KINCOPPAL-ROSE BAY
Cor Unum

2008

HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 60

- Attempt Questions 1 – 5
- All questions are of equal value

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Attempt Questions 1-5
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\dfrac{x^2-5x}{x-4} \leq 3$ 3

(b) Twelve people are going to the Easter Show.
Five of them are going by car and the rest are going by train.

(i) How many different groups of five people can be found to fill the car? 1

(ii) In one of these groups of five, it is found that only 2 of the people can drive. In how many ways can the seats be filled in this group under this condition? 1

(c) When $P(x) = x^3 + bx + c$ is divided by $x-1$, the remainder is -4
When $P(x) = x^3 + bx + c$ is divided by $x+2$, the remainder is 11.
Find the values of b and c. 2

(d) If α, β and γ are the roots of $x^3 - 5x^2 - 3x + 2 = 0$, find the values of:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ 1

(e) Find the size of the acute angle between the curves $y = \log_e x$ and $y = x^2 - 4x + 3$ at the point $(1, 0)$, to the nearest degree. 3

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Use the Factor Theorem to show that $x+1$ is a factor of the polynomial $x^3-x^2-3x-1=0$ **1**

(ii) Hence, or otherwise, factorise the polynomial $P(x)=x^3-x^2-3x-1$ into its two factors. **2**

(b) The point $Q\ (a,b)$ divides the interval joining $A\ (-1,5)$ and $B\ (6,-4)$, externally, in the ratio 3: 2. Find the values of a and b. **2**

(c) Find the definite integral $\int_0^1 xe^{x^2+1}dx$, using the substitution $u=x^2+1$. Leave your answer in terms of e. **3**

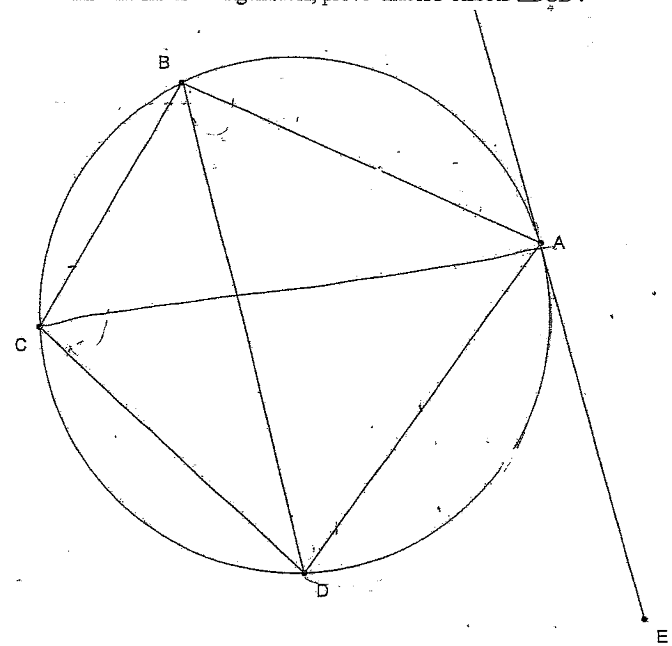
(d) (i) Find $\frac{d}{dx}(\sin x+\cos x)$ **1**

(ii) Hence, prove that $\int_0^{\frac{\pi}{4}} \frac{\cos x-\sin x}{\sin x+\cos x}dx=\frac{1}{2}\ln 2$ **3**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \sin^2 3x\,dx$ **3**

(b) $ABCD$ is a cyclic quadrilateral.
If $AB=AD$ and AE is a tangent at A, prove that AC bisects $\angle BCD$. **4**



(c) (i) Using the expansions of $\cos 2\theta$, **3**

show that $\sqrt{\frac{1+\cos x}{1-\cos x}}=\cot\frac{x}{2}$, where $0<x<\pi$.

