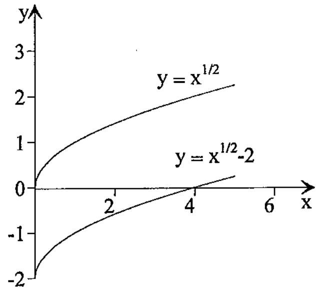
Topic 7. Graphs.
Level 1.
Problem GRA1_01.
Sketch (showing critical points) the graphs of: a) $y = x^{1/2}$; b) $y = x^{1/2} - 2$.

Solution:

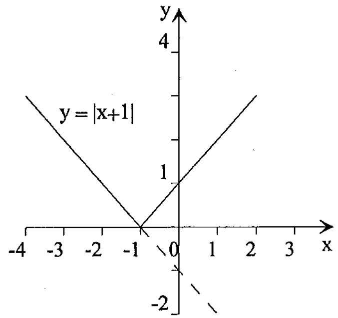


a) $y = x^{1/2}$
Domain $\{x : x \geq 0\}$
$\frac{dy}{dx} = \frac{1}{2}x^{-1/2},\ x > 0.$
$\frac{dy}{dx}$ is not defined at x = 0
$\frac{dy}{dx} \to \infty$ as $x \to 0^+$
$\Rightarrow$ the tangent line at the critical point $(0, 0)$ is vertical.

b) $y = x^{1/2} - 2$
Domain $\{x : x \geq 0\}$
$\frac{dy}{dx} = \frac{1}{2}x^{-1/2},\ x > 0.$
$\frac{dy}{dx}$ is not defined at x = 0
$\frac{dy}{dx} \to \infty$ as $x \to 0^+$
$\Rightarrow$ the tangent line at the critical point $(0, -2)$ is vertical.

Problem GRA1_02.
Sketch (showing critical points) the graphs of: a) $y = |x+1|$; b) $y = |x| + 1$.

Solution:
a) $y = |x+1|$

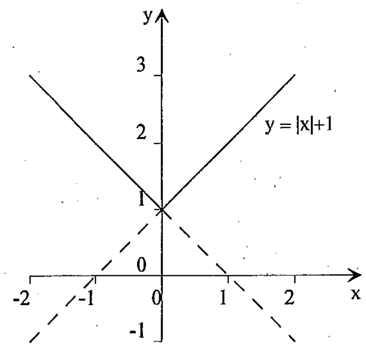


$$y = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x < -1 \end{cases} \quad \frac{dy}{dx} = \begin{cases} 1, & x > -1 \\ -1, & x < -1 \end{cases}$$

$\frac{dy}{dx} \to 1$ as $x \to -1^+$ $\quad \frac{dy}{dx} \to -1$ as $x \to -1^-$

$\Rightarrow \frac{dy}{dx}$ is not defined at $x = -1$, and $(-1, 0)$ is a critical point.

b) $y = |x| + 1$



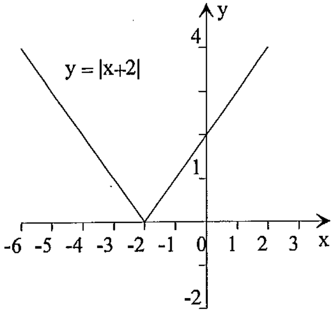
$$y = \begin{cases} x+1, & x \geq 0 \\ -x+1, & x < 0 \end{cases} \quad \frac{dy}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$\frac{dy}{dx} \to 1$ as $x \to 0^+$ $\quad \frac{dy}{dx} \to -1$ as $x \to 0^-$

$\Rightarrow \frac{dy}{dx}$ is not defined at $x = 0$, and $(0, 1)$ is a critical point.

Problem GRA1_03.
Sketch (showing critical points) the graph of $y = |x+2|$.

Solution:



$y=|x+2|$ $\quad y=\begin{cases} x+2, & x\ge -2 \\ -x-2, & x<-2 \end{cases}$ $\quad \dfrac{dy}{dx}=\begin{cases} 1, & x>-2 \\ -1, & x<-2 \end{cases}$

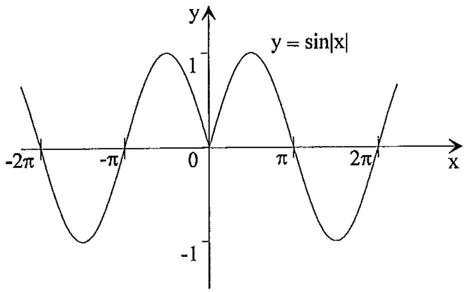
$\dfrac{dy}{dx}\to 1$ as $x\to -2^{+}$ $\quad \dfrac{dy}{dx}\to -1$ as $x\to -2^{-}$ $\quad \Rightarrow \dfrac{dy}{dx}$

is not defined at $x=-2$, and $(-2,\,0)$ is a critical point.

