KINCOPPAL-ROSE BAY
Cor Unum

2008
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) $\displaystyle\int \frac{2x}{\sqrt{1-x^4}}\,dx$ 2

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ 3

(c) $\displaystyle\int_1^{e^2} 3x^2 \ln x \; dx$ 3

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2-x+1}}$ 2

(e) (i) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

(ii) Use this property to show that $\displaystyle\int_0^1 x^3(1-x)^6\,dx = \frac{1}{840}$ 3

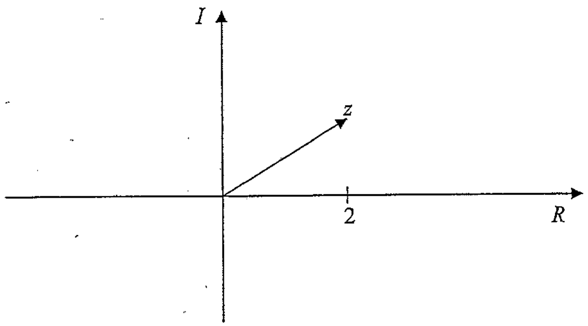
Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A complex number z is given by $z = \sqrt{3} + i$

(i) Evaluate $\bar{z}$. Verify that $z\bar{z}$ is real. 2

(ii) Find $\frac{1}{z}$ in the form $a + ib$, where a and b are real. 1

(b) A point z on the Argand Diagram is given below:



Copy this diagram into your examination booklet and use it to plot the following points. Clearly label each point.

(i) $\bar{z}$ 1

(ii) $2iz$ 1

(iii) $\frac{1}{z}$ 1

(c) Express $i - 1$ in modulus argument form, and hence simplify $(i-1)^5$ 2

Question 2 continues on page 4

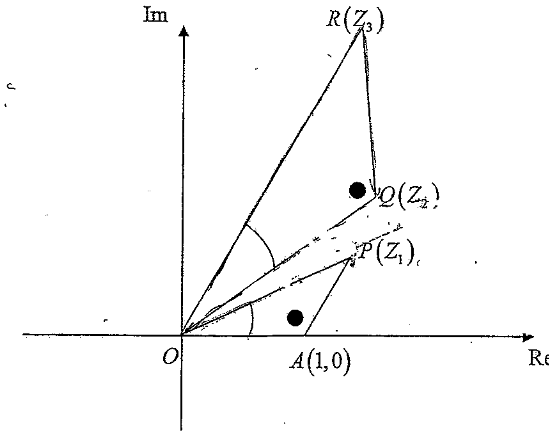
Question 2 (continued)

(d) Sketch the locus and state its equation:

(i) $|z-2| = |z-2i|$ 2

(ii) $z\bar{z} - 3(z + \bar{z}) \le 0$ 2

(e)



In the figure above, the points P, Q and A represent the complex numbers Z_1, Z_2 and $(1,0)$ respectively. Given $\angle OAP = \angle OQR$ and $\angle AOP = \angle QOR$.

Explain why $R(Z_3)$ represents the complex number $Z_1 Z_2$. 3
You must support your answer with clear and complete mathematical reasons.

End of Question 2

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) If $P(x)=x^4+x^3-3x^2-5x-2$, show that $P(x)=0$ has a multiple root, find this root and its multiplicity. 3

(ii) Hence factorise $P(x)=x^4+x^3-3x^2-5x-2$ into its linear factors. 1

(b) The equation $x^3+2x-1=0$ has roots α,β,γ. Find the monic equations with roots

(i) $\alpha^2,\beta^2,\gamma^2$. 2

(ii) $\alpha\beta,\ \beta\gamma,\ \alpha\gamma$ 3

(iii) Evaluate $\alpha^3+\beta^3+\gamma^3$ 2

(c) A point $P\left(ct,\frac{c}{t}\right)$ lies on the rectangular hyperbola $xy=c^2$.

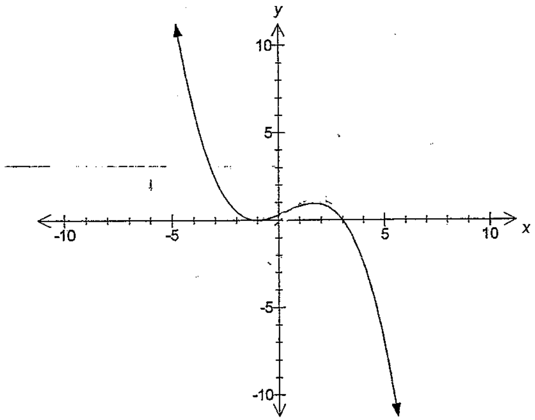
(i) Show that the equation of the tangent at the point $P\left(ct,\frac{c}{t}\right)$ on the rectangular hyperbola is given by $x+t^2y=2ct$. 2

(ii) Prove that the area bounded by the tangent and the asymptotes of the rectangular hyperbola is a constant. 2

End of Question 4

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph of $f(x)=\frac{1}{10}(x+1)^2(3-x)$ is drawn above.

