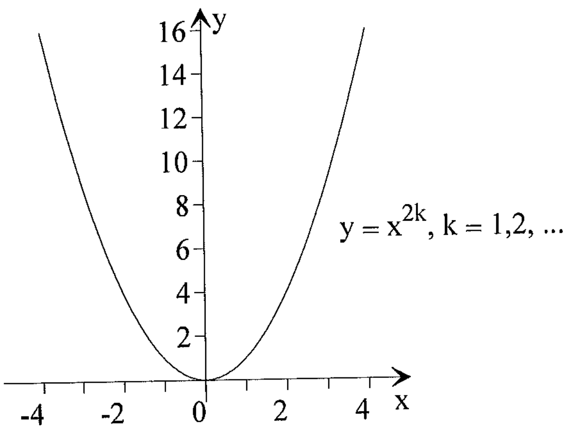
Topic 7. Graphs.
Level 2.
Problem GRA2_01.

For $n \geq 2$, an even positive integer, sketch the graphs of a) $y = x^n$; b) $y = x^{-n}$, c) $y = x^{1/n}$; d) $y = x^{-1/n}$.

Solution:

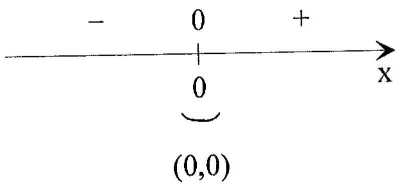
a)



$y = x^n$, $n = 2k$, $k = 1,2,...$

$\frac{dy}{dx} = 2kx^{2k-1}$

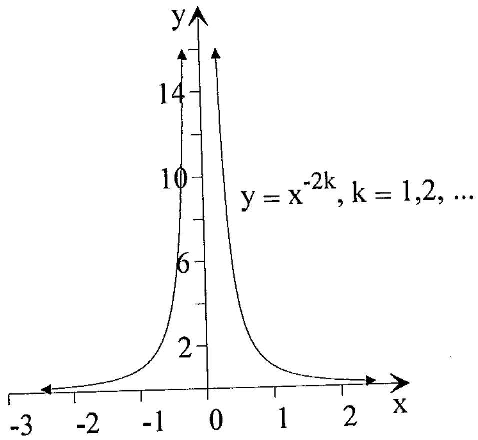
Sign of $\frac{dy}{dx}$



The minimum turning point is (0, 0).

$(-x)^{2k} = x^{2k} \Rightarrow$ the graph $y = x^{2k}$ is the graph of an even function $\Rightarrow$ the graph $y = x^{2k}$ is symmetric about y-axis.

b)

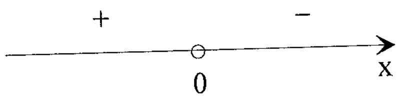


$y = x^{-n}, n = 2k, k = 1,2,...$

Domain $\{x : x \neq 0\}$

$$\frac{dy}{dx} = -2kx^{-2k-1}$$

Sign of $\frac{dy}{dx}$

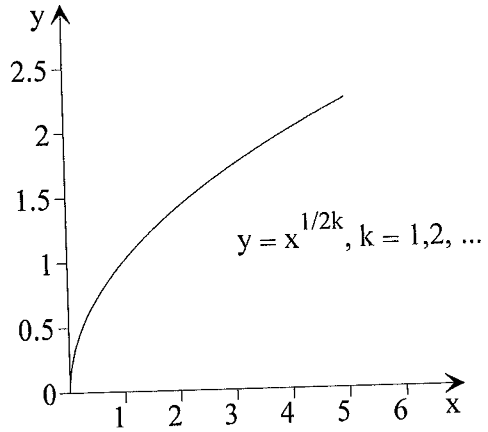


$y \to \infty$ as $x \to 0 \Rightarrow$ the line $x = 0$ is a vertical asymptote.

$y \to 0$ as $x \to \infty \Rightarrow$ the line $y = 0$ is a horizontal asymptote.