Problem GRA1_04.

Sketch the graph of $y=\sin|x|$.

Solution:



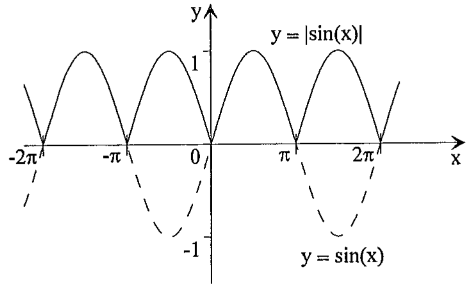
$y=\sin|x|=\begin{cases} \sin(-x), \; x<0 \\ \sin x, \; x\ge 0 \end{cases}$

Hence the section of the graph of $\sin|x|$ for $x<0$ is a reflection of $y=\sin x,\; x\ge 0$, in the y-axis.

Problem GRA1_05.

Sketch the graph of $y=|\sin x|$.

Solution:



$y=|\sin x|=\begin{cases} \sin x, \; \sin x\ge 0 \\ -\sin x, \; \sin x<0 \end{cases}$

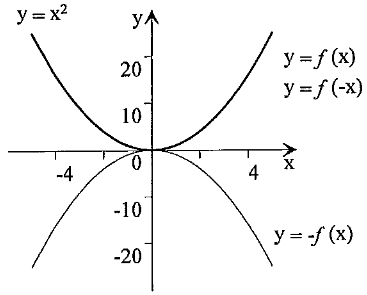
Hence the section of the graph of $y=\sin x$ which lie below the x-axis are reflected in the x-axis.

Problem GRA1_06.

For the function $f(x)=x^2$ (an even function) sketch the graphs of:

a) $y=f(x)$, b) $y=f(-x)$, c) $y=-f(x)$.

Solution:



b) $f(x)=x^2 \Rightarrow f(-x)=(-x)^2=x^2=f(x) \Rightarrow$ the graphs of $y=f(x)$ and $y=f(-x)$ coincide.

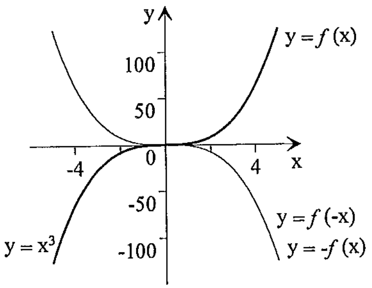
c) The graph $y=-f(x)$ is a reflection of $y=f(x)$ in the x-axis.

Problem GRA1_07.

For the function $f(x)=x^3$ (an odd function) sketch the graphs of:

a) $y=f(x)$, b) $y=-f(x)$, c) $y=f(-x)$.

Solution:



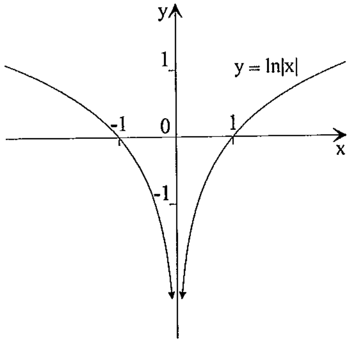
b) The graph $y=-f(x)$ is a reflection of $y=f(x)$ in the x-axis.

c) $f(x)=x^3 \Rightarrow f(-x)=(-x)^3=-x^3=-f(x) \Rightarrow$ the graphs of $y=f(-x)$ and $y=-f(x)$ coincide.

Problem GRA1_08.

Use the graph of $y=\ln x$ to sketch the graph of $y=\ln|x|$.

Solution:



$y=\ln|x|$, domain $\{x:\ x\neq 0\}$.

$$y=\ln|x|=\begin{cases}\ln x, & x>0\\ \ln(-x), & x<0\end{cases}$$

Hence the section of the graph of $y=\ln|x|$ for $x<0$ is a reflection of $y=\ln x$ in the y-axis.

