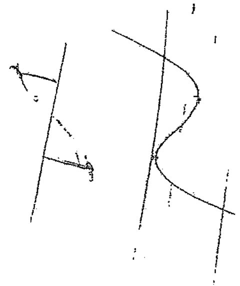
KINCOPPAL-ROSE BAY
Cor Unum

2008
HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics Extension 2



General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new page for each question

Total marks – 60

- Attempt Questions 1 – 4
- All questions are of equal value

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Attempt Questions 1-4
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the square roots of $5-12i$. Give your answers in the form $a+ib$ **3**

(b) Sketch on the Argand diagram the locus $|z-2|=|z+2i|$ **1**

(c) Sketch the region in the Argand diagram that satisfies both conditions **2**

$$-\frac{\pi}{2}\le \arg(z-2)\le 0 \text{ and } \operatorname{Im}(z)\le -1$$

(d) Let $z=1-i$ and $w=-1+i\sqrt{3}$

(i) Find $\arg z$ and $\arg w$ **1**

(ii) Hence find $\arg(wz)$ **1**

(iii) Hence prove that $\sin\frac{5\pi}{12}=\frac{\sqrt{2}(1+\sqrt{3})}{4}$ **2**

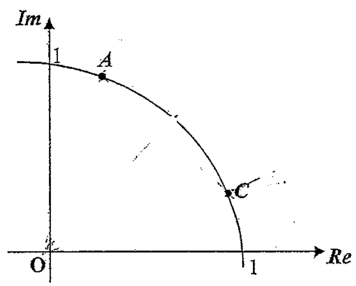
(e) (i) Let $z=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$. Find z^6 in simplest form. **1**

(ii) On the Argand diagram, plot and label all complex numbers that satisfy both the conditions $z^6=1$ and $\operatorname{Re}(z)\le 0$ **2**

Question 1 continues on page 3

Question 1 (continued)

(f)



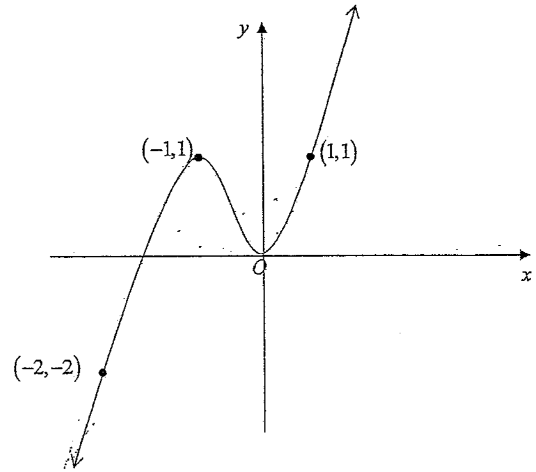
In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin and radius 1. They represent the complex numbers z_1 and z_2 respectively.

(i) Copy the above diagram. Then mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$ **1**

(ii) Explain why AC is perpendicular to OB. **1**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following. You must indicate important features and the new coordinates of the points indicated as appropriate.

(i) $y = |f(x)|$ **1**

(ii) $y = f(|x|)$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **3**

Question 2 continues on page 5

Question 2 (continued)

(b) (i) Sketch the graph of $y = x^3 - 12x$ 1

(ii) Use this graph, and techniques of calculus, to find the set of values of the real number k for which the equation $x^3 - 12x + k = 0$ has exactly one real root. 2

(c) (i) Given $x^3 + y^3 = 1$, show that $\frac{dy}{dx} = -\left(\frac{x}{y}\right)^2$ 1

(ii) Determine the coordinates of any vertical and/or horizontal asymptotes of the graph $x^3 + y^3 = 1$. 2

(iii) Sketch the graph of $x^3 + y^3 = 1$ 1

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the zeros of $P(x) = x^4 - 2x^2 - 3$

(i) Over $\mathbb{Q}$ the rational numbers 1

(ii) Over $\mathbb{C}$ the complex numbers 1

(b) Given the polynomial $P(x) = x^3 - 5x + 3$ with roots α, β and γ find the:

(i) cubic polynomial with integer coefficients whose roots are $3\alpha, 3\beta$ and 3γ 2

(ii) the polynomial with rational coefficients whose roots are α^2, β^2 and γ^2 2

(c) The equation $x^3 + 4x + 3 = 0$ has roots α, β and γ. Evaluate

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha^2 + \beta^2 + \gamma^2$ 2

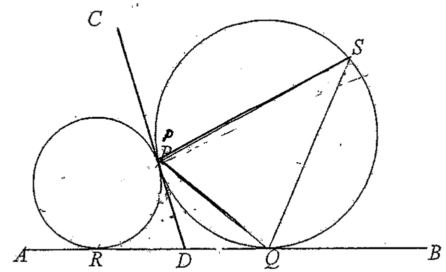
(iii) $\alpha^3 + \beta^3 + \gamma^3$ 2

(iv) What does your answer to part (ii) suggest about the roots to the polynomial equation $x^3 + 4x + 3 = 0$? 1

(d) Express $\frac{2x^2 + 3x - 7}{(x-5)(x^2+4)}$ as the sum of the partial fractions $\frac{A}{x-5} + \frac{Bx + C}{x^2 + 4}$ 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) AB and CD are tangents to both circles and they intersect at the point D. The tangent AB touches the smaller circle at R and the larger circle at Q. The tangent CD touches the two circles at P.



Copy the diagram into your examination booklet.

(i) Prove that $\angle RPQ$ is a right angle. 3
Hint: assign pronumerals to appropriate angles to help you in your proof.

(ii) Is quadrilateral $RPSQ$ a cyclic quadrilateral? You must support your answer with geometric reasons. 2

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$, $u_2 = 7$, and $u_n = 7u_{n-1} - 12u_{n-2}$ for $n \geq 3$. 4
Use the method of Mathematical Induction to show that $u_n = 4^n - 3^n$ for $n \geq 1$

(c) (i) Given that $x = a$ is a root of multiplicity n of the polynomial $P(x) = Q(x)(x-a)^n$, show that $x = a$ is also a root of $P'(x)$. 2

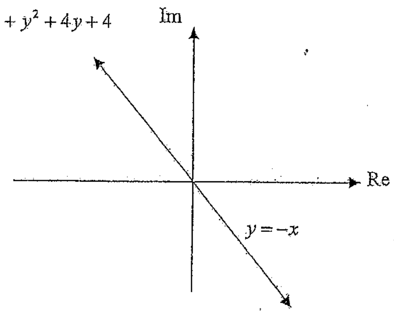
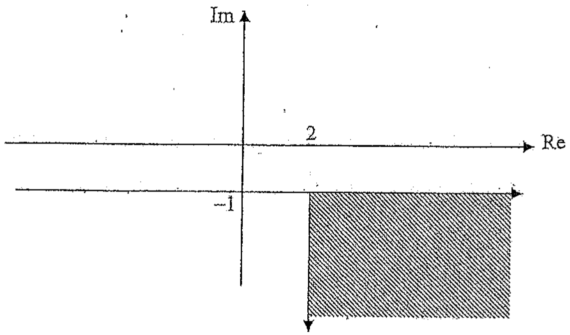
(ii) The equation $x^n + px - q = 0$ has a double root.

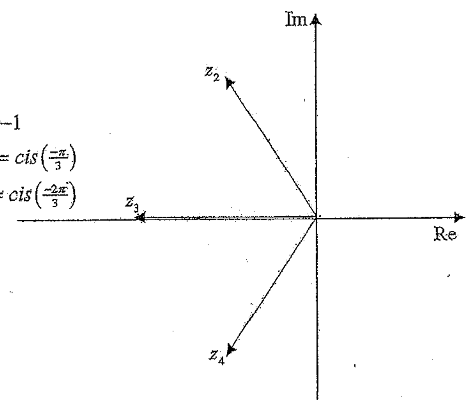
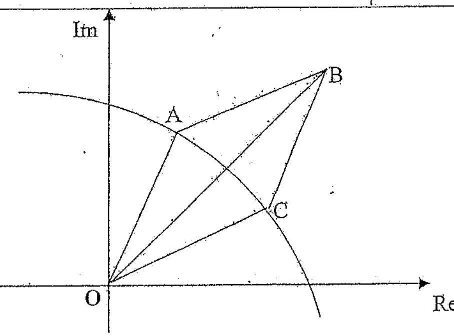
(α) Show that $x^{n-1} = -\frac{p}{n}$ 1