$(-x)^{-2k} = x^{-2k} \Rightarrow$ the function $y = x^{-2k}$ is even and the graph $y = x^{-2k}$ is symmetric about y-axis.

c)



$y = x^{\frac{1}{n}}, n = 2k, k = 1,2,...$

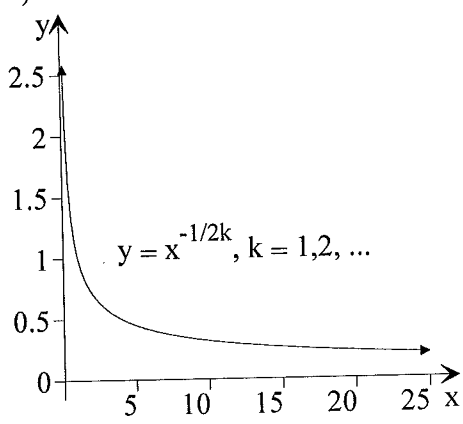
Domain $\{x : x \geq 0\}$

$$\frac{dy}{dx} = \frac{1}{n} x^{\frac{1}{n}-1}$$

$\frac{dy}{dx}$ is not defined at $x = 0$.

$\frac{dy}{dx} \to \infty$ as $x \to 0^+ \Rightarrow$ the tangent line at the critical point (0, 0) is vertical.

d)



$y = x^{-\frac{1}{n}}$, $n = 2k$, $k = 1,2,\ldots$

Domain $\{x : x > 0\}$

$y \to +\infty$ as $x \to 0^+ \Rightarrow$ the line $x = 0$ is a vertical asymptote.

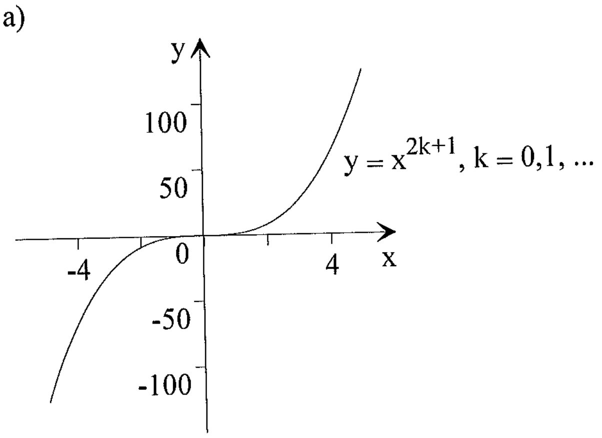
$y \to 0^+$ as $x \to +\infty \Rightarrow$ the line $y = 0$ is a horizontal asymptote.

Problem GRA2_02.

For $n \geq 3$, an odd positive integer, sketch the graphs of a) $y = x^n$; b) $y = x^{-n}$, c) $y = x^{\frac{1}{n}}$; d) $y = x^{-\frac{1}{n}}$.

Solution:

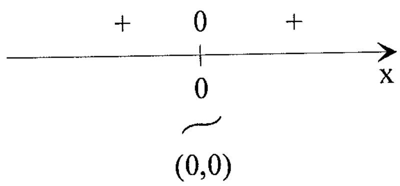
a)



$y = x^n$, $n = 2k + 1$, $k = 0,1,\ldots$

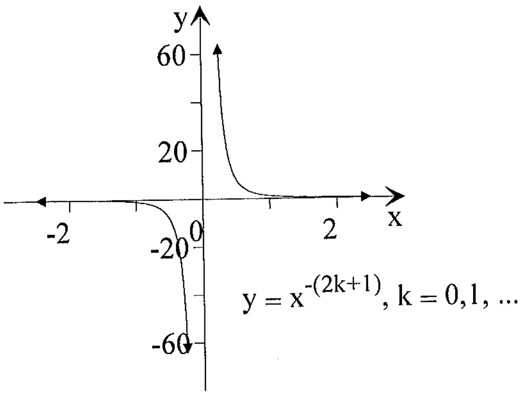
$\frac{dy}{dx} = nx^{n-1} \equiv (2k+1)x^{2k}$