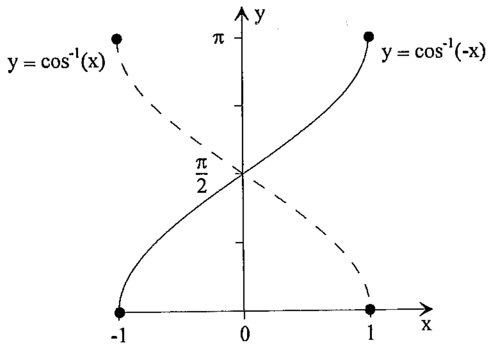
Problem GRA1_09.

Use the graph of $y=\cos^{-1}x$ to sketch the graphs of:

a) $y=\cos^{-1}(-x)$ b) $y=-\cos^{-1}x$.

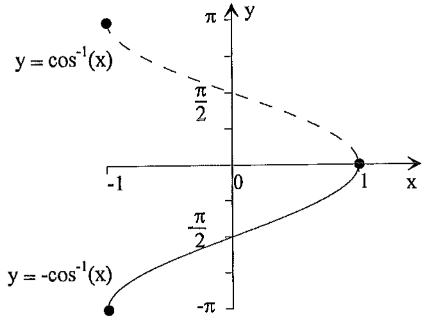
Solution:

a)



The graph $y=\cos^{-1}(-x)$ is a reflection of $y=\cos^{-1}x$ in the y-axis.

b)

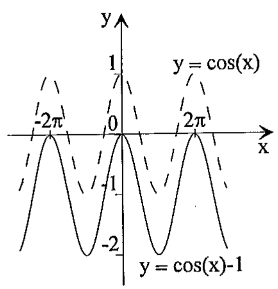


The graph $y=-\cos^{-1}x$ is a reflection of $y=\cos^{-1}x$ in the x-axis.

Problem GRA1_10.

Use the graph of $y=\cos x$ to sketch the graph of $y=\cos x-1$.

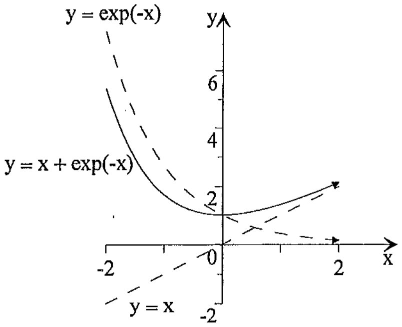
Solution:



The graph $y = \cos x - 1$ is obtained by translating the graph $y = \cos x$ through one unit down.

Problem GRA1_11.
Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = x + e^{-x}$.

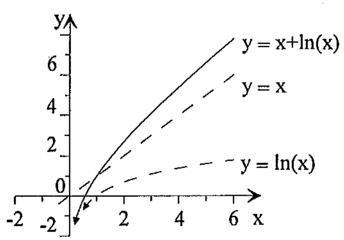
Solution:



The ordinates of the graph $y = x + e^{-x}$ are obtained by summing the ordinates of the graphs $y = x$ and $y = e^{-x}$.

Problem GRA1_12.
Use the graphs of $y = x$ and $y = \ln x$ to sketch the graph of $y = x + \ln x$.

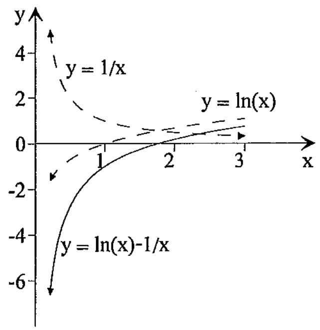
Solution:



The ordinates of the graph $y = x + \ln x$ are obtained by summing the ordinates of the graphs $y = x$ and $y = \ln x$.

Problem GRA1_13.
Use the graphs of $y = \ln x$ and $y = \frac{1}{x}$ to sketch the graph of $y = \ln x - \frac{1}{x}$.

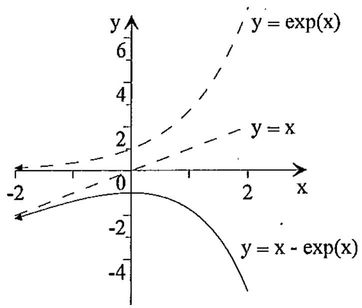
Solution:



In order to sketch the graph of $y = \ln x - \frac{1}{x}$ we apply the procedure of subtraction of ordinates of the graphs $y = \ln x$ and $y = \frac{1}{x}$.

