VIRILE AGITUR

Student Number

Knox Grammar School

2008

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time - 2 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 84

- Attempt Questions 1 – 7
- Answer each question in a separate writing booklet
- All questions are of equal value

Subject Teachers

Mr I. Bradford
Mr M. Vuletich
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Mr J. Harnwell

This paper MUST NOT be removed from the examination room

Number of Students in Course: 66

Number of Writing Booklets Per Student (Four Page) 7

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available

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Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int_{-1}^{1} \frac{2}{\sqrt{4-x^2}}\,dx$. **2**

(b) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$. **2**

(c) The interval AB has end points A (2, 4) and B (x, y). The point P (-1, 1) divides AB internally in the ratio 3:4. Find the coordinates of B. **2**

(d) Find the size of the acute angle between the tangents to the curve $y = \tan^{-1} x$ at the points where $x = 1$ and $x = \sqrt{3}$. **3**

Give your answer correct to the nearest minute.

(e) Solve $\frac{2}{1+2x} \geq 1$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A monic polynomial $P(x)$ of degree 3 has a double root at $x = 1$ and $P(2) = 13$. **2**

Write $P(x)$ as a product of its factors.

(b) Find the general solution to $2\sin x - 1 = 0$ in terms of π. **2**

(c) Use the substitution $u = \tan x$, to evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$. **3**

(d) (i) Sketch the graph of $y = 2\cos^{-1}\left(\frac{x}{\pi}\right)$. **2**

(ii) Consider the region bounded by the curve between $x = 0$, $y = 0$ and $y = \frac{\pi}{2}$. **3**

Show that $x = \pi\cos\left(\frac{y}{2}\right)$, hence find the area of this region.

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the coefficient of x^{12} in the expansion of $\left(2x^2+\frac{1}{x^2}\right)^{12}$ **2**

Leave your answer in the form ${}^{12}C_r 2^k$.

(b) (i) Express $\sqrt{3}\cos 2t-\sin 2t$ in the form $R\cos(2t+\alpha)$, where $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Hence or otherwise find all positive solutions of $\sqrt{3}\cos 2t-\sin 2t=0$ for $0\le t\le \pi$. **2**

(c) Consider the functions $f(x)=2\cos\frac{\pi x}{3}$ and $g(x)=2-x$

(i) Sketch the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes in the domain $0\le x\le 6$. **2**

(ii) Use your graph to find the number of solutions for the equation $2\cos\frac{\pi x}{3}+x-2=0$ in the domain $0\le x\le 6$. **1**

(iii) Use one application of Newton's method to find a further approximation of the root near $x=4$, for $2\cos\frac{\pi x}{3}+x-2=0$. **3**

Give your answer correct to two significant figures.

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x)=\frac{4-x^2}{1+x^2}$.

(i) Determine the coordinates and nature of any turning points, x and y intercepts and any asymptotes. Sketch the graph of $y=f(x)$ showing these important features. **4**

(ii) What is the largest domain containing the value $x=2$ for which $f(x)$ has an inverse function $f^{-1}(x)$? **1**

(iii) Give the equation of the inverse function, $f^{-1}(x)$ in terms of x. **2**

(b) A particle P moves in a straight line in simple harmonic motion. The acceleration in metres per second per second is given by

$$\ddot{x}=2-3x$$

where x metres, is the displacement of the particle from the origin.
Initially the particle is at $x=1$ moving with a velocity of $\sqrt{5}\ \text{ms}^{-1}$.

(i) Using integration show that the velocity v ms^{-1} of the particle is given by **2**

$v^2=4+4x-3x^2$.

(ii) Find the amplitude of motion. **1**

(iii) Find the centre of motion. **1**

(iv) Find the maximum speed of the particle. **1**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

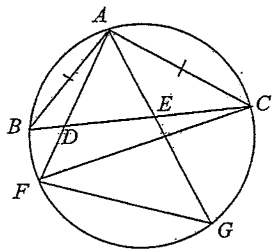
(a) The point $A(2ap,\ ap^2)$ lies on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal at A is $x + py = 2ap + ap^3$. **2**

(ii) Given that the normal at A also passes through the point $R(-6a, 9a)$, show that $p^3 - 7p + 6 = 0$. **1**

(iii) Hence, find the values of p on this parabola at which the normals to the parabola intersect at R. **2**

(b)



NOT TO SCALE

The diagram shows an isosceles triangle ABC inscribed in a circle with $AB = AC$. D and E are two points on the base BC of the triangle. AD and AE are produced to meet the circle at the points F and G respectively.