On separate diagrams, draw a neat sketch showing the main features of each of the following:

(i) $y=f(x-1)$ 1

(ii) $y=f(|x|)$ 1

(iii) $y=\{f(x)\}^2$ 2

(iv) $y=xf(x)$ 2

(v) $y^2=f(x)$ 2

(vi) $y=e^{f(x)}$ 2

(b) Given that $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \; dx$ show that :

(i) $I_n=\frac{1}{n-1}\left((\sqrt{2})^{n-2}+(n-2)I_{n-2}\right)$ 4

(ii) Hence or otherwise evaluate I_4 1

End of Question 3

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) ABC is an equilateral triangle, inscribed in a circle. X is a point on the minor arc BC.

(i) Prove that $\Delta BDX \,|||\, \Delta ACX$ — 3

(ii) Prove that $XB + XC = XA$ — 3

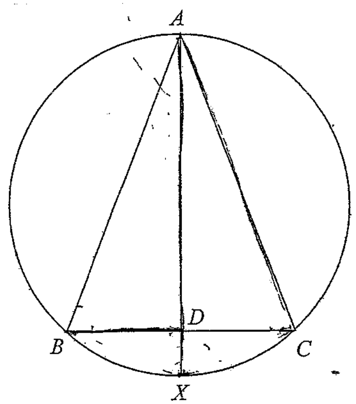


DIAGRAM NOT TO SCALE

(b) State whether each of the following are true or false giving brief reasons for your answers:

(i) $\int_0^{\pi} \sin 9x \; dx = 0$

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \; dx = 0$ — 2

(c) Find the equation of the tangent to the curve $\cos 2x + \sin y = 1$ at the point $x = \dfrac{\pi}{6}$. — 3

(d) Use the substitution $x = a\sin\theta$ to show that — 3

$$\int \sqrt{(a^2 - x^2)} \; dx = \frac{1}{2} a^2 \sin^{-1}\frac{x}{a} + \frac{1}{2} x\sqrt{(a^2 - x^2} + C$$

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given $a\alpha^2 + b\alpha + c = 0$ where $a,b,c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$, prove that $a\left(\overline{\alpha}\right)^2 + b\overline{\alpha} + c = 0$ — 2

(ii) A polynomial $P(x)$ with real coefficients, has two of its zeros $3i$ and $1+2i$. Find in expanded form, a possible polynomial $P(x)$. — 3

(b) Use De Moivres Theorem and binomial expansion to find an expression for $\cos 4\theta$ in terms of $\cos\theta$. — 3

(c) (i) Given $z = \cos\theta + i\sin\theta$, prove $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ — 2

(ii) Hence by considering the expansion $\left(z + \dfrac{1}{z}\right)^4$ show that — 3

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4\theta \; d\theta$ — 2

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of the polynomial $p(x)=x^3+ax^2+bx+c=0$ are three consecutive terms of an arithmetic series. Prove that the relationship between the coefficients is given by $2a^3-9ab+27c=0$ **4**
Hint: make an appropriate choice for the roots in arithmetic progression.

(b) A point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>0$ and $b>0$.
The equation of the normal at the point $P(a\cos\theta, b\sin\theta)$ is given by
$xa\sin\theta - yb\cos\theta = (a^2-b^2)\sin\theta\cos\theta$.

(i) Show that the ellipse intersects the rectangular hyperbola $xy=c^2$ in four points if $ab>2c^2$ **3**

(ii) Show that for $0<\theta<\frac{\pi}{2}$, the normal at P on the ellipse intersects the hyperbola in two distinct points, say A and B. **3**

(iii) If M is the mid-point of AB, show that the coordinates of M are given by **2**

$$\left(\frac{(a^2-b^2)\cos\theta}{2a}, -\frac{(a^2-b^2)\sin\theta}{2b}\right)$$

(iv) Hence find the locus of M as θ varies. **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the function $y=\cos^{-1}(e^x)$,

(i) Find the domain and the range. **2**

(ii) Draw a neat sketch the graph of $y=\cos^{-1}(e^x)$. **2**

(iii) Hence draw a neat sketch of the curve $y=\frac{1}{(\cos^{-1}(e^x))}$ **2**

(b) (i) Using induction, show that for each positive integer n, there are unique positive integers p_n and q_n such that: $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$ **4**