(ii) Hence prove that $\cot 67\frac{1}{2}°=\sqrt{2}-1$ **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

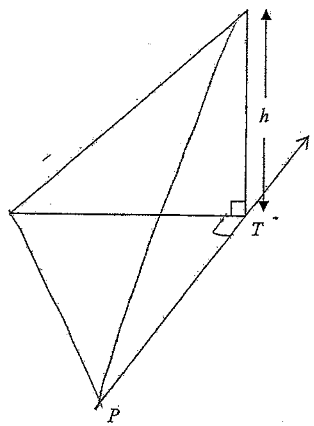
(a) Show, by the process of mathematical induction that, for $n \geq 1$ 3

$$\log_e 2 + \log_e\left(\frac{3}{2}\right) + \log_e\left(\frac{4}{3}\right) + \ldots + \log_e\left(\frac{n+1}{n}\right) = \log_e(n+1)$$

(b) At position T, a hill is h metres high. The angular elevation of the hill from a place P, due south of the hill, is $37°$. The angular elevation of the hill from a place Q, due west of the hill, is $23°$. The distance from P to Q is 3 km.

(i) Copy and complete the diagram showing all relevant information. 1

(ii) Find the height (h) of the hill in kilometres, to 1 decimal place. 3



(c) (i) Show that the equation of the tangent to the curve $y = e^{\frac{x}{2}}$ at the point $x = 2$, is $y = \frac{e}{2}x$. 2

(ii) Draw a sketch showing the curve and its tangent. 3

Evaluate in exact form, the area of the region bounded by the curve, its tangent and the y-axis.

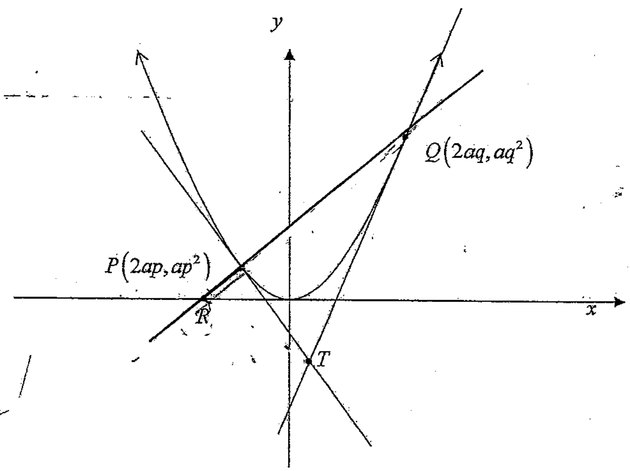
Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The letters of the word CIRCLE are written at random on the circumference of a circle.

(i) How many different arrangements are possible? 2

(ii) In how many of these arrangements will the "C"s be separated? 2

(b)



$P(2ap, ap^2)$ and . $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The line PQ meets the x-axis at R. The tangents at P and Q meet at T.

(i) Given that the equation of the secant PQ is $2y = x(p+q) - 2apq$, determine the coordinates of the point R in terms of p and q. 1

(ii) Show that the equation of the tangent at $P(2ap, ap^2)$ is given by $y = px - ap^2$. 2

(iii) Hence show that the coordinates of the point T are given by $(a(p+q), apq)$ 2
Note - you do not have to derive the equation of the tangent at Q, you may quote it.

(iii) The secant PQ varies in position, but maintains a constant gradient of m. Prove that M, the midpoint of RT moves on a straight line and that this line is parallel to PQ. 3

End of paper

Solutions Mathematics Extension 1 Half Yearly 2008

Question 1 (12 marks)	Marks
(a) $\frac{x^2-5x}{x-4} \le 3$ $x \ne 4$ $x^2-5x=3x-12$ $x^2-8x+12=0$ $(x-6)(x-2)=0$ $x=6,\ 2$ Test 0 ✓, 3 X, 5 ✓, 7 X (number line: 2, 4, 6) $Solution: x \le 2,\ 4 < x \le 6$	✓ $x \ne 4$ ✓ check ✓ solution
(b) (i) $^{12}C_5 = 792$ (ii) $2 \times {^4C_4} = 48$	✓ (i) ✓ (ii)
(c) $P(1)=1+b+c=-4$ (1) $P(-2)=-8-2b+c=11$ (2) from (2) $\Rightarrow 8+2b-c=-11$ (3) (1)+(3) $\Rightarrow 9+3b=-15$ $\underline{b=-8}$ $\underline{c=3}$	✓ using remainder theorem ✓ solutions
(d) (i) $\alpha+\beta+\gamma=\frac{-b}{a}=5$ (ii) $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2=\alpha\beta\gamma(\alpha+\beta+\gamma)$ $=2\times5$ $=10$	✓ Part (i) ✓ Part (ii)
(e) $y=\log_e x$ $y=x^2-4x+3$ $y'=\frac{1}{x}$ $y'=2x-4$ $m_1=1$ $m_2=-2$ $\tan\theta=\left\|\frac{1+2}{1-2}\right\|=\|-3\|$ $\theta\approx72°$	✓ derivatives ✓ gradients ✓ correct use of formula