(i) Copy this diagram into your writing booklet and show that $\angle ADE = \angle ACF$. **2**

(ii) Show that $DEGF$ is a cyclic quadrilateral. **2**

(c) Use mathematical induction to prove that for all integers $n \geq 1$ **3**

$$\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}.$$

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If α, β, γ are the roots of the equation $x^3 + 2x^2 - x - 5 = 0$, find the value of **2**

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

(b) The population N, of a particular species of bears in a region, after t years can be expressed as:

$$N = \frac{A}{15} + Ae^{-(\ln 2)t} \text{ where } A \text{ is a constant.}$$

Given that the initial population was 600 bears,

(i) find the value of A. **1**

(ii) Find the population of bears after 10 years. **1**

(iii) Find the time required for the population to decrease to 42 bears. **2**

(b) The velocity v ms^{-1} of a particle at position x metres from the origin can be calculated using the equation $v = \pm\sqrt{x^3(4-x)}$.

(i) Show that the acceleration $\ddot{x}$ is equal to $2x^2(3-x)$. **2**

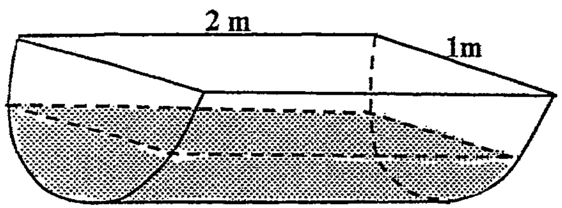
(ii) Initially the particle is 4 metres to the right of the origin. In what direction will the particle travel immediately after leaving its initial position? **1**

(iii) Find the maximum speed of the particle and state where it occurs. **2**

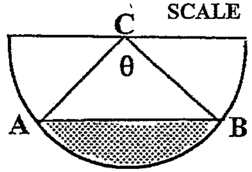
(iv) Write a brief description of the motion of this particle as it moves from $x = 4$ to $x = 0$. **1**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows an aquarium tank 2 metres long with a semi-circular cross-section of diameter 1 metre as shown.



Tank



Cross-section

NOT TO SCALE

In the diagram of the cross-section, C is at the centre of the top edge, AB represents the water level and $\angle ACB = \theta$ where θ is measured in radians.

(i) Show that the volume of the water in the tank is given by **1**

$$V = \frac{1}{4}(\theta - \sin\theta).$$

(ii) Show that the depth, d, of the water is given by **1**

$$d = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2}\right).$$

(iii) Water is poured into the tank at the rate of 0.1 m^3/min. Find the exact rate at which the water level is rising when the depth of water is 0.2 m. **3**

Question 7 Continued on page 8

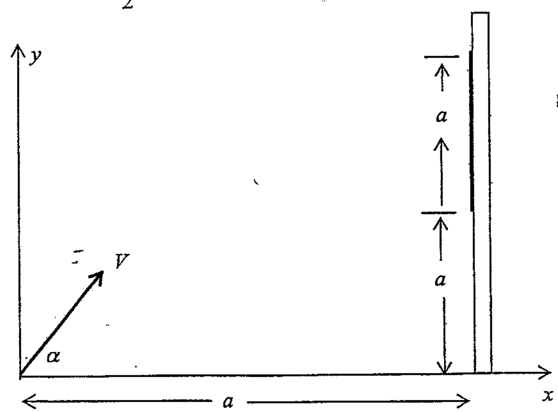
Question 7 (continued) **Marks**

(b) A cannon can fire a projectile with velocity $V = \sqrt{kga}$ where k and a are positive constants and at an angle α to the horizontal.

The cannon is placed on horizontal ground, a metres from a vertical building which has a large target fixed to it. The target is a metres tall with its lower edge set a metres above the ground.

Using axes as shown in the diagram, you may assume the position of the projectile, t seconds, after being fired is given by

$$x = Vt\cos\alpha, \quad y = -\frac{1}{2}gt^2 + Vt\sin\alpha,$$ where g is the acceleration due to gravity.