(ii) Show also that $p_n^{\,2} - 2q_n^{\,2} = (-1)^n$. **1**

(c) If $f(xy)=f(x)+f(y)$, for all $x, y \neq 0$, prove that

(i) $f(1)=f(-1)=0$ **2**

(ii) $f(x)$ is an even function. **2**

End of paper

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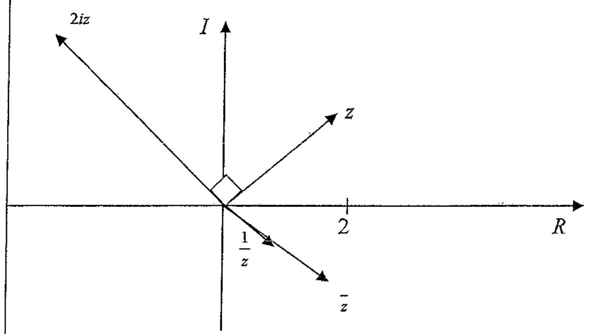
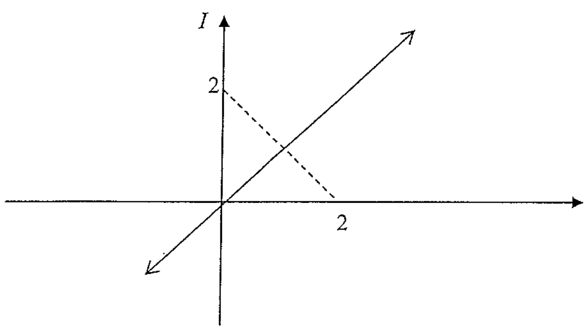
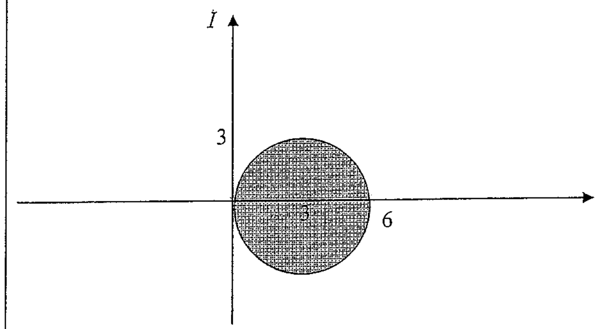
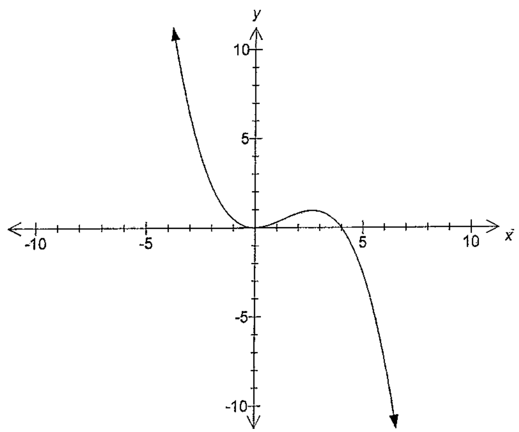
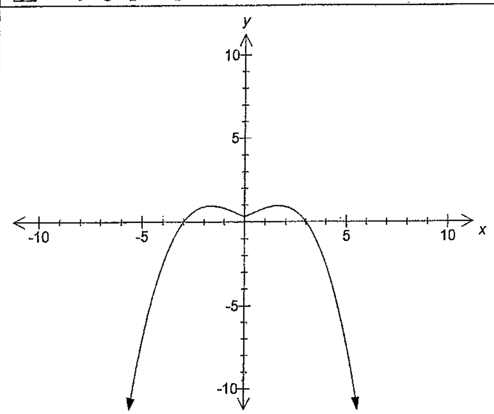
Question	Criteria	Marks	Bands
1(a)	$\int \frac{2x}{\sqrt{1-x^4}}\,dx$ *Let* $u=x^2$ $\therefore \frac{du}{dx}=2x$ *or* $dx=\frac{du}{2x}$ ☑ $\therefore \int \frac{2x}{\sqrt{1-x^4}}\,dx = \int \frac{2x}{\sqrt{1-u^2}}\cdot\frac{du}{2x}$ $= \int \frac{1}{\sqrt{1-u^2}}\,du$ $=\sin^{-1}u+C$ $=\sin^{-1}x^2+C$ ☑	2	
1(b)	$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ *Let* $t=\tan\frac{x}{2}$ $\therefore \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]=\frac{1}{2}\left[1+t^2\right]$ *or* $dx=\frac{2\,dt}{1+t^2}$ *and* $\cos\theta=\frac{1-t^2}{1+t^2}$ ☑ $\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$ since $t=\tan\frac{\frac{\pi}{2}}{2}=1$ *and* $t=\tan\frac{0}{2}=0$ $=\int_0^1 \frac{1}{\frac{2(1+t^2)}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$ $=\int_0^1 \frac{1+t^2}{3+t^2}\cdot\frac{2\,dt}{1+t^2}$ ☑ $=\int_0^1 \frac{2}{3+t^2}\,dt$ $=2\int_0^1 \frac{1}{3+t^2}\,dt$ $=2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$ ☑ $=\frac{\pi}{3\sqrt{3}}$	3	