Sign of $\frac{dy}{dx}$



Clearly the curve is symmetric about the point (0, 0) as the transformation $y \to -y$ and $x \to -x$ leaves the Cartesian equation of the curve unchanged.

b)



$y = x^{-n}$, $n = 2k+1$, $k = 0,1,\ldots$

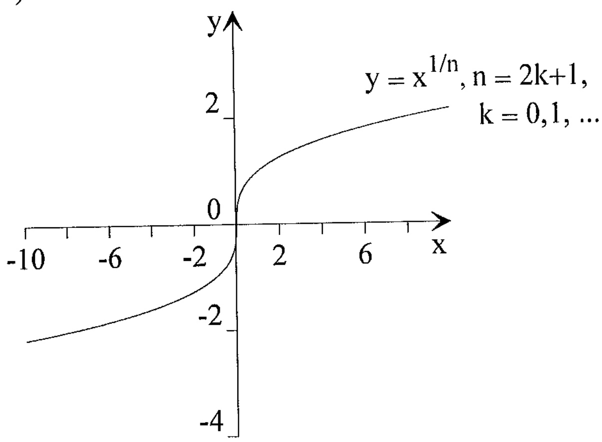
Domain $\{x : x \neq 0\}$

As $x \to 0^-$, $y \to -\infty$, and as $x \to 0^+$, $y \to +\infty \Rightarrow$ the line $x = 0$ is a vertical asymptote.

As $x \to \infty$, $y \to 0 \Rightarrow$ the line $y = 0$ is a horizontal asymptote.

Clearly the curve is symmetric about the point (0, 0) as the transformation $y \to -y$ and $x \to -x$ leaves the Cartesian equation of the curve unchanged.

c)



$y = x^{\frac{1}{n}}$, $n = 2k+1$, $k = 0,1,\ldots$

$$\frac{dy}{dx} = \frac{1}{n} x^{\frac{1}{n}-1}$$

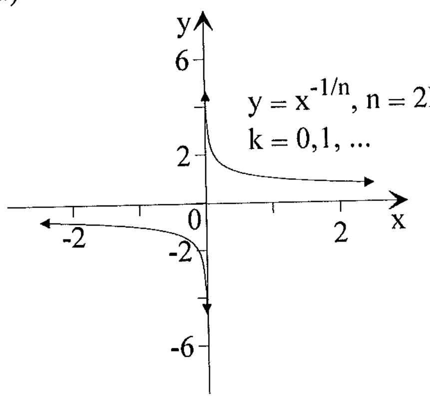
$\frac{dy}{dx}$ is not defined at $x = 0$.

$\Rightarrow$ (0, 0) is a critical point.

$\frac{dy}{dx} \to \infty$ as $x \to 0 \Rightarrow$ the tangent line at (0, 0) is vertical.

Clearly the curve is symmetric about the point (0, 0) as the transformation $y \to -y$ and $x \to -x$ leaves the Cartesian equation of the curve unchanged

d)



$y = x^{-\frac{1}{n}}$, $n = 2k+1$, $k = 0,1,...$

Domain $\{x : x \neq 0\}$

As $x \to 0^-$, $y \to -\infty$, and as $x \to 0^+$, $y \to +\infty \Rightarrow$ the line $x = 0$ is a vertical asymptote.

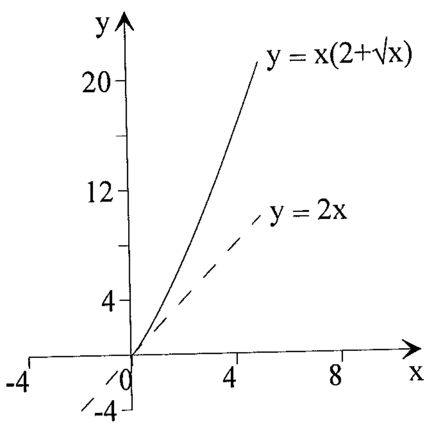
As $x \to \infty$, $y \to 0 \Rightarrow$ the line $y = 0$ is a horizontal asymptote.