NOT TO SCALE

(i) Show that the Cartesian equation of the particle's position can be written as: **2**

$$y = x\tan\alpha - \frac{x^2}{2ka}\sec^2\alpha.$$

(ii) Show that the projectile will hit the base of the target if **3**
$\tan^2\alpha - 2k\tan\alpha + (2k+1) = 0$
and hence show that if $k < 1 + \sqrt{2}$ then the projectile will always hit the building below the target.

(iii) Given that $k = 3$, show that the target will be hit only if $3 - \sqrt{2} \le \tan\alpha \le 3 + \sqrt{2}$. **2**

End of paper

SOLUTIONS

KNOX EXTENSION 1 MATHEMATICS TRIAL 2008.

QUESTION 1

a) $\int_{-1}^{1} \frac{2}{\sqrt{4-x^2}}\,dx = 2\int_{-1}^{1} \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= 2\left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1}$ (1)

$= 2\left[\sin^{-1}(\tfrac{1}{2}) - \sin^{-1}(-\tfrac{1}{2})\right]$

$= 2\left[\frac{\pi}{6} + \frac{\pi}{6}\right]$

$= \frac{2\pi}{3}$ (1)

b) $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2}$ (1)

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin\pi - (0 - \frac{1}{2}\sin 0)\right]$

$= \frac{\pi}{4}$ (1)

c) $A(2,4)$ $B(x,y)$ $P(-1,1)$

$\frac{4(2)+3(x)}{3+4} = -1$, $\frac{4(4)+3(y)}{3+4} = 1$

$3x+8 = -7$ $\quad$ $3y+16 = 7$

$x = -5$ $\quad$ $y = -3$

$\therefore B(-5,-3)$

(1) (1)

d) $y = \tan^{-1}x$, $x=1$, $x=\sqrt{3}$.

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$\therefore m_1 = \frac{1}{2}$ $\quad m_2 = \frac{1}{4}$ (1)

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$\tan\theta = \left|\frac{\frac{1}{2}-\frac{1}{4}}{1+\frac{1}{2}\times\frac{1}{4}}\right|$ (1)

$\tan\theta = \left|\frac{\frac{1}{4}}{\frac{9}{8}}\right|$

$\tan\theta = \frac{2}{9}$

$\theta = 12^\circ 32'$ (nearest min). (1)

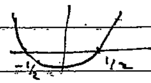
e) $\frac{2}{1+2x} \geq 1$ $\quad (x \neq -\frac{1}{2})$

$2(1+2x) \geq (1+2x)^2$

$(1+2x)^2 - 2(1+2x) \leq 0$ (1)

$(1+2x)\left[(1+2x)-2\right] \leq 0$

$(1+2x)(2x-1) \leq 0$ (1)



$\therefore -\frac{1}{2} < x \leq \frac{1}{2}$

(1)

QUESTION 2

(a) $P(x)$, monic, degree 3

double root $x=1$.

$\therefore P(x) = (x-1)^2(x-a)$ (1)

$P(2) = 2-a$

$\therefore 2-a = 13$

$a = -11$

$\therefore P(x) = (x-1)^2(x+11)$ (1)

(b) $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$ (1)

$\therefore x = n\pi + (-1)^n \sin^{-1}(\tfrac{1}{2})$

$x = n\pi + (-1)^n \frac{\pi}{6}$ (1)

(c) $\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

For $x=0$, $u=0$

$x = \frac{\pi}{4}$, $u = 1$

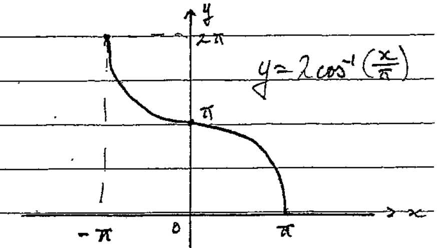
$\therefore \int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \int_0^1 \frac{du}{\sqrt{1-u^2}}$ (2)

$= \left[\sin^{-1}u\right]_0^1$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2}$ (1)

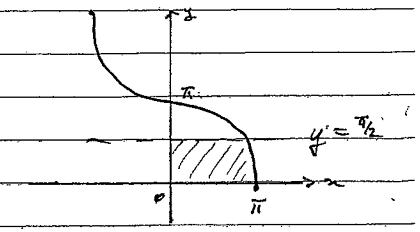
d) i)