1(c)	$\int_1^{e^2} 3x^2\ln x\ dx$ *Let* $u=\ln x$ $\frac{dv}{dx}=3x^2$ $\frac{du}{dx}=\frac{1}{x}$ $v=x^3$ $\therefore \int_1^{e^2} 3x^2\ln x\ dx = uv - \int v\ du$ ☑ $=\left[x^3 \ln x\right]_1^{e^2} - \int x^3\frac{dx}{x}$ $=\left[x^3 \ln x\right]_1^{e^2} - \int x^2\ dx$ $=\left[x^3 \ln x\right]_1^{e^2} - \left[\frac{x^3}{3}\right]_1^{e^2}$ ☑ $=\left[e^6\ln e^2-1^3\ln 1\right]-\left[\frac{e^6}{3}-\frac{1}{3}\right]_1^{e^2}$ $=2e^6-\frac{e^6}{3}+\frac{1}{3}$ $=\frac{5e^6+1}{3}$ ☑	3	
1(d)	$\int \frac{dx}{\sqrt{x^2-x+1}} = \int \frac{1}{\sqrt{x^2-x+(\frac{1}{2})^2+1-(\frac{1}{2})^2}}\,dx$ ☑ $=\int \frac{1}{\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}}\,dx$ $=\log\lvert(x-\frac{1}{2})+\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}\rvert+C$ ☑	2	
1(e)(i)	$\int_0^a f(a-x)\ dx$ *Let* $u=a-x$ $\therefore \frac{du}{dx}=-1$ *If* $x=a$ *then* $u=a-a=$ $x=0$ *then* $u=a-0=a$ ☑ $\int_0^a f(u)\ .-du = -\int_a^0 f(u)\ du$ $=\int_0^a f(u)\ du$ ☑	2	
1(e)(ii)	$\int_0^1 x^3(1-x)^6\ dx = \int_0^1 (1-x)^3(1-(1-x))^6\ dx$ ☑ $=\int_0^1 (1-3x+3x^2-x^3)x^6\ dx$ $=\int_0^1 x^6-3x^7+3x^8-x^9\ dx$ ☑ $=\left[\frac{x^7}{7}-\frac{3x^8}{8}+\frac{x^9}{3}-\frac{x^{10}}{10}\right]_0^1$ ☑ $=\frac{1}{840}$	3	

Question	Criteria	Marks	Bands
2(a)(i)	$z\bar{z}=\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)=4$ $\therefore z\bar{z}$ is real.	2	
2(a)(ii)	$\frac{1}{z}=\frac{1}{\left(\sqrt{3}+i\right)}\cdot\frac{\left(\sqrt{3}-i\right)}{\left(\sqrt{3}-i\right)}=\frac{\left(\sqrt{3}-i\right)}{4}$	1	
2(b) (i)-(iii)		3	
2(c)	$i-1=\sqrt{2}cis\frac{3\pi}{4}$ $(i-1)^5=\left(\sqrt{2}\right)^5 cis\frac{15\pi}{4}=4\sqrt{2}cis\frac{7\pi}{4}$ $=4\sqrt{2}\left\{\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right\}$ $=4-4i$	1 1	
2(d)(i)	 $y=x$	1 1	

2(d)(ii)	 $(x-3)^2+y^2\le 9$	1 1	
2(e)	$\angle AOR=\angle AOQ+\angle QOR$ $=\angle AOQ+\angle AOP$ *i.e.* $\arg z_3=\arg z_2+\arg z_1$ ① the triangles ORQ and OPA are equiangular and hence similar $\therefore$ their sides are proportional $\frac{OR}{OP}=\frac{OQ}{OA}=\frac{\lvert z_3\rvert}{\lvert z_1\rvert}=\frac{\lvert z_2\rvert}{\lvert 1\rvert}$ ① $\lvert z_3\rvert=\frac{\lvert z_2\rvert\lvert z_1\rvert}{1}$ $=\lvert z_2\rvert\lvert z_1\rvert$ $\therefore z_3=z_2.z$ *i.e* R represents the complex number z_2z_1 ①	1 1 1	

Question	Criteria	Marks	Bands
3(a)(i)	 ☑ *shift graph* 1 *place to the right*	1	
3(a)(ii)	 ☑ *reflection in y axis*	1	

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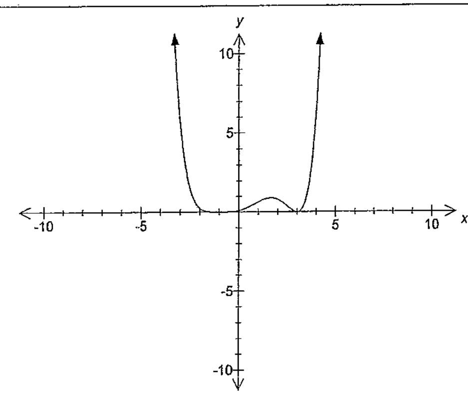
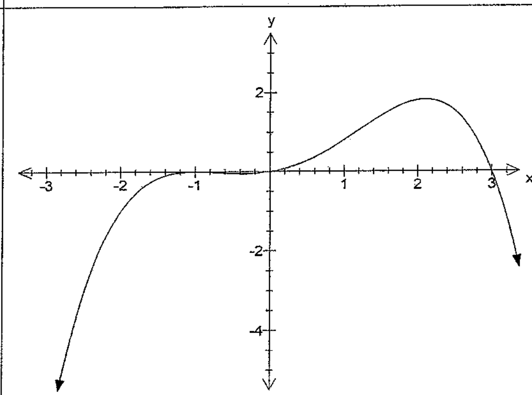
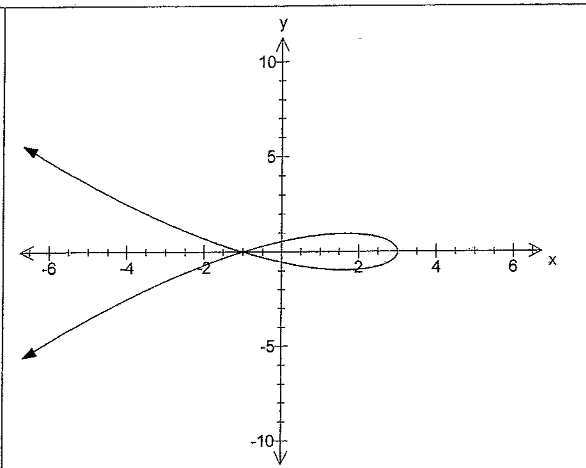
3(a)(iii)	 ☑ graph above x-axis ☑ graph $0 < y < 1$ (less steep gradient) and graph $y > 1$ steeper gradient	2	
3(a)(iv)	 ☑ 3 roots at -1, 0 and 3 ☑ 2 stationary points	2	