Question 2 (12 marks)	Marks
(a) (i) $P(-1)=(-1)^3-(-1)^2-3(-1)-1$ $=-1-1+3-1$ $=0$ (ii) $(x^3-x^2-3x-1)\div(x+1)=x^2-2x-1$ x^3+x^2 $-2x^2-3x$ $-2x^2-2x$ $-x-1$ $-x-1$ 0 $P(x)=(x+1)(x^2-2x-1)$	(i) ✓ (ii) ✓ correct use of division algorithm ✓ written as factors
(b) $(-1,5)$ $(6,-4)$ $-3:2$ $a=\frac{-2-18}{-3+2}$ $b=\frac{10+12}{-3+2}$ $\underline{a=20}$ $\underline{b=-22}$	✓ correct use of formula ✓ solution
(c) $u=x^2+1$ $\frac{du}{dx}=2x$ $xdx=\frac{1}{2}du$ when $x=0,\ u=1$ when $x=1,\ u=2$ $\int_0^1 xe^{x^2+1}dx=\frac{1}{2}\int_1^2 e^u du=\frac{1}{2}\left[e^u\right]_1^2$ $=\frac{1}{2}\left[e^2-e\right]$ $=\frac{e}{2}\left[e-1\right]$	✓ xdx ✓ correct substitution ✓ correct integral

(d) (i) $\frac{d}{dx}(\sin x+\cos x)=\cos x-\sin x$

(ii) $\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{\sin x+\cos x}dx=\ln[\sin x+\cos x]_0^{\frac{\pi}{4}}$

$$=\ln\left[\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-(\sin 0+\cos 0)\right]$$

$$=\ln\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-\ln 1$$

$$=\ln\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$=\ln\sqrt{2}$$

$$=\frac{1}{2}\ln 2$$

Marks:
(i) ✓
(ii) ✓ *integration*
✓ *substitution*
✓ *working to show*

Question 3 (12 marks)	Marks

(a) $\cos 6x=1-2\sin^2 3x$

$\sin^2 3x=\frac{1}{2}(1-\cos 6x)$

$\int\sin^2 3x\,dx=\frac{1}{2}\int(1-\cos 6x)dx$

$$=\frac{1}{2}\left(x-\frac{1}{6}\sin 6x\right)+c$$

$$=\frac{1}{6}(3x-\sin 6x)+c$$

Marks:
✓ $\sin^2 3x$
✓ *integral*
✓ *+c*

(b) *Let* $\angle EAD=\theta$

then $\angle DBA=\theta$ *(angle in alternate segment)*

now □ *ABC is isosceles (given,* $AB=AD$*)*

$\therefore\angle BDA=\theta$ *(base angle isosceles* □ *ABC)*

then $\angle BAD=180-2\theta$ *(straight angle)*

$\therefore\angle BCD=180-(180-2\theta)$ *(opp angles cyclic quadrilatera*

$=2\theta$

but $\angle ACD=\theta$ *(angle in alternate segment)*

$\therefore\angle BCD=\theta$

$\therefore$ *AC biscets* $\angle BCD$

Marks:
✓ *angles in alt segment*
✓ *cyclic quad*
✓ *reasoning to get to dou*
✓ *conclusion*

Question 3 cont (12 marks)	Marks

(c) (i) $\cos 2\theta=2\cos^2\theta-1\Rightarrow 2\cos^2\theta=\cos 2\theta+1$

$\cos 2\theta=1-2\sin^2\theta\Rightarrow 2\sin^2\theta=1-\cos 2\theta$

$$\sqrt{\frac{1+\cos x}{1-\cos x}}=\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

$$=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Marks:
(i) ✓ *expressions fo*
✓ *correct substit*