(1) SHAPE

(1) LABEL

ii)



$y = 2\cos^{-1}\left(\frac{x}{\pi}\right)$

$\frac{y}{2} = \cos^{-1}\left(\frac{x}{\pi}\right)$

$\frac{x}{\pi} = \cos\left(\frac{y}{2}\right)$

$\therefore x = \pi\cos\left(\frac{y}{2}\right)$ (1)

$\text{Area} = \int_0^{\pi/2} x\,dy$

$= \pi\int_0^{\pi/2} \cos\frac{y}{2}\,dy$

$= 2\pi\left[\sin\frac{y}{2}\right]_0^{\pi/2}$ (1)

$= 2\pi\left[\sin\frac{\pi}{4} - \sin 0\right]$

$= 2\pi \times \frac{1}{\sqrt{2}}$

$= \sqrt{2}\,\pi$ units^2. (1)

QUESTION 3

(a) $\left(2x^2+\frac{1}{x^2}\right)^{12}$

$= \sum_{k=0}^{12} {}^{12}C_k \, (2x^2)^k \left(\frac{1}{x^2}\right)^{12-k}$

$= \sum_{k=0}^{12} {}^{12}C_k \, 2^k \, x^{2k} \, . \, x^{-24+2k}$

$= \sum_{k=0}^{12} {}^{12}C_k \, . \, 2^k \, . \, x^{4k-24}$ (1)

$\therefore 4k-24 = 12$

$k = 9$

Coefficient of x^{12} is ${}^{12}C_9 \times 2^9$ (1)

(b) i) $\sqrt{3}\cos 2t - \sin 2t$

$R = \sqrt{(\sqrt{3})^2+1^2}$

$R = 2$ $\tan\alpha = \frac{1}{\sqrt{3}}, \alpha = \frac{\pi}{6}$

$\therefore 2\left(\frac{\sqrt{3}}{2}\cos 2t - \frac{1}{2}\sin 2t\right)$

$= 2\cos\left(2t+\frac{\pi}{6}\right)$ (2)

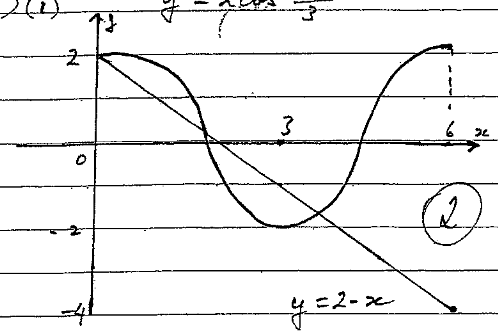
ii) $2\cos\left(2t+\frac{\pi}{6}\right) = 0$

$2t + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ (1)

$2t = \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$

For $0 \le t \le \pi$, $t = \frac{\pi}{6}$ or $\frac{2\pi}{3}$ (1)

(c)(i) $y = 2\cos\frac{\pi x}{3}$



(2)

ii) Number of solutions $\therefore 2\cos\frac{\pi x}{3} + x - 2 = 0$

is 3 solutions. (1)

iii) Let $h(x) = 2\cos\frac{\pi x}{3} + x - 2$

If $x_1 = 4$.

$x_2 = 4 - \frac{h(4)}{h'(4)}$ (1)

$h(4) = 2\cos\frac{4\pi}{3} + 2$

$h'(x) = -\frac{2\pi}{3}\sin\frac{\pi x}{3} + 1$

$h'(4) = -\frac{2\pi}{3}\sin\frac{4\pi}{3} + 1$

$\therefore x_2 = 4 - \frac{2\cos\frac{4\pi}{3}+2}{-\frac{2\pi}{3}\sin\frac{4\pi}{3}+1}$

$x_2 = 3.6$ (2 S.F.)