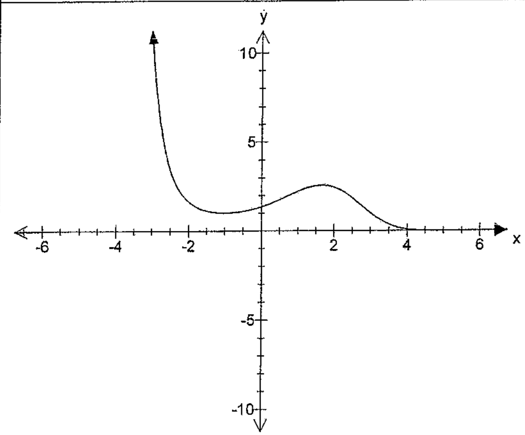
3(a)(v) — 2 marks



☑ sketch only +ve section of orginal graph and reflect in x - axis

☑ *for $-1<y<1$ the sketch is less steep and more steep after $y<-1$ and $y>1$*

3(a)(vi) — 2 marks



☑ *where cuts $x-axis$ now becomes $(-1,1)$ and $(3,1)$*

☑ anything that was negative now becomes an asymptote above the $x-axis$.

3(b)(i) — 4 marks

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \ dx$$

$$\int \sec^n x \ dx = \int \sec^{n-2} x. \ \sec^2 x \, dx$$ ☑

where $u = \sec^{n-2} x$ $\quad \dfrac{dv}{dx} = \sec^2 x$

$\dfrac{du}{dx} = (n-2)\sec^{n-3} x \sec x \tan x$ $\quad v = \tan x$

$$\int \sec^{n-2} x. \ \sec^2 x \, dx = uv - \int v \ du$$

$$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan^2 x \ dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \ dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1) \ dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x - \sec^{n-2} dx$$ ☑

$$\therefore \int \sec^n x \ dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x - (n-2)\int \sec^{n-2} dx$$

$$\int \sec^n x \ dx + (n-2)\int \sec^n x = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} dx$$

$$(n-1)\int \sec^n x \ dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} dx$$

$$\int \sec^n x \ dx = \frac{1}{n-1}\sec^{n-2} x \tan x - \frac{n-2}{n-1}\int \sec^{n-2} dx$$ ☑

$$I_n = \frac{1}{n-1}\left[\left[\sec^{n-2} x \tan x\right]_0^{\frac{\pi}{4}} - (n-2)I_{n-2}\right]$$

$$I_n = \frac{1}{n-1}\left[\left[\sec^{n-2}(\tfrac{\pi}{4})\tan(\tfrac{\pi}{4}) - \sec^{n-2}(0)\tan(0)\right] - (n-2)I_{n-2}\right]$$ ☑

$$I_n = \frac{1}{n-1}\left[\left(\sqrt{2}\right)^{n-2} - (n-2)I_{n-2}\right]$$

3(b)(ii) — 1 mark

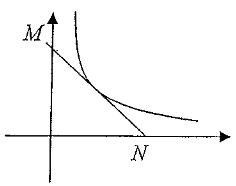
$$I_4 = \frac{1}{3}\left[\left(\sqrt{2}\right)^2 - 2I_2\right]$$

$$I_2 = \frac{1}{2}\left[\left(\sqrt{2}\right)^0 - 0I_0\right] = \frac{1}{2}$$

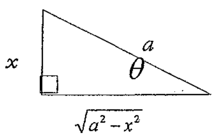
$$\therefore \ I_4 = \frac{1}{3}\left[\left(\sqrt{2}\right)^2 - 2\left(\frac{1}{2}\right)\right]$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$ ☑