(β) Hence show that $\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$. 3

End of paper

Mathematics Extension 2 HSC Half Yearly Exam Solutions 2008

Question	Criteria	Marks	Bands
1(a)	Let $z=x+iy$ $z^2=5-12i=(x+iy)^2$ $x^2-y^2=5$ and $2xy=-12$ by inspection: $x=\pm3$ $y=\mp2$ since $x\in\mathbb{R}$ $\therefore$ square roots are $3-2i,-3+2i$	1 1 1	
1(b)	$\|z-2\|=\|z+2i\|$ $(x-2)^2+y^2=x^2+(y+2)^2$ $x^2-4x+4+y^2=x^2+y^2+4y+4$ $y=-x$ 	1	
1(c)		2	
1(d)(i)	$\arg(z)=\frac{-\pi}{4}$ $\arg(w)=\frac{2\pi}{3}$	1	
1(d)(ii)	$\arg(zw)=\arg w+\arg z=\frac{-\pi}{4}+\frac{2\pi}{3}=\frac{5\pi}{12}$	1	

1(d)(iii)	$wz=\left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)$ Using part (ii) $\arg(zw)=\frac{5\pi}{12}$ $\|z\|=\sqrt{(1)^2+(-1)^2}=\sqrt{2}$ $\|w\|=\sqrt{(-1)^2+(\sqrt{3})^2}=2$ $\sin\frac{5\pi}{12}=\frac{\operatorname{Im}(wz)}{\|wz\|}=\frac{\operatorname{Im}(wz)}{\|w\|\|z\|}$ $\sin\frac{5\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}=\frac{\sqrt{2}(1+\sqrt{3})}{4}$	1 1	
1(e)(i)	$z^6=(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^6$ $=(\cos 6\left(\frac{\pi}{3}\right)+i\sin 6\left(\frac{\pi}{3}\right))$ $=(\cos 2\pi+i\sin 2\pi)$ $=1$	1	
1(e)(ii)	$z_0=0$ $z_1=cis\frac{\pi}{3}$ $z_2=cis\frac{2\pi}{3}$ $z_3=cis\pi=-1$ $z_4=cis\frac{4\pi}{3}=cis\left(\frac{-\pi}{3}\right)$ $z_5=cis\frac{5\pi}{3}=cis\left(\frac{-2\pi}{3}\right)$ 	1 1	
1(f)(i)		1	
1(f)(ii)	*OABC* is a parallelogram. *OA*=*OC* (equal radii of circle) $\therefore$ *OABC* is a rhombus. By definition of a rhombus the diagonals bisect each other at right angles. $\therefore OB\perp AC$	1	

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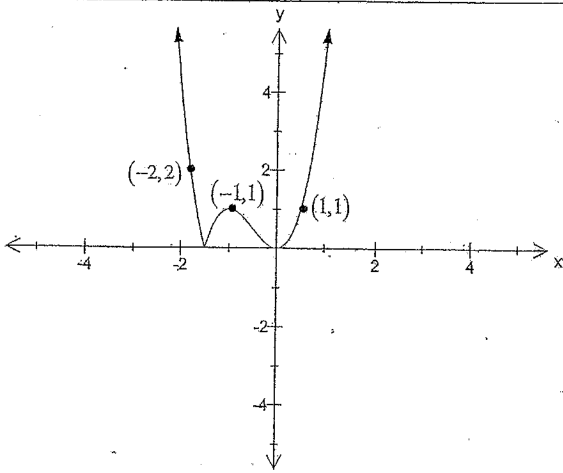
<tr><th>Question</th><th>Criteria</th><th>Marks</th><th>Bands</th></tr>
<tr><td>2(a)(i)</td><td>
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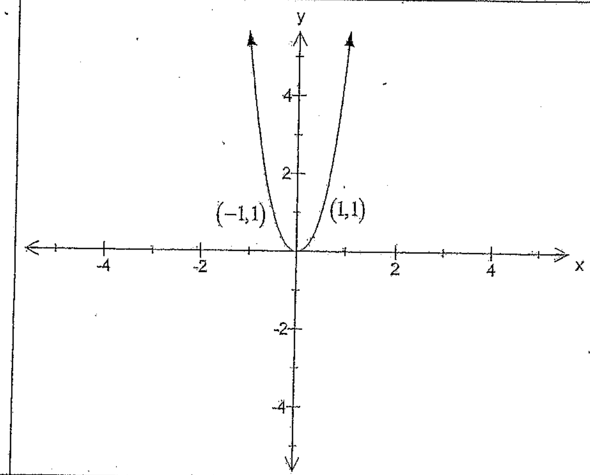
<tr><td>2(a)(ii)</td><td>
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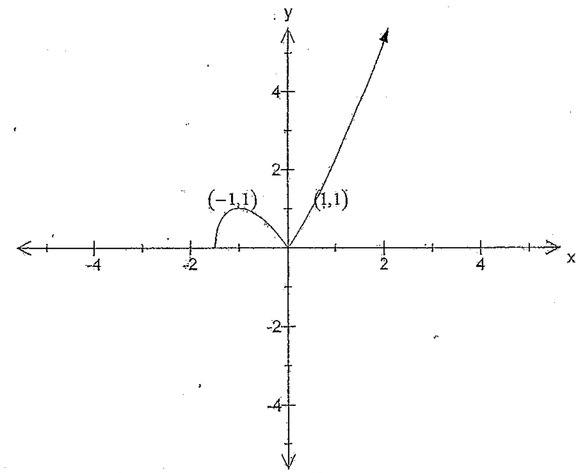
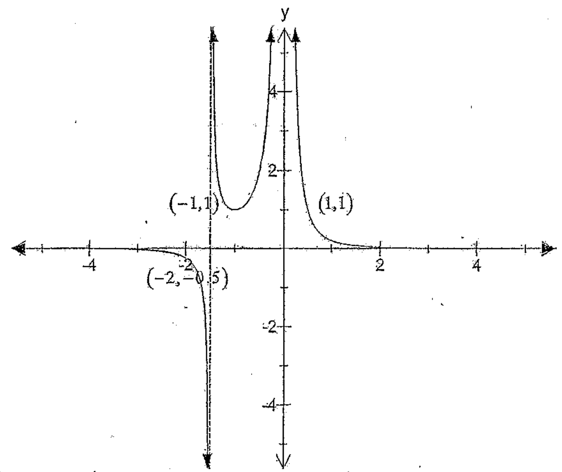
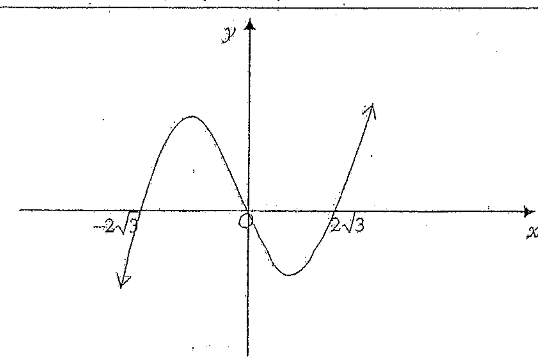
<tr><td>2(a)(iii)</td><td>
</td><td>2</td><td></td></tr>
<tr><td>2(a)(iv)</td><td>
</td><td>3</td><td></td></tr>
<tr><td>2(b)(i)</td><td>
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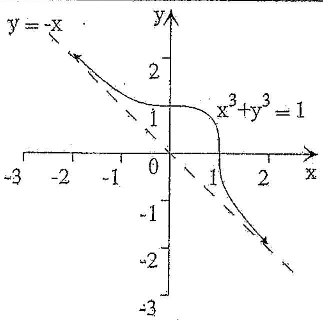
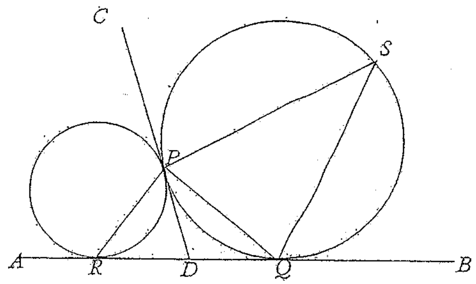
2(b)(ii)	$y'=3x^2-12$ $3x^2-12=0$ $x=\pm 2$ $y=\pm 16$ stationary point at $(-2,16)$ and $(2,-16)$ $\therefore k>16$ or $k<-16$	1 1	
2(c)(i)	$x^3+y^3=1$ $3x^2+3y^2\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{3x^2}{3y^2}=-\frac{x^2}{y^2}$	1	
2(c)(ii)	$\frac{dy}{dx}=-\left(\frac{x}{y}\right)^2$ vertical $\Rightarrow y=0$ $\therefore(1,0)$ $\frac{dy}{dx}=-\left(\frac{x}{y}\right)^2$ horizontal $\Rightarrow x=0$ $\therefore(0,1)$	2	
2(c)(iii)	 curve is symmetric about $y=x$, since the transformation $y\leftrightarrow x$ leaves the Cartesian equation of the curve unchanged. $x^3+y^3=1 \Rightarrow y=-x\left(1-\frac{1}{x^3}\right)^{\frac{1}{3}}$. By expansion for the large values of x we have $y=-x\left(1-\frac{1}{3x^3}+...\right)\Rightarrow y=-x+0\left(\frac{1}{x}\right)$. Hence the curve has an oblique asymptote $y=-x$ as $x\to\pm\infty$.	1	

Question	Criteria	Marks	Bands
3(a)(i)	$P(x)=(x^2-3)(x^2+1)$ no zeros over rationals $\mathbb{Q}$	1	
3(a)(ii)	$P(x)=(x-\sqrt{3})(x-\sqrt{3})(x+i)(x-i)$ $x=\pm\sqrt{3},\pm i$	1	
3(b)(i)	$\left(\frac{x}{3}\right)^3-5\left(\frac{x}{3}\right)+3=0$ $\frac{x^3}{27}-\frac{5x}{3}+3=0$ $x^3-45x+81=0$	1 1	
3(b)(ii)	$(\sqrt{x})^3-5\sqrt{x}+3=0$ $x\sqrt{x}-5\sqrt{x}=-3$ $\sqrt{x}(x-5)=-3$ $x(x^2-10x+25)=9$ $x^3-10x^2+25x-9=0$	1 1	
3(c)(i)	$\alpha+\beta+\gamma=0$	1	
3(c)(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $=0-2(4)$ $=-8$	1 1	
3(c)(iii)	$\alpha^3+4\alpha+3=0$ $\beta^3+4\beta+3=0$ $\gamma^3+4\gamma+3=0$ $\therefore\alpha^3+\beta^3+\gamma^3=-4(\alpha+\beta+\gamma)-9$ $=-4(0)-9$ $=-9$	1 1	
3(c)(iv)	Since the sum of the **squares of the roots** to the polynomial is -8, (i.e. negative) there must be a pair of complex conjugate roots.	1	
3(d)	$2x^2+3x-7=A(x^2+4)+(Bx+C)(x-5)$ $x=5 \quad 50+15-7=29A$ $58=29A$ $\therefore A=2$ $x=0 \quad -7=8-5(C)$ $-15=-5C$ $\therefore C=3$ $x=1 \quad 2+3-7=2(5)+(-4)(B+3)$ $-2=-4B-2$ $\therefore B=0$ $\therefore\frac{2x^2+3x-7}{(x-5)(x^2+4)}=\frac{2}{x-5}+\frac{3}{x^2+4}$	1 1 1	

Question	Criteria	Marks	Bands
4(a)(i)	 let $\angle DRP = \alpha$ $PD = RD$ (tangents from external point D are equal) $\therefore \Delta RDP$ is isosceles $\angle DRP = \angle DPR = \alpha$ (equal base angles of isosceles triangle) let $\angle DQP = \beta$ $PD = DQ$ (tangents from external point D are equal) $\therefore \Delta DPQ$ is isosceles $\angle DPQ = \angle DQP = \beta$ (equal base angles of isosceles triangle) $\angle RPQ = \alpha + \beta$ $2\alpha + 2\beta = 180^\circ$ (angle sum of ΔRPQ) $\therefore \alpha + \beta = 90^\circ$	1 1 1	
4(a)(ii)	$\angle DQP = \angle QSP = \beta$ (angle between a tangent and the chord of contact is equal to the angle in the alternate segment) $\angle QRP = \alpha$ $\angle QRP + \angle QSP = \alpha + \beta$ but $\alpha + \beta = 90^\circ$ $\therefore RPSQ$ is not a cyclic quadrilateral since the opposite angles are not supplementary	1 1	

4(b)	$n=1 \quad u_1 = 4-3 = 1$ $n=2 \quad u_2 = 4^2 - 3^2 = 16 - 9 = 7$ $\therefore$ true for $n = 1, 2$ Assume true for $n = k$ and $n = k+1$ $u_k = 4^k - 3^k$ $u_{k+1} = 4^{k+1} - 3^{k+1}$ Prove true for $n = k+2$ i.e. $u_{k+2} = 4^{k+2} - 3^{k+2}$ $u_{k+2} = 7u_{k+1} - 12u_k$ $= 7\left(4^{k+1} - 3^{k+1}\right) - 12\left(4^k - 3^k\right)$ $= 28.4^k - 21.3^k - 12.4^k + 12.3^k$ $= 16.4^k - 9.3^k$ $= 4^2.4^k - 3^2 3.^k$ $= 4^{k+2} - 3^{k+2}$ $\therefore$ true for $n = k+2$ Now we have proved it true for $n = 1, 2$ $\therefore$ by step 3, it is true for $n = 1+2$ i.e. $n = 3$, and so on for all integers	1 1 1 1	
4(c)(i)	$P'(x) = Q(x) \times n(x-a)^{n-1} + Q'(x)(x-a)^n$ $= (x-a)^{n-1}\{nQ(x) + Q'(x)(x-a)\}$ $\therefore x = a$ is also a root of $P'(x)$	1 1	
4(c)(ii) (α)	Let $P(x) = x^n + px - q$. Then $P'(x) = nx^{n-1} + p$. Hence $P'(x) \Rightarrow x^{n-1} = -p/n$.	1	
4(c)(ii) (β)	α is a double root of $P(x) = 0$ if and only if $P(\alpha) = 0$ and $P'(\alpha) = 0$, but $P''(\alpha) \neq 0$ Furthermore, $P(x) = 0 \Leftrightarrow x(x^{n-1} + p) = q$. Substituting $x^{n-1} = -\frac{p}{n}$, we obtain $x\left(\frac{-p}{n} + p\right) = q \Rightarrow x = \frac{nq}{(n-1)p}$. But $x^{n-1} = -\frac{p}{n}$. $\left\{\frac{nq}{(n-1)p}\right\}^{n-1} = \frac{-p}{n} \Rightarrow \left(\frac{n}{p}\right)^{n-1}\left(\frac{q}{(n-1)}\right)^{n-1} = -\frac{p}{n}$. $\therefore \left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$.	1 1 1	