(2)

QUESTION 4

(a) $f(x) = \frac{4-x^2}{1+x^2}$

i) $f'(x) = \frac{-2x(1+x^2) - 2x(4-x^2)}{(1+x^2)^2}$

$= \frac{-2x - 2x^3 - 8x + 2x^3}{(1+x^2)^2}$

$= \frac{-10x}{(1+x^2)^2}$

$f'(x) = 0$ when $x = 0$

$f'(0-\epsilon) > 0$

$f'(0+\epsilon) < 0$

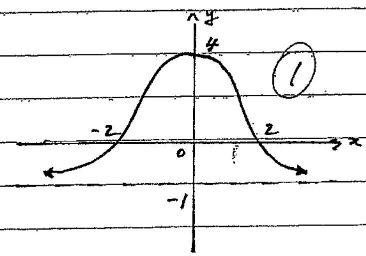
$\therefore (0,4)$ is a rel. maximum (2)

when $x = 0 \Rightarrow y = 4$

when $y = 0 \Rightarrow 4 - x^2 = 0$

$x = \pm 2.$

Horizontal asymptote at $y = -1$. (1)



(1)

ii) $x \geq 0$ (1)

iii) $y = \frac{4-x^2}{1+x^2}$

$\therefore x = \frac{4-y^2}{1+y^2}$

$x + xy^2 = 4 - y^2$

$xy^2 + y^2 = 4 - x$

$y^2(x+1) = 4 - x$

$y^2 = \frac{4-x}{x+1}$

$\therefore y = \sqrt{\frac{4-x}{x+1}}$ $y \geq 0$ for inverse.

$\therefore f^{-1}(x) = \sqrt{\frac{4-x}{x+1}}$ (2)

b) $\ddot{x} = 2 - 3x$

i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 3x$

$\frac{1}{2}v^2 = 2x - \frac{3x^2}{2} + c$

$v^2 = 4x - 3x^2 + k$

when $x = 1$, $v = \sqrt{5}$

$\therefore 5 = 4(1) - 3(1) + k$

$k = 4$

$\therefore v^2 = 4x - 3x^2 + 4$

$v^2 = 4 + 4x - 3x^2$ (2)

ii) when $v = 0$, $3x^2 - 4x - 4 = 0$

$(3x+2)(x-2) = 0.$

$x = -\frac{2}{3}$ or $x = 2$

Amplitude $= \frac{2-\left(-\frac{2}{3}\right)}{2} = \frac{4}{3}$ m. (1)

iii) Centre of motion $x = \frac{2}{3}$ (1)

iv) Max. speed when $x = \frac{2}{3}$

$\therefore$ Max. speed $= \frac{4\sqrt{3}}{3}$ m/s

QUESTION 5

(a) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

at $x = 2ap \Rightarrow \frac{dy}{dx} = p$

Gradient normal $= -\frac{1}{p}$.

Eqn. normal,

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ (2)

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

ii) $(-6a, 9a)$

$\therefore -6a + 9ap = 2ap + ap^3$

$ap^3 - 7ap + 6a = 0$

$p^3 - 7p + 6 = 0 \quad a \neq 0$. (1)

iii) $p^3 - 7p + 6 = 0$

Let $P(p) = p^3 - 7p + 6$

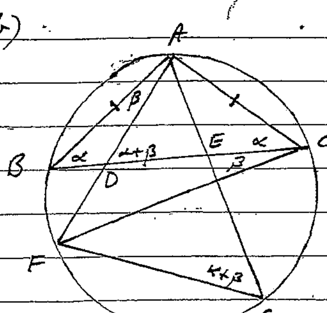
$P(1) = 0 \Rightarrow p = 1$ is a root.

$P(p) = (p-1)(p^2 + p - 6)$

$= (p-1)(p+3)(p-2)$

$\therefore$ Values of p are $1, 2$ or -3. (2)

(b)



i) To prove $\angle ADE = \angle ACF$

Let $\angle ABD = \alpha$, $\angle BAF = \beta$

$\angle ACE = \alpha$, given ΔABC is isosceles.

$\angle BCF = \beta$, angles in same segment on arc BF

$\therefore \angle ACF = \alpha + \beta$

$\angle ADE = \alpha + \beta$ exterior angle ΔBA

$\therefore \angle ADE = \angle ACF$. (2)

ii) $\angle ACF = \alpha + \beta$

$\angle AGF = \alpha + \beta$, angles in same segment, arc AF.

$\therefore \angle ADE = \angle AGF$

Since the exterior angle of Quad. DEGF is equal to interior opposite then DEGF is a cyclic quad (2)

c) To prove $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$, $n \geq 1$

For $n = 1$

LHS $= \sum_{r=1}^{1} \frac{r}{2^r} = \frac{1}{2^1} = \frac{1}{2}$ RHS $= 2 - \frac{1+2}{2^1} = \frac{1}{2}$.