Question	Criteria	Marks	Bands
4(a)(i)	$P'(x)=4x^3+3x^2-6x-5$ $P''(x)=12x^2+6x-6$ $12x^2+6x-6=0 \Rightarrow x=-1,-\frac{1}{2}$ $P'(-1)=0$ $\therefore x=-1$ is a root of multiplicity 3	1 1 1	
4(a)(ii)	$x^4+x^3-3x^2-5x-2=(x+1)^3(x-2)$	1	
4(b)(i)	$\left(\sqrt{x}\right)^3+2\sqrt{x}-1=0$ $x\sqrt{x}+2\sqrt{x}=1$ $\left(\sqrt{x}(x+2)\right)^2=1$ $x\left(x^2+4x+4\right)=1$ $x^3+4x^2+4x-1=0$	1 1	
4(b)(ii)	$\alpha\beta,\alpha\gamma,\beta\gamma=\frac{\alpha\beta\gamma}{\gamma},\frac{\alpha\beta\gamma}{\beta},\frac{\alpha\beta\gamma}{\alpha}=\frac{1}{\gamma},\frac{1}{\beta},\frac{1}{\alpha}$ $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)-1=0$ $\frac{1}{x^3}+\frac{2}{x}-1=0$ $x^3-2x^2-1=0$	1 1 1	
4(b)(iii)	$\alpha^3+2\alpha-1=0$ $\beta^3+2\beta-1=0$ $\gamma^3+2\gamma-1=0$ $\left(\alpha^3+\beta^3+\gamma^3\right)+2(\alpha+\beta+\gamma)-3=0$ $\therefore \alpha^3+\beta^3+\gamma^3=3-2(0)$ $=3$	1 1	
4(c)(i)	$y'=-\frac{c}{x^2}$ at $\left(ct,\frac{c}{t}\right)$ $y'=-\frac{1}{t^2}$ $y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$ $yt^2-ct=-x+ct$ $yt^2+x=2ct$	1 1	
4(c)(ii)	$M: x=0 \Rightarrow y=\frac{2c}{t}$ $N: y=0 \Rightarrow x=2ct$ area $=\frac{1}{2}(2ct)\left(\frac{2c}{t}\right)$ $=2c^2$ which is a constant 	1 1	

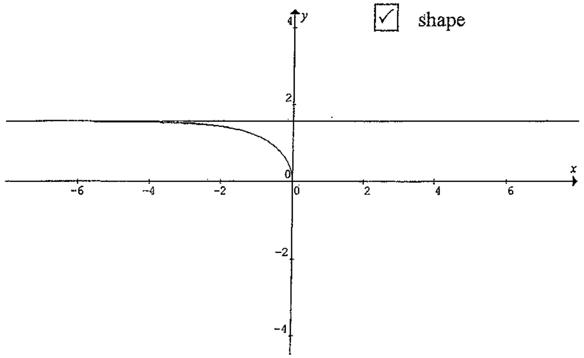
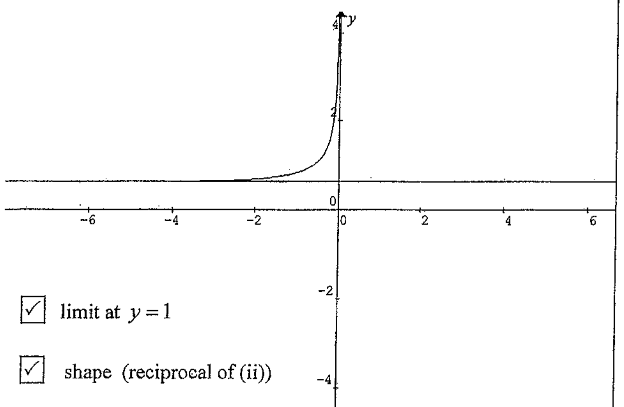
Question	Criteria	Marks	Bands
5(a)(i)	$\angle BAC=60$ (ΔABC *is equliateral* Δ) $\angle BXC=120$ (opposite angles in cyclic quad ABXC are supplementary) $\angle AXC=\angle ABC=60$ ($\angle$*'s on smae arc at circumference are equal*) $\therefore \angle AXC=\angle AXB=60$ *in* ΔBDX *and* ΔACX $\angle DXC=\angle AXB=60$ (*proved above*) ☑ $\angle DXB=\angle CAX$ ($\angle$*'s at circumference on same arc*) ☑ $\therefore \Delta BDX \vert\vert\vert \Delta ACX$ (*eqiangular*) ☑	3	
5(a)(ii)	$\Delta CDX \vert\vert\vert \Delta ABX$ (*as proved in (i) above*) since $\Delta BDX \vert\vert\vert \Delta ACX$ $\therefore \frac{BD}{AC}=\frac{BX}{AX}=\frac{DX}{CX}$ *and* $\frac{CD}{AB}=\frac{CX}{AX}=\frac{DX}{BX}$ $\therefore BD=\frac{BX.AC}{AX}$ *and* $CD=\frac{CX.AB}{AX}$ ☑ since $BC=BD+DC$ *hence* $BC=\frac{BX.AC}{AX}+\frac{CX.AB}{AX}$ ☑ $BC.AX=(BX.AC)+(CX.AB)$ *and as* $BC=AC=AB$ (*equaliteral* Δ) $\therefore \div$ *LHS and RHS by BC* ☑ $\therefore AX=BX+CX$	3	
5(b)(i)	$y=\sin 9x$ *when* $0<x<\pi$ *it has* $4\frac{1}{2}$ *cycles,* *more area above* $x-$*axis then below* $\therefore \int_0^\pi \sin 9x\,dx \neq 0$ (*false*) ☑	1	
5(b)(ii)	$x\sin x$ *is an even function* ☑ $\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx=2\int_0^{\frac{\pi}{2}} x\sin x\,dx \neq 0$ ☑ (*false*)	2	