Clearly the curve is symmetric about the point (0, 0) as the transformation $y \to -y$ and $x \to -x$ leaves the Cartesian equation of the curve unchanged.

Problem GRA2_03.

Sketch (showing critical points) the graph of $y = x(2+\sqrt{x})$.

Solution:



$y = x(2+\sqrt{x})$

Domain $\{x : x \geq 0\}$.

$\frac{dy}{dx} = 2 + \frac{3}{2}\sqrt{x}$, $x > 0$. $\frac{dy}{dx} \to 2$ as $x \to 0^+$ $\Rightarrow$

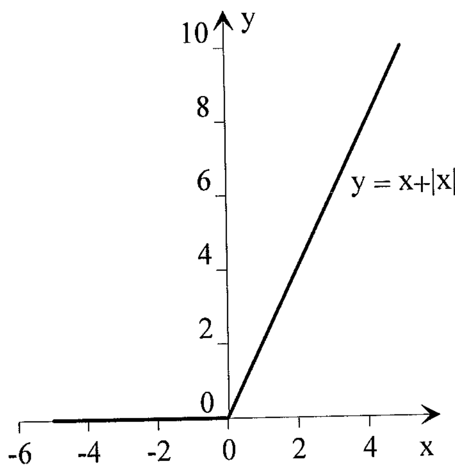
the tangent line at the critical point $(0, 0)$ is $y = 2x$.

$y = x(2 + \sqrt{x})$

Problem GRA2_04.

Sketch (showing critical points) the graph of $y = x + |x|$.

Solution:



$y = x + |x|$

$$y = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \frac{dy}{dx} = \begin{cases} 2, & x > 0 \\ 0, & x < 0 \end{cases}$$

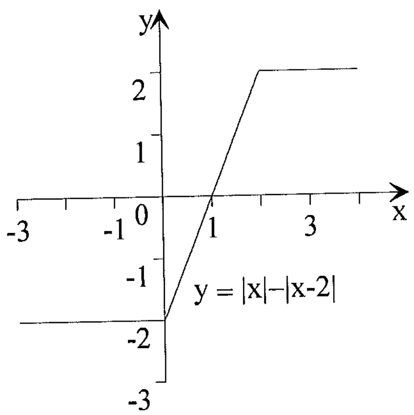
$\frac{dy}{dx} \to 2$ as $x \to 0^{+}$ $\frac{dy}{dx} \to 0$ as $x \to 0^{-}$

$\Rightarrow \frac{dy}{dx}$ is not defined at $x = 0$, and $(0, 0)$ is a critical point.

Problem GRA2_05.

Sketch (showing critical points) the graph of $y = |x| - |x - 2|$.

Solution:



$y = |x| - |x-2|$

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$, $|x-2| = \begin{cases} x-2, & x \ge 2 \\ -x+2, & x < 2 \end{cases}$

Note that,

if $x \ge 2$, $y = x - x - 2 = 2$

if $2 > x \ge 0$, $y = x + x - 2 = 2x - 2$

if $x < 0$, $y = -x + x - 2 = -2$

$\Rightarrow y = \begin{cases} 2, & x \ge 2 \\ 2x-2, & 2 > x \ge 0 \\ -2, & x < 0 \end{cases}$ $\qquad \dfrac{dy}{dx} = \begin{cases} 0, & x > 2 \\ 2, & 0 < x < 2 \\ 0, & x < 0 \end{cases}$

$\dfrac{dy}{dx} \to 0$ as $x \to 2^+$ $\quad \dfrac{dy}{dx} \to 2$ as $x \to 2^-$

$\Rightarrow \dfrac{dy}{dx}$ is not defined at $x = 2$, and $(2, 2)$ is a critical point.