Assume $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$ when $n = k$.

ie $\sum_{r=1}^{k} \frac{r}{2^r} = 2 - \frac{k+2}{2^k}$

When $n = k+1$

$\sum_{r=1}^{k+1} \frac{r}{2^r} = \sum_{r=1}^{k} \frac{r}{2^r} + \frac{k+1}{2^{k+1}}$

$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

$= 2 - \left[\frac{2k+4-(k+1)}{2^{k+1}}\right]$

$= 2 - \left[\frac{k+3}{2^{k+1}}\right]$

$= 2 - \frac{(k+1)+2}{2^{k+1}}$ (3)

$=$ RHS. when $n = k+1$ etc.

QUESTION 6.

(a) $x^3 + 2x^2 - x - 5 = 0$

$\alpha + \beta + \gamma = -2$

$\alpha\beta + \beta\gamma + \alpha\gamma = -1$

$\alpha\beta\gamma = 5$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$

$= -\frac{1}{5}$ (2)

(b) $N = \frac{A}{15} + Ae^{-(\ln 2)t}$

i) $N = 600$, $t = 0$

$600 = \frac{A}{15} + Ae^0$

$600 = \frac{16A}{15}$

$A = \frac{1125}{2} = 562.5$ (1)

ii) when $t = 10$

$N = \frac{A}{15} + Ae^{-(\ln 2)\times 10}$

$N \doteq 38$ (whole number) (1)

iii) when $N = 42$

$42 = \frac{A}{15} + Ae^{-(\ln 2)t}$

$\frac{42 - \frac{A}{15}}{A} = e^{-\ln 2\, t}$

$\ln\left[\frac{1}{A}\left(42 - \frac{A}{15}\right)\right] = -\ln 2\, t$

$t = -\frac{\ln\left[\frac{1}{A}\left(42 - \frac{A}{15}\right)\right]}{\ln 2}$

$t \doteq 6.966$

$t \doteq 7$ years. (2)

(b). $v = \pm\sqrt{x^3(4-x)}$

(i) $v^2 = x^3(4-x)$

$\frac{1}{2}v^2 = \frac{1}{2}x^3(4-x)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\left[3x^2(4-x) + (-1)x^3\right]$

$= \frac{1}{2}\left[12x^2 - 3x^3 - x^3\right]$

$= \frac{1}{2}\left[12x^2 - 4x^3\right]$

$= \frac{1}{2} \cdot 4\left[3x^2 - x^3\right]$

$= 2x^2(3-x)$

$\therefore \ddot{x} = 2x^2(3-x)$ (2)

ii) $t = 0$, $x = 4 \Rightarrow v = 0$

$\ddot{x} = -32$

$\therefore$ travels to the left after leaving from initial position. (1)

iii) Max. speed when $\ddot{x} = 0$

$\ddot{x} = 0$ at $x = 0$ and $x = 3$.

at $x = 0 \Rightarrow v = 0$

at $x = 3 \Rightarrow v = -\sqrt{27}$

$\therefore$ Max speed is $\sqrt{27}$ m/s at $x = 3$. (2)

(iv) The particle starts 4m to the right of the origin and accelerates left, reaching its max speed at $x=3$ before slowing down until it reaches $x=0$ where it momentarily stops and then accelerates to the right. (1)

QUESTION 7

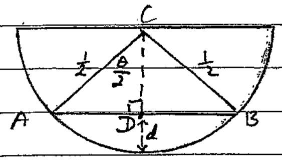
a) i) Area of minor segment AB $= \frac{1}{2}\left(\frac{1}{2}\right)^2(\theta - \sin\theta)$

$= \frac{1}{8}(\theta - \sin\theta)$

$\therefore$ Volume $= 2 \times \text{Area}$

$= \frac{1}{4}(\theta - \sin\theta)$ (1)

ii)



$CD = \frac{1}{2}\cos\frac{\theta}{2}$

$\therefore d = \frac{1}{2} - \frac{1}{2}\cos\frac{\theta}{2}$ (1)

iii) $\frac{dV}{dt} = 0\cdot1$, $\frac{dd}{dt} = ?$

$\frac{dd}{dt} = \frac{dd}{d\theta}\cdot\frac{d\theta}{dt}$

$\frac{d\theta}{dt} = \frac{d\theta}{dV}\cdot\frac{dV}{dt}$

$= \frac{4}{1-\cos\theta} \times 0\cdot1$

$= \frac{2}{5(1-\cos\theta)}$ (1)