Problem GRA1_14.
Use the graphs of $y = x$ and $y = e^{x}$ to sketch the graph of $y = x - e^{x}$.

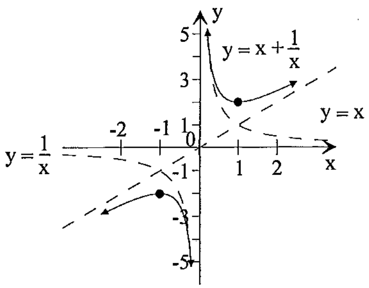
Solution:



In order to sketch the graph $y = x - e^{x}$ we apply the procedure of subtraction of ordinates of the graphs $y = x$ and $y = e^{x}$.

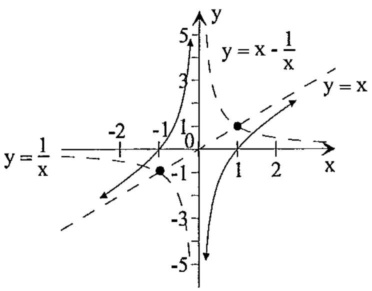
Problem GRA1_15.
Sketch the graphs of a) $y = x + \frac{1}{x}$, b) $y = x - \frac{1}{x}$.

Solution:
a)



The ordinates of the graph $y = x + \frac{1}{x}$ are obtained by summing the ordinates of the graphs $y = x$ and $y = \frac{1}{x}$.

b)



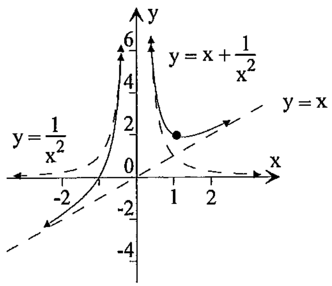
In order to sketch the graph of $y = x - \frac{1}{x}$ we apply the procedure of subtraction of ordinates of the graphs $y = x$ and $y = \frac{1}{x}$.

Problem GRA1_16.

Sketch the graphs of a) $y = x + \frac{1}{x^2}$, b) $y = x - \frac{1}{x^2}$.

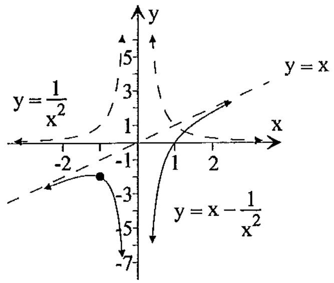
Solution:

a)



The ordinates of the graph $y = x + \frac{1}{x^2}$ are obtained by summing the ordinates of the graphs $y = x$ and $y = \frac{1}{x^2}$.

b)

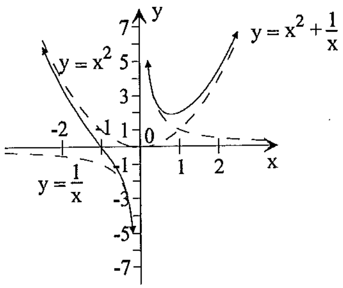


The ordinates of the graph $y = x - \frac{1}{x^2}$ are obtained by subtraction of the ordinates of the graphs $y = x$ and $y = \frac{1}{x^2}$.

Problem GRA1_17.

Sketch the graph of $y = x^2 + \frac{1}{x}$.

Solution:

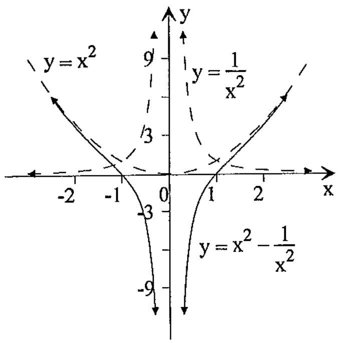


The ordinates of the graph $y = x^2 + \frac{1}{x}$ are obtained by summing the ordinates of the graphs $y = x^2$ and $y = \frac{1}{x}$.

Problem GRA1_18.

Sketch the graph of $y = x^2 - \frac{1}{x}$.