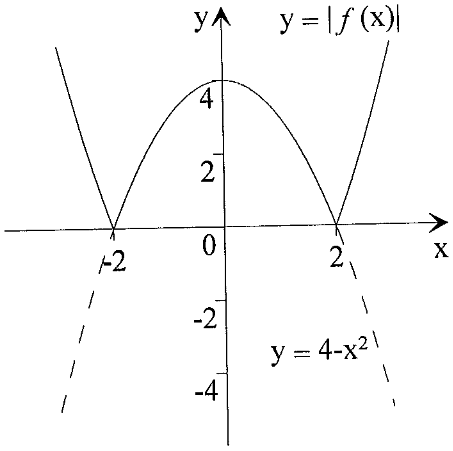
$\dfrac{dy}{dx} \to 2$ as $x \to 0^+$ $\quad \dfrac{dy}{dx} \to 0$ as $x \to 0^-$

$\Rightarrow \dfrac{dy}{dx}$ is not defined at $x = 0$, and $(0, -2)$ is a critical point.

Problem GRA2_06.

Use the graph of $f(x) = 4 - x^2$ (an even function) to sketch (showing critical points) the graph of $y = |f(x)|$.

Solution:



$f(x) = 4 - x^2 \Rightarrow f(-x) = 4 - (-x)^2 = 4 - x^2 \Rightarrow f(-x) = f(x) \Rightarrow$ the graph of $y = f(x)$ is symmetric about the y-axis.

$y = |4 - x^2| = \begin{cases} 4 - x^2, & 4 - x^2 \ge 0 \\ x^2 - 4, & 4 - x^2 < 0 \end{cases}$

The sections of $y = 4 - x^2$ which lie below the x-axis are reflected in the x-axis. The graph of $y = |4 - x^2|$ is also symmetric about the y-axis, as $|4 - (-x)^2| = |4 - x^2|$.

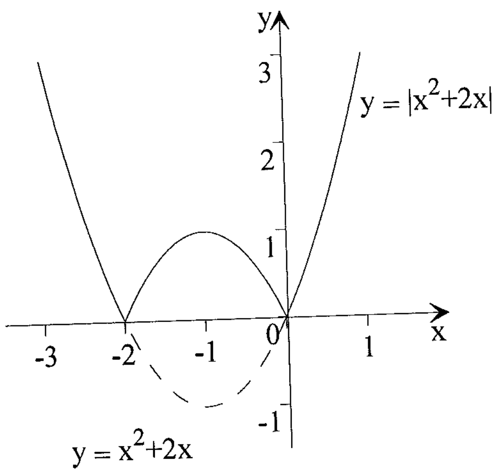
$$y = |4 - x^2| = \begin{cases} x^2 - 4, & x < -2 \\ 4 - x^2, & -2 \le x < 2 \\ x^2 - 4, & x \ge 2 \end{cases} \qquad \frac{dy}{dx} = \begin{cases} 2x, & x < -2 \\ -2x, & -2 < x < 2 \\ 2x, & x > 2 \end{cases}$$

$\frac{dy}{dx} \to 4$ as $x \to 2^+$, $\frac{dy}{dx} \to -4$ as $x \to 2^- \Rightarrow \frac{dy}{dx}$ is not defined at $x = 2$, and (2, 0) is a critical point. Hence the symmetric about the y-axis point (-2,0) is also critical.

Problem GRA2_07.

Use the graph $y = x(x+2)$ to sketch showing critical points the graph of $y = |x(x+2)|$.

Solution:



Those sections of $y = x(x+2)$ which lie below x-axis are reflected in the x-axis.

$$y = |x(x+2)| = \begin{cases} x(x+2), & x < -2 \\ -x(x+2), & -2 \le x < 0 \\ x(x+2), & 0 \le x \end{cases} \qquad \frac{dy}{dx} = \begin{cases} 2x+2, & x < -2 \\ -2x-2, & -2 < x < 0 \\ 2x+2, & 0 < x \end{cases}$$

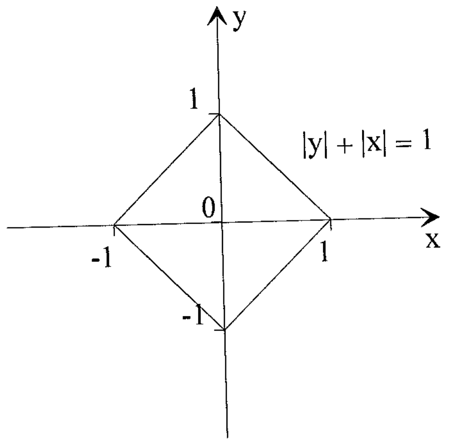
$\frac{dy}{dx} \to -2$ as $x \to -2^-$, $\frac{dy}{dx} \to 2$ as $x \to -2^+$ $\Rightarrow \frac{dy}{dx}$ is not defined at $x = -2$, and $(-2, 0)$ is a critical point.

$\frac{dy}{dx} \to -2$ as $x \to 0^-$, $\frac{dy}{dx} \to 2$ as $x \to 0^+$ $\Rightarrow \frac{dy}{dx}$ is not defined at $x = 0$, and $(0, 0)$ is a critical point.

Problem GRA2_08.

Sketch the graph of $|x| + |y| = 1$.

Solution:



$|x| + |y| = 1$

Clearly $|x|, |y| \geq 0 \Rightarrow$ domain $\{x : -1 \leq x \leq 1\}$, range $\{y : -1 \leq y \leq 1\}$.

If $y \geq 0$, then $y = \begin{cases} 1 + x, & -1 \leq x < 0 \\ 1 - x, & 0 \leq x \leq 1 \end{cases}$, $\dfrac{dy}{dx} = \begin{cases} 1, & -1 < x < 0 \\ -1, & 0 < x < 1 \end{cases}$

$\Rightarrow \dfrac{dy}{dx} \to +1 \text{ as } x \to 0^-$ and $\dfrac{dy}{dx} \to -1 \text{ as } x \to 0^+$

$\Rightarrow \dfrac{dy}{dx}$ is not defined at $x = 0$, and (0, 1) is a critical point.

The curve is symmetric about x-axis as the transformation $y \to -y$ leaves the Cartesian equation of the curve unchanged. Hence the symmetric point (0, -1) is also critical.

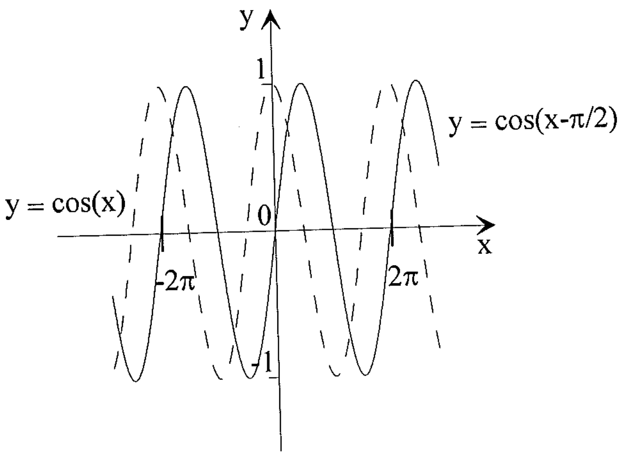
If $y \leq 0$, then $y = \begin{cases} -1 - x, & -1 \leq x < 0 \\ -1 + x, & 0 \leq x \leq 1 \end{cases}$.

The curve is also symmetric about the line $y = x$ as the transformation $(x, y) \to (y, x)$ leaves the Cartesian equation of the curve unchanged. Hence the points (1, 0) and (-1, 0) is also critical.

Problem GRA2_09.

Use the graph of $y = \cos x$ to sketch the graph of $y = \cos(x - \frac{\pi}{2})$.

Solution:

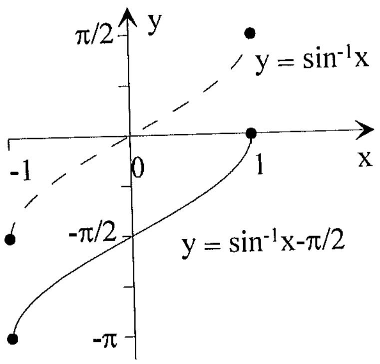


The graph $y=\cos(x-\frac{\pi}{2})$ is obtained by translating the graph $y=\cos x$ through $\frac{\pi}{2}$ units to the right.

Problem GRA2_10.

Use the graph of $y=\sin^{-1}x$ to sketch the graph of: $y=\sin^{-1}x-\frac{\pi}{2}$.

Solution:

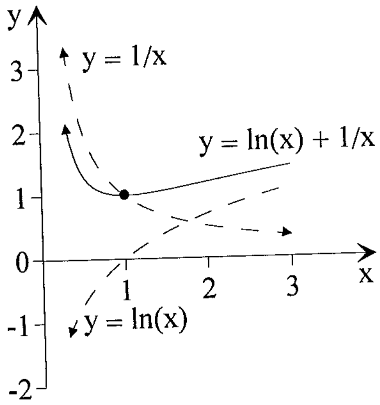


The graph $y=\sin^{-1}x-\frac{\pi}{2}$ is obtained by translating the graph $y=\sin^{-1}x$ through $\frac{\pi}{2}$ units down.

Problem GRA2_11.

Use the graphs of $y=\ln x$ and $y=\frac{1}{x}$ to sketch the graph of $y=\ln x+\frac{1}{x}$.

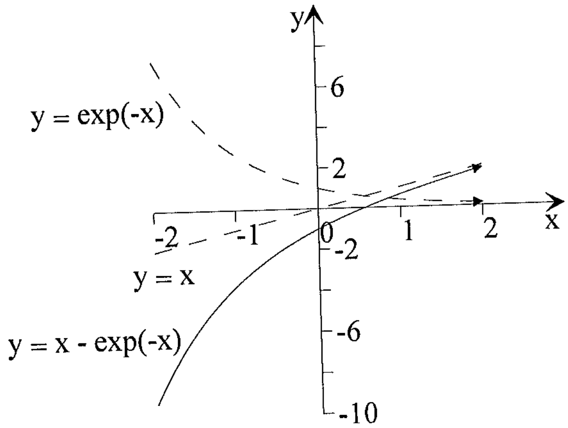
Solution:



The ordinates of the graph $y = \ln x + \frac{1}{x}$ are obtained by summing the ordinates of the graphs $y = \ln x$ and $y = \frac{1}{x}$.

Problem GRA2_12.
Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = x - e^{-x}$.

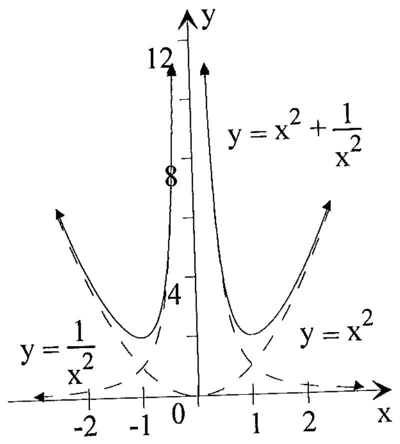
Solution:



In order to sketch the graph of $y = x - e^{-x}$ we apply the procedure of subtraction of ordinates of the graphs $y = x$ and $y = e^{-x}$.

Problem GRA2_13.
Sketch the graph of $y = x^2 + \frac{1}{x^2}$.

Solution:

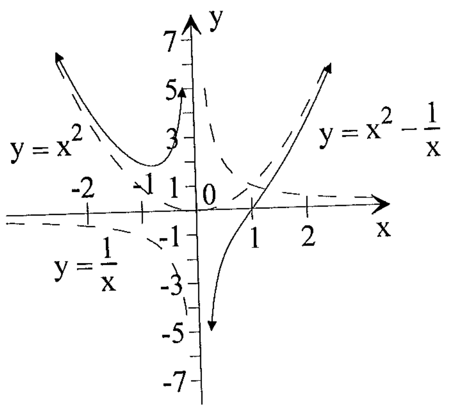


The ordinates of the graph $y = x^2 + \frac{1}{x^2}$ are obtained by summing the ordinates of the graphs $y = x^2$ and $y = \frac{1}{x^2}$. Clearly the function $x^2 + \frac{1}{x^2}$ is even, and hence the graph $y = x^2 + \frac{1}{x^2}$ is symmetric about y-axis.

Problem GRA2_14.

Sketch the graph of $y = x^2 - \frac{1}{x}$.

Solution:

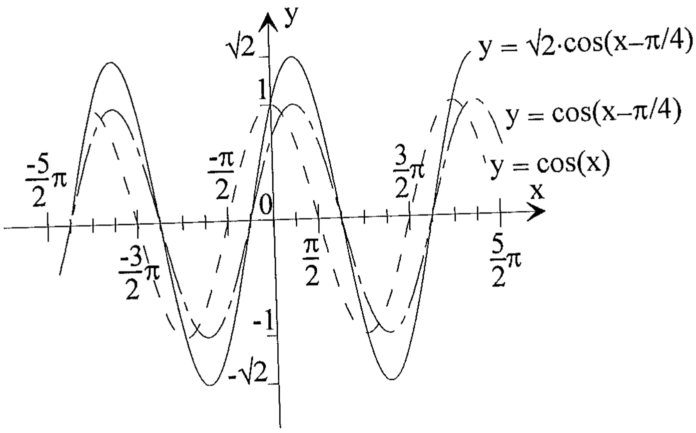


The ordinates of the graph $y = x^2 - \frac{1}{x}$ are obtained by applying the procedure of subtraction of ordinates of the graphs $y = x^2$ and $y = \frac{1}{x}$.

Problem GRA2_15.

Sketch the graph of $y = \cos x + \sin x$.

Solution:



$$y = \cos x + \sin x = \sqrt{2}(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) \Rightarrow y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right).$$

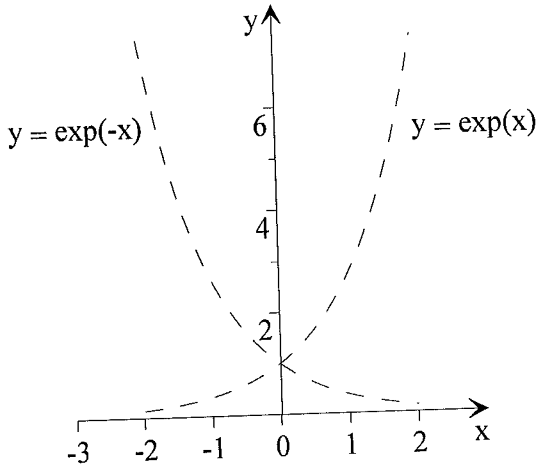
The graph $y = \cos\left(x - \frac{\pi}{4}\right)$ is obtained by translating the graph $y = \cos x$ through $\frac{\pi}{4}$ units to the right.

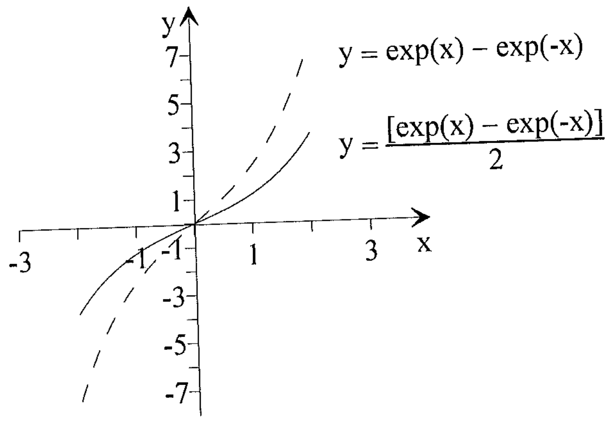
The graph $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ is obtained by enlarging $y = \cos\left(x - \frac{\pi}{4}\right)$ by a factor $\sqrt{2}$ in the direction parallel to the y-axis.

Problem GRA2