$\therefore \frac{dd}{dt} = \frac{1}{4}\sin\frac{\theta}{2} \times \frac{2}{5(1-\cos\theta)}$

$= \frac{\sin\theta/2}{10(1-\cos\theta)}$

when $d = 0\cdot2$,

$0\cdot2 = \frac{1}{2} - \frac{1}{2}\cos\frac{\theta}{2}$

$0\cdot4 = 1 - \cos\frac{\theta}{2}$

$\cos\theta/2 = 0\cdot6$

$\theta = 2\cos^{-1}(0\cdot6)$ (1)

$\therefore \frac{dd}{dt} = \frac{\sin\theta/2}{10(1-\cos\theta)}$ when $\theta = 2\cos^{-1}(0\cdot6)$

$= \frac{\sin[\cos^{-1}(0\cdot6)]}{10[1-\cos(2\cos^{-1}0\cdot6)]}$

$= \frac{\sin[\cos^{-1}(0\cdot6)]}{10[1-(2\cos^2(\cos^{-1}(0\cdot6))-1)]}$

$= \frac{0\cdot8}{10(1-(0\cdot72-1))}$

$= \frac{1}{16}$ m/min. (1)

(b)

i) $x = Vt\cos\alpha$, $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

$V = \sqrt{kga}$

$t = \frac{x}{V\cos\alpha}$

$\therefore y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + \frac{Vx}{V\cos\alpha}\sin\alpha$

$= -\frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} + x\tan\alpha$

$= x\tan\alpha - \frac{1}{2}\cdot\frac{gx^2}{kga}\cdot\sec^2\alpha$

$= x\tan\alpha - \frac{x^2}{2ka}\sec^2\alpha$. (2)

ii) To hit base of target, $x = a$, $y = a$

$\therefore a = a\tan\alpha - \frac{a}{2k}\sec^2\alpha$

$1 = \tan\alpha - \frac{1}{2k}\sec^2\alpha$

$\frac{1}{2k}\sec^2\alpha - \tan\alpha + 1 = 0$

$\sec^2\alpha - 2k\tan\alpha + 2k = 0$

$(1+\tan^2\alpha) - 2k\tan\alpha + 2k = 0$

$\tan^2\alpha - 2k\tan\alpha + (2k+1) = 0$ (2)

PTO

QUESTION 7 cont'd.

$\tan^2\alpha - 2k\tan\alpha + 2k + 1 = 0$

$\Delta = 4k^2 - 4\cdot1\cdot(2k+1)$

$= 4k^2 - 8k - 4$

$= 4(k^2 - 2k - 1)$

Below target, $\Delta < 0$

$k^2 - 2k - 1 < 0$

$k = \frac{2 \pm \sqrt{8}}{2}$

$k = 1 \pm \sqrt{2}$

$\therefore$ below target if (1)

$1-\sqrt{2} < k < 1+\sqrt{2}$

iii) $k = 3$, $x = a$, $a \le y \le 2a$

$\therefore a \le a\tan\alpha - \frac{a}{2k}\sec^2\alpha \le 2a$

$1 \le \tan\alpha - \frac{1}{2k}\sec^2\alpha \le 2$ $(a>0)$

$1 \le \tan\alpha - \frac{1}{6}(1+\tan^2\alpha) \le 2$ $(k=3)$

$6 \le 6\tan\alpha - 1 - \tan^2\alpha \le 12$

$-12 \le \tan^2\alpha - 6\tan\alpha + 1 \le -6$ (1)

$\therefore \tan^2\alpha - 6\tan\alpha + 13 \ge 0$ or $\tan^2\alpha - 6\tan\alpha + 7 \le 0$

No solution

$\tan\alpha = \frac{6 \pm \sqrt{36 - 4.1.7}}{2}$

$\tan\alpha = \frac{6 \pm \sqrt{8}}{2}$

$\tan\alpha = 3 \pm \sqrt{2}$

$\therefore$ Hits target if (1)

$3 - \sqrt{2} \le \tan\alpha \le 3 + \sqrt{2}$