(c) (ii) $\cot 67\frac{1}{2}°=\sqrt{\frac{1+\cos 135°}{1-\cos 135°}}$

$$=\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}}$$

$$=\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}}$$

$$=\sqrt{(\sqrt{2}-1)^2}$$

$$=\sqrt{2}-1$$

Marks:
(i) ✓ *first line of sol*
✓ *exact values*
✓ *simplification*

Nb marks may be reorganised for this question

Question 4 (12 marks)	Marks
(a) *Step 1: Prove true for $n=1$*	✓ *Steps 1 + 2*
$LHS = \log_e 2$	✓ *Step 3*
$RHS = \log_e(1+1) = \log_e 2 = LHS$	✓ *Step 4*

Step 2: Assume true for $n=k$

$$\text{i.e. } \log_e 2 + \log_e\left(\frac{3}{2}\right) + \log_e\left(\frac{4}{3}\right) + \ldots + \log_e\left(\frac{k+1}{k}\right) = \log_e(k+1)$$

Step 3: Required to prove true for $n=k+1$.

$$\text{i.e. } \log_e 2 + \log_e\left(\frac{3}{2}\right) + \log_e\left(\frac{4}{3}\right) + \ldots + \log_e\left(\frac{k+1}{k}\right) + \log_e\left(\frac{k+2}{k+1}\right) = \log_e(k+2)$$

$$LHS = \log_e 2 + \log_e\left(\frac{3}{2}\right) + \log_e\left(\frac{4}{3}\right) + \ldots + \log_e\left(\frac{k+1}{k}\right) + \log_e\left(\frac{k+2}{k+1}\right)$$

$$= \log_e(k+1) + \log_e\left(\frac{k+2}{k+1}\right)$$

$$= \log_e\left[(k+1)\left(\frac{k+2}{k+1}\right)\right]$$

$$= \log_e(k+2)$$

$$= RHS$$

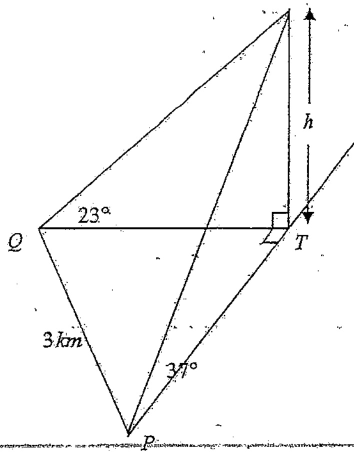
Step 4:

Since the result is true for $n=1$ and $n=k+1$,

then it is true for $n=2,3$ etc.

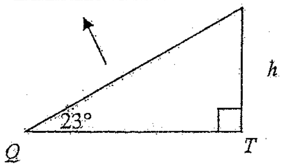
The result is true for all $n \geq 1$

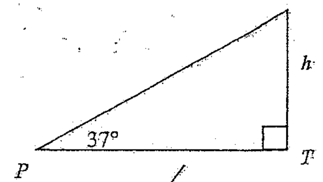
(b) (i)