Solution:

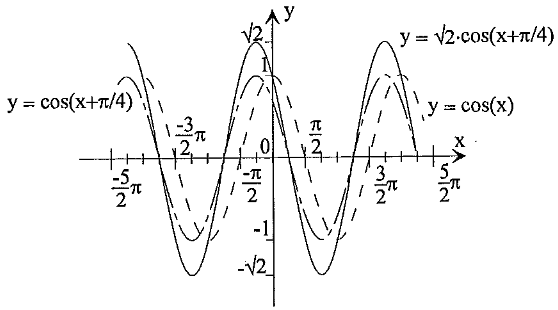


The ordinates of the graph $y = x^2 - \frac{1}{x}$ are obtained by applying the procedure of subtraction of ordinates of the graphs $y = x^2$ and $y = \frac{1}{x}$. Clearly the function $x^2 - \frac{1}{x^2}$ is even, and hence the graph $y = x^2 - \frac{1}{x^2}$ is symmetric about y-axis.

Problem GRA1_19.

Sketch the graph of $y = \cos x - \sin x$.

Solution:



$$y = \cos x - \sin x = \sqrt{2}(\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) \Rightarrow \ y = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right).$$

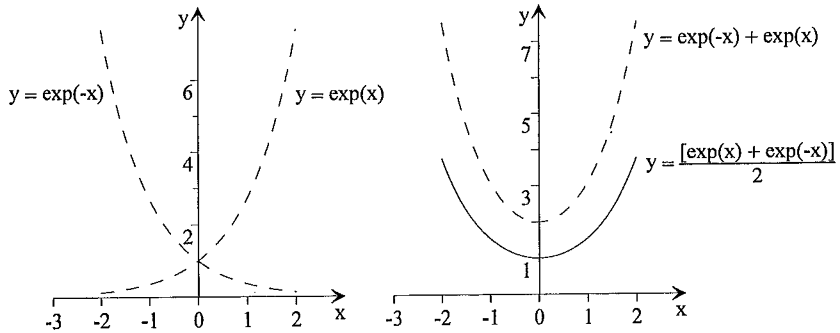
The graph $y = \cos\left(x + \frac{\pi}{4}\right)$ is obtained by translating the graph $y = \cos x$ through $\frac{\pi}{4}$ units to the left.

The graph $y = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ is obtained by enlarging $y = \cos\left(x + \frac{\pi}{4}\right)$ by a factor $\sqrt{2}$ in the direction parallel to the y-axis.

Problem GRA1_20.

Sketch the graph of $y = \frac{1}{2}\left(e^x + e^{-x}\right)$.

Solution:

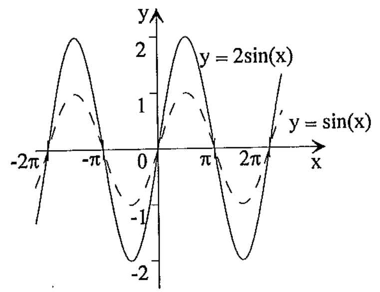


The graph of $y = e^x + e^{-x}$ is obtained by summing of ordinates of the graphs $y = e^x$ and $y = e^{-x}$.

The graph of $y=\frac{1}{2}\left(e^{x}+e^{-x}\right)$ is obtained by enlarging $y=e^{x}+e^{-x}$ by a factor $\frac{1}{2}$ in the direction parallel to the y-axis.

Problem GRA1_21.
Use the graph of $y=\sin x$ to sketch the graph of $y=2\sin x$.

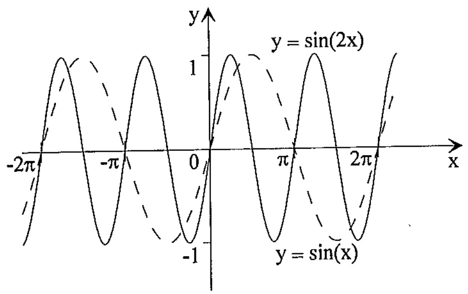
Solution:



The graph of $y=2\sin x$ is obtained by enlarging $y=\sin x$ by a factor 2 parallel to the y-axis.

Problem GRA1_22.
Use the graph of $y=\sin x$ to sketch the graph of $y=\sin 2x$.

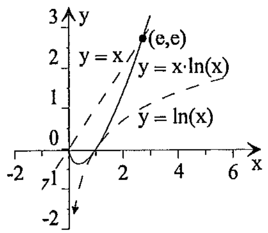
Solution:



The graph of $y=\sin 2x$ is obtained by enlarging $y=\sin x$ by a factor $\frac{1}{2}$ parallel to the x-axis.

