the other faces costs 13.5 cents per square centimetre.

(i) **Show that the total cost of materials,** C**, for the box can be written as** **1**

$C=27x^2+54xh$

$SA=x^2+4xh$

$\therefore C=27x^2+54xh$

(ii) **If the cost of making each box is \$65.61, find an expression for** h **in terms of** x**.** **1**

$C=27x^2+54xh$

$6561=27x^2+54xh$

$54xh=6561-27x^2$

$54xh=6561-27x^2$

$h=\frac{243}{2x}-\frac{x^2}{2}$

(iii) **Show that the formula for the volume of the box can be expressed as** **2**

$V=Ah$

$V=x^2h$

$V=x^2\left(\frac{243}{2x}-\frac{x^2}{2}\right)$

$V=\frac{1}{2}(243x-x^3)$

(iv) **Find the maximum volume of a box than can be produced for \$65.61.** **3**

Maximum volume when $V'=0$

$V'=\frac{1}{2}(243-3x^2)$

$0=\frac{1}{2}(243-3x^2)$

$3x^2=243$

$x^2=81$

$x=\pm 9, as\ x>0$

$x=9$

$V''=\frac{1}{2}(-6x)$

$V''=-3x$

when $x=9$

$V''=-27<0$

Therefore a maximum volume at $x=9$

$V=\frac{1}{2}(243\times 9-9^3)=729\text{ cm}^3$

(b) Ally borrowed \$20 000 from a bank to go on a vacation. Interest on the loan is calculated monthly at a rate of 0.5% per month. Her monthly repayments are \$$M$ and the loan is to be paid off after five years.

Let A_n be the amount of money owing on the loan after the n^{th} repayment has been made.

(i) Show that $A_3 = 20000 \times 1.005^3 - M(1+1.005+1.005^3)$ **2**

$A_1 = 20000\times(1.005) - M$
$A_2 = (20000\times(1.005) - M)(1.005) - M$
$A_2 = 20000(1.005)^2 - M(1.005) - M$
$A_3 = 20000(1.005)^3 - M(1.005)^2 - M(1.005) - M$
$A_3 = 20000(1.005)^3 - M(1 + 1.005 + 1.005^2)$

(ii) Show that $A_n = 20000 \times 1.005^n - 200M(1.005^n - 1)$ **1**

$A_n = 20000(1.005)^n - M(1 + 1.005 + 1.005^2 + \cdots \ldots + 1.005^{n-1})$

$S_n = \dfrac{1(1.005^n - 1)}{1.005 - 1}$

$S_n = \dfrac{1.005^n - 1}{0.005}$

$\therefore A_n = 20000(1.005)^n - M(\dfrac{1.005^n - 1}{0.005})$

$A_n = 20000(1.005)^n - 200M(1.005^n - 1)$

(iii) Calculate the value of the monthly repayments required by the bank to pay off the loan in the 5 years. **2**

$A_n = 0$
$-20000(1.005)^{60} = -200M(1.005^{60} - 1)$

$M = \dfrac{-20000(1.005)^{60}}{-200(1.005^{60} - 1)}$

$M = 386.66$

The monthly repayments are \$386.66

(iv) **After costing her trip, Ally found that she would actually need to borrow \$30 000 for her vacation. If the monthly repayments remained the same, how long would it take Ally to pay off her loan?** **3**

$0 = 30000(1.005)^n - 200M(1.005^n - 1)$
$200M(1.005^n) - 200M = 30000(1.005)^n$
$200M(1.005^n) - 30000(1.005^n) = 200M$
$(1.005^n)(200M - 30000) = 200M$

$(1.005^n) = \dfrac{200M}{(200M - 30000)}$

$(1.005^n) = \dfrac{200\times386.66}{(200\times386.66 - 30000)}$

$(1.005^n) = 1.6338$
$n = \log_{1.005} 1.6338$

$n = \dfrac{\log_{10} 1.6338}{\log_{10} 1.005}$

$n = 98.43$
Therefore it will take 99 months (8 years 3 months) to pay off the loan.