5(c)	slope tangent $=\frac{dy}{dx}$: (differentiating implicitly)	3	
	$-2\sin 2x+\cos y\frac{dy}{dx}=0$		
	$\cos y\frac{dy}{dx}=2\sin 2x$		
	$\frac{dy}{dx}=\frac{2\sin 2x}{\cos y}$ ☑		
	At $(\frac{\pi}{6},\frac{\pi}{6})$, slope of tangent = 2 ☑		
	Equation of tangent is		
	$y-\frac{\pi}{6}=2(x-\frac{\pi}{6})$		
	$y=2x-\frac{\pi}{6}$ ☑		
5(d)	Let $x=a\sin\theta \quad dx=a\sin\theta\, d\theta$	3	
	$\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta d\theta=a^2\int\cos^2\theta\, d\theta$ ☑		
	$=\frac{a^2}{2}\int\cos 2\theta+1\, d\theta=$		
	$\frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta+\theta\right]+C=\frac{a^2}{2}[\sin\theta\cos\theta+\theta]+C$ ☑		
	From this triangle :		
			
	$\frac{x}{a}=\sin\theta$		
	$\cos\theta=\frac{\sqrt{a^2-x^2}}{a}$		
	$\theta=\sin^{-1}a$		
	$\frac{a^2}{2}[\sin\theta\cos\theta+\theta]+C=$ ☑		
	$\frac{a^2}{2}\left[\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}+\sin^{-1}\frac{x}{a}\right]+C=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$		

Question	Criteria	Marks	Bands
6(a)(i)	$\overline{a\alpha^2+b\alpha+c}=\overline{0}$	1	
	$\overline{a\alpha^2}+\overline{b\alpha}+\overline{c}=0$		
	$a\overline{\alpha^2}+b\overline{\alpha}+c=0$		
	$a\overline{\alpha}^2+b\overline{\alpha}+c=0$	1	
	$\therefore\overline{\alpha}$ is a solution		
6(a)(ii)	$(x-3i)(x+3i)(x-(1+2i))(x-(1-2i))$	1	
	$(x^2+9)(x^2-2x+5)$	1	
	$x^4-2x^3+14x^2-18x+45$	1	
6(b)	$\cos 4\theta+i\sin 4\theta=(\cos\theta+i\sin\theta)^4$		
	$=\cos^4\theta+4i\cos^3\theta\sin\theta-6\cos^2\theta\sin^2\theta-4i\cos\theta\sin^3\theta+\sin^4\theta$	1	
	equate real part:	1	
	$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$		
	$=\cos^4\theta-6\cos^2\theta(1-\cos^2\theta)+(1-\cos^2\theta)^2$		
	$=8\cos^4\theta-8\cos^2\theta+1$	1	
6(c)(i)	$z^n=\cos n\theta+i\sin n\theta$		
	$z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$	1	
	$=\cos n\theta-i\sin n\theta$		
	$z^n+z^{-n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$	1	
	$=2\cos n\theta$		
6(c)(ii)	$\left(z+\frac{1}{z}\right)^4=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$	1	
	$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$		
	$=2\cos 4\theta+4(2\cos 2\theta)+6$	1	
	$\left(z+\frac{1}{z}\right)^4=(2\cos\theta)^4=16\cos^4\theta$	1	
	$\therefore 16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$		
	$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$		
6(c)(iii)	$\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}d\theta$		
	$=\left[\frac{1}{8}\frac{\sin 4\theta}{4}+\frac{1}{2}\frac{\sin 2\theta}{2}+\frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$	1	
	$=\frac{1}{32}\sin 2\pi+\frac{1}{4}\sin\pi+\frac{3\pi}{16}-0$		
	$=\frac{3\pi}{16}$	1	

Question	Criteria	Marks	Bands
7(a)	Let roots be: $\alpha-d,\alpha,\alpha+d$ sum of roots: $3\alpha=-a \quad \Rightarrow \alpha=-\frac{a}{3}$ $\alpha=-\frac{a}{3}$ is a root to: $x^3+ax^2+bx+c=0$ $\left(-\frac{a}{3}\right)^3+a\left(-\frac{a}{3}\right)^2+b\left(-\frac{a}{3}\right)+c=0$ $-\frac{a^3}{27}+\frac{a^3}{9}-\frac{ab}{3}+c=0$ $-a^3+3a^3-9ab+27c=0$ $2a^3-9ab+27c=0$	1 1 1 1	
7(b)(i)	Consider the intersection of the two curves: $y=\frac{c}{x^2}$ $\frac{x^2}{a^2}+\frac{c^4}{x^2b^2}=1 \quad x^4b^2-x^2a^2b^2+a^2c^4=0$ Solving for x^2: $\Delta=a^4b^4-4a^2b^2c^4$ for the roots to be real and distinct: $\Delta>0$ $a^4b^4-4a^2b^2c^4>0$ $a^2b^2>4c^2 \quad or \quad ab>2c^2$ If $ab>2c^2, x^2$ has two distinct values and hence x has 4 values corresponding to 4 points of intersection.	1 1 1	
7(b)(ii)	$y=\frac{c^2}{x} \quad xa\sin\theta-\frac{c^2}{x}b\cos\theta=\left(a^2-b^2\right)\sin\theta\cos\theta$ $x^2a\sin\theta-c^2b\cos\theta=\left(a^2-b^2\right)x\sin\theta\cos\theta$ $x^2a\sin\theta-\left(a^2-b^2\right)x\sin\theta\cos\theta-c^2b\cos\theta=0$ $\Delta=\left[\left(a^2-b^2\right)^2\sin\theta cos\theta\right]^2+4ac^2b\cos\theta\sin\theta$ If $0<\theta<\frac{\pi}{2}$, $0<\sin\theta<1, 0<\cos\theta<1, \quad a,b>0$ $\therefore \Delta>0$ and this gives two values for x.	1 1 1	

7(b)(iii)	$x_1+x_2=\frac{2\left(a^2-b^2\right)\sin\theta\cos\theta}{2a\sin\theta}=\frac{\left(a^2-b^2\right)\sin\theta\cos\theta}{a\sin\theta}$ $\frac{x_1+x_2}{2}=\frac{\left(a^2-b^2\right)\sin\theta\cos\theta}{2a\sin\theta}$ $x=\frac{\left(a^2-b^2\right)\cos\theta}{2a} \quad (1)$ sub into normal to find y: $a\frac{\left(a^2-b^2\right)\cos\theta}{2a}\sin\theta-yb\cos\theta=\left(a^2-b^2\right)\sin\theta\cos\theta$ $\frac{\left(a^2-b^2\right)}{2}\sin\theta-yb=\left(a^2-b^2\right)\sin\theta \qquad \cos\theta\neq 0$ $y=\frac{-\left(a^2-b^2\right)}{2b}\sin\theta \quad (2)$	1 1	
7(b)(iv)	Eliminate θ: From (1) $\quad \cos\theta=\frac{2ax}{a^2-b^2}$ From (2) $\quad \sin\theta=-\frac{2by}{a^2-b^2}$ $\sin^2\theta+\cos^2\theta=1$ $\frac{4a^2x^2}{\left(a^2-b^2\right)^2}+\frac{4b^2y^2}{\left(a^2-b^2\right)^2}=1$ $\frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2}+\frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2}=1$	1 1 1	

Question	Criteria	Marks	Bands
8(a)(i)	Domain: $e^x > 0$ *for all* x for $\cos^{-1}(e^x): -1 \le e^x \le 1$ only if $0 \le e^x \le 1$, i.e. if $x \le 0$. ☑ Range: For this domain range will be : $0 \le y \le \frac{\pi}{2}$ ☑	2	
8(a)(ii)	☑ limit at $y=1$ ☑ shape 	2	
8(a)(iii)	 ☑ limit at $y=1$ ☑ shape (reciprocal of (ii))	2	

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8(b)(i)	*prove true for* $n=1$ $\therefore (1+\sqrt{2})^1 = 1+\sqrt{2}$ *true where* $p_n = 1$ *and* $q_n = 1$ ☑ *assume true for* $n=k$ $\therefore (1+\sqrt{2})^k = p_k + q_k\sqrt{2}$ *prove true for* $n=k+1$ $\therefore (1+\sqrt{2})^{k+1} = (1+\sqrt{2})^k(1+\sqrt{2})^1$ $= (p_k + q_k\sqrt{2})(1+\sqrt{2})$ (*by assumption above*) $= p_k + p_k\sqrt{2} + q_k\sqrt{2} + 2q_k$ $= (p_k + 2q_k) + (p_k + q_k)\sqrt{2}$ ☑ since p_k and q_k are integers $\therefore p_k + 2q_k$ is an integer $= p_{k+1}$ $\therefore p_k + q_k$ is an integer $= q_{k+1}$ *hence* $(1+\sqrt{2})^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$ ☑ If true for $n=k$ and $n=k+1$ and since true for $n=1,2,3...$ $\therefore$ *true for* $\forall\ n$ positive integers	3	
8(b)(ii)	$p_1^2 - 2q_1^2 = 1 - 2\times 1^2 = -1 = (-1)^1$ *If* $p_k^2 - 2q_k^2 = (-1)^k$ *then when* $n=k+1$ $(p_{k+1})^2 - (q_{k+1})^2 = (p_k + 2q_k)^2 - 2(p_k + q_k)^2$ (*from above*) ☑ $= p_k^2 + 4p_kq_k + 4q_k^2 - 2p_k^2 - 4p_kq_k - 2q_k^2$ $= 2q_k^2 - p_k^2$ $= -1(p_k^2 - 2q_k^2)$ $= -1\times(-1)^k$ $= (-1)^{k+1}$ ☑ if true for $n=k$ and $n=k+1$ and since true for $n=1,2,3...$ $\therefore$ *true for* $\forall\ n$ positive integers	2	
8(c)(i)	$f(xa) = f(x) + f(a)$ *if* $x=1$ *then* $f(a) = f(1) + f(a) \Rightarrow f(1) = 0$ *if* $x=a$ *then* $f(a^2) = f(a) + f(a)$ $f(a^2) = 2f(a)$ ☑ $\therefore 2f(-1) = f(-1^2) = f(1) = 0$ ☑	2	
8(c)(ii)	since $2f(a) = f(a^2)$ $\therefore 2f(-a)) = f((-a))^2$ $= f(a^2)$ $= 2f(a)$ ☑ since $2f(a) = 2f(-a))$ $\therefore f(a) = f(-a)$ *even function* ☑	2	