Problem GRA1_23.
Use the graphs of $y=x$ and $y=\ln x$ to sketch the graph of $y=x\ln x$.

Solution:

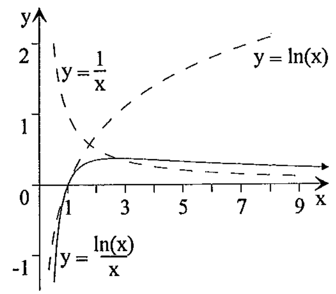


The graph $y=x\ln x$ is obtained by multiplication of ordinates of $y=x$ and $y=\ln x$.
Features of $y=x\ln x$:

- $y=0$ when $x=1$
- $y=e$ when $x=e$
- $y=x\ln x$ lies above $y=\ln x$ touching it at $x=1$.
- As $x\to 0^{+}$, $\ln x\to-\infty$ more slowly than $\frac{1}{x}\to+\infty$ and hence $x\ln x\to 0^{+}$.

Problem GRA1_24.
Use the graphs of $y=\ln x$ and $y=\frac{1}{x}$ to sketch the graph of $y=\frac{\ln x}{x}$.

Solution:



The graph of $y=\frac{\ln x}{x}$ is obtained by multiplication of ordinates of $y=\frac{1}{x}$ and $y=\ln x$.

Features of $y=\frac{\ln x}{x}$:

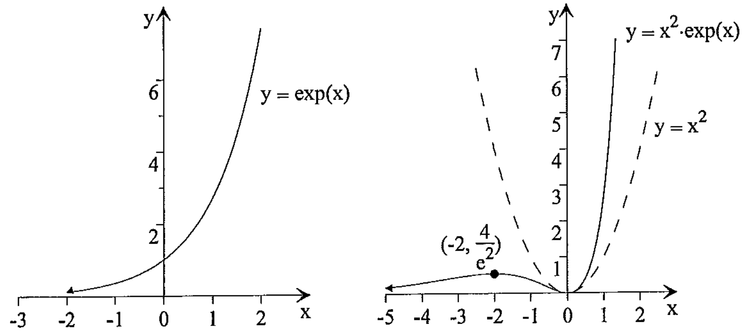
- $y=0$ when $x=1$.
- $y=\frac{\ln x}{x}$ lies below $y=\ln x$ for all $x>0$.
- $y=\frac{\ln x}{x}$ lies above $y=\frac{1}{x}$ only for $x>e$.

• As $x \to \infty$, $\ln x \to \infty$ more slowly than any power of x and hence $\frac{\ln x}{x} \to 0$.

Problem GRA1_25.

Sketch the graph of $y = x^2 e^x$.

Solution:



The graph $y = x^2 e^x$ is obtained by multiplication of ordinates $y = x^2$ and $y = e^x$.

Features:

• $y = 0$ when $x = 0$

• $y = x^2 e^x$ lies above $y = x^2$ for $x > 0$ (where $e^x > 1$).

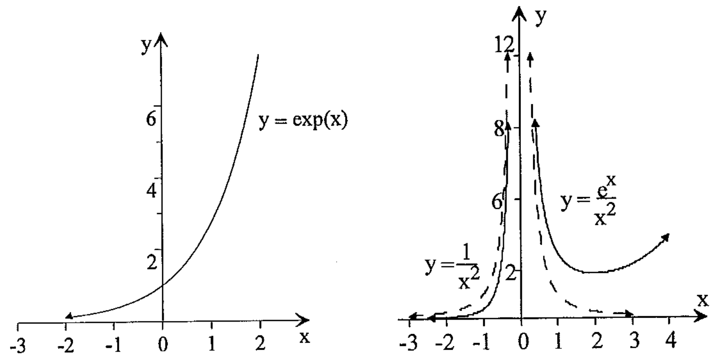
• $y = x^2 e^x$ lies below $y = x^2$ for $x < 0$ (where $e^x < 1$).

• As $x \to -\infty$, $x^2 \to +\infty$ more slowly than e^{-x} and hence $x^2 e^x \to 0^+$.

Problem GRA1_26.

Sketch the graph of $y = \frac{e^x}{x^2}$.

Solution:



The graph $y = \frac{e^x}{x^2}$ is obtained by multiplication of ordinates $y = \frac{1}{x^2}$ and $y = e^x$.