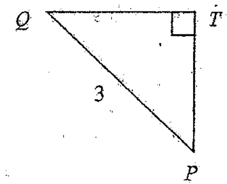
☑ *completed diagram*

$$QT = \frac{h}{\tan 23}$$





$$PT = \frac{h}{\tan 37}$$



$$3^2 = \left(\frac{h}{\tan 23}\right)^2 + \left(\frac{h}{\tan 37}\right)^2$$

$$3^2 = h^2\left(\frac{1}{\tan^2 23} + \frac{1}{\tan^2 37}\right)$$

$$\frac{3}{\sqrt{\left(\frac{1}{\tan^2 23} + \frac{1}{\tan^2 37}\right)}} = h$$

$$h \approx 1.1095 \text{ km}$$

$$= 1.1 \text{ km (1dp)}$$

✓ *QT and PT*

✓ *tying together into □QPT*

✓ *correct solution*

(c) (i) $y=e^{\frac{x}{2}}$

$y'=\frac{1}{2}e^{\frac{x}{2}}$

when $x=2$,

$m=\frac{e}{2}$ and $y=e$

$y-e=\frac{e}{2}(x-2)$

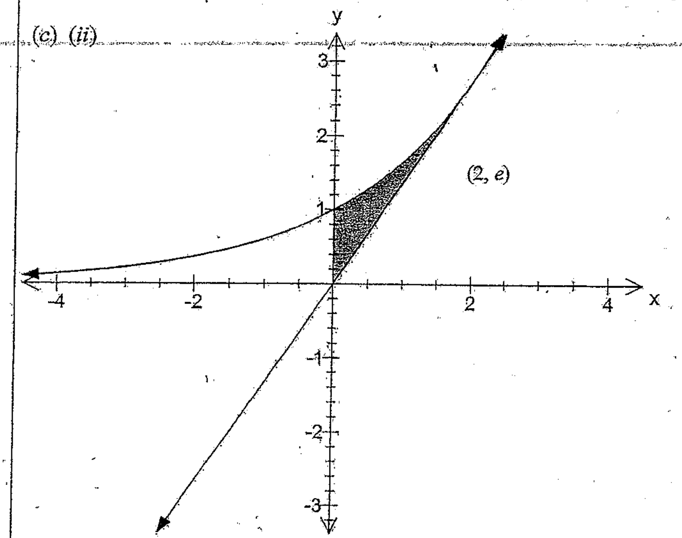
$2y-2e=ex-2e$

$2y=ex$

$y=\frac{e}{2}x$

Marks: ☑ derivative and gradient ☑ working to get eqn

(c) (ii)



Marks: ☑ sketch showing region

$A=\int_0^2 e^{\frac{x}{2}}dx-\frac{1}{2}\times 2\times e$

$=2\left[e^{\frac{x}{2}}\right]_0^2-e$

$=2\left[e^1-e^0\right]-e$

$=2e-2-e$

$=(e-2)$ units^2

Marks: ☑ integral ☑ answer

Question 5 (12 marks)	Marks

(a) (i) $\frac{5!}{2!}=60$

Marks: (i) ☑ 5! ☑ division by same elements

(ii) "e's" together $=4!$

$\therefore$ apart $\Rightarrow \frac{5!}{2!}-4!=36$

or

"e's" can go in 3 ways apart

rest in 4! ways

$\therefore \frac{3\times 4!}{2!}=36$

Marks: (ii) ☑ "e's" together ☑ subtraction

(b)(i) When $y=0$,

$0=x(p+q)-2apq$

$2apq=x(p+q)$

$x=\frac{2apq}{p+q}$

$R\left(\frac{2apq}{p+q},0\right)$

Marks: (i) ☑

(b) (ii) $y=\frac{x^2}{4a}$

$y'=\frac{2x}{4a}=\frac{x}{2a}$

at $x=2ap$, $m=p$

$y-ap^2=p(x-2ap)$

$y=px-2ap^2+ap^2$

$y=px-ap^2$

Marks: ☑ derivative and gradient ☑ working to eqn

(b) (iii) eqn at $Q: y = qx - aq^2$ solving sim: $px - ap^2 = qx - aq^2$ $px - qx = ap^2 - aq^2$ $x(p-q) = a(p-q)(p+q)$ nb: $p \neq q$ $\underline{x = a(p+q)}$ $y = p[a(p+q)] - qp^2$ $y = ap^2 + apq - ap^2$ $\underline{y = apq}$	☑ eqn at Q and working for x value ☑ working for y value
(b) (iv) Since PQ has constant gradient m $m = \frac{p+q}{2}$ i.e. $\boxed{p+q = 2m}$	☑ midpoint ☑ $(p+q)$ and apq ☑ substitution and resolution to (A) ☑ conclusion
let M be (X, Y) then $X = \frac{a(p+q) + \frac{2apq}{p+q}}{2}$ $= \frac{a(p+q)^2 + 2apq}{2(p+q)}$ and $Y = \frac{apq}{2} \Rightarrow \boxed{apq = 2Y}$ $\therefore X = \frac{a(2m)^2 + 2 \times 2Y}{4m}$ $4mX = 4am^2 + 4Y$ $4Y = 4mX - 4am^2$ $Y = mX - am^2$ (A) Which is a straight line with gradient m. $\therefore$ the locus of M is a straight line parallel to PQ	