Features:

• Domain $\{x : x \neq 0\}$

• $y = \frac{e^x}{x^2}$ lies above $y = \frac{1}{x^2}$ for $x > 0$ (where $e^x > 1$).

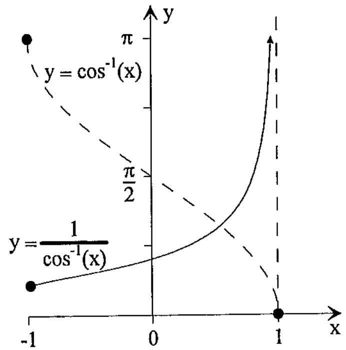
• $y = \frac{e^x}{x^2}$ lies below $y = \frac{1}{x^2}$ for $x < 0$ (where $e^x < 1$).

• As $x \to 0$, $\frac{e^x}{x^2} \to +\infty$.

Problem GRA1_27.

Sketch the graph of $y = \frac{1}{\cos^{-1} x}$.

Solution:



The graph of $y=\frac{1}{\cos^{-1}x}$ are constructed by considering the features of $y=\cos^{-1}x$.

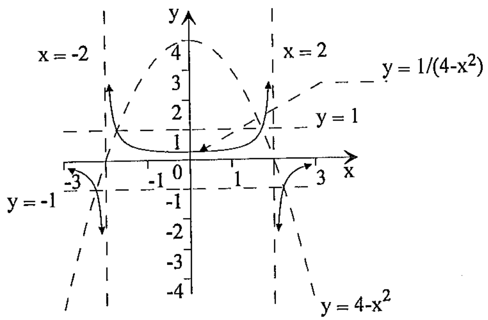
Features of $y=\frac{1}{\cos^{-1}x}$:

- $\cos^{-1}x$ decreases $\Rightarrow \frac{1}{\cos^{-1}x}$ increases.
- $\cos^{-1}x=0$ when $x=1 \Rightarrow$ the line $x=1$ is the vertical asymptote of $y=\frac{1}{\cos^{-1}x}$.

Problem GRA1_28.

Use the graph of $f(x)=4-x^2$ (an even function) to sketch the graph of $y=\frac{1}{f(x)}$. Is this the graph of an even function?

Solution:



$\frac{1}{f(-x)}=\frac{1}{4-(-x)^2}=\frac{1}{4-x^2}=\frac{1}{f(x)} \Rightarrow y=\frac{1}{4-x^2}$ is an even function.

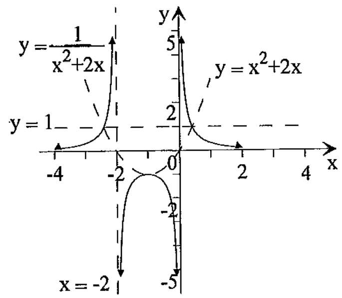
Features:

- $f(x), \frac{1}{f(x)}$ have the same sign.
- $f(x)=0$ when $x=\pm 2 \Rightarrow$ the lines $x=-2$ and $x=+2$ correspond to vertical asymptotes of $y=\frac{1}{f(x)}$.
- As $x\to\infty, f(x)\to-\infty \Rightarrow \frac{1}{f(x)}\to 0^-$.
- Maximum turning point of $y=f(x)$ is $(0, 4) \Rightarrow$ minimum turning point of $y=\frac{1}{f(x)}$ is $(0, \frac{1}{4})$.

Problem GRA1_29.

Use the graph $y=x(x+2)$ to sketch the graph of $y=\frac{1}{x(x+2)}$.

Solution:



$y=\frac{1}{f(x)}$ where $f(x)=x(x+2)$.

Features of $y=\frac{1}{f(x)}$:

- $f(x), \frac{1}{f(x)}$ have the same sign.
- $f(x)=0$ when $x=-2$ or $x=0$ $\Rightarrow$ the lines $x=-2$ and $x=0$ correspond to vertical asymptotes of $y=\frac{1}{f(x)}$.
- As $x\to\infty, f(x)\to+\infty \Rightarrow \frac{1}{f(x)}\to 0^+$.
- Minimum turning point of $y=f(x)$ is $(-1, 1)$ $\Rightarrow$ maximum turning point of $y=\frac{1}{f(x)}$ is $(-1, 1)$.

Problem GRA1_30.

Use the graph of $y=1-x^2$ to sketch the graph of $y=(1-x^2)^2$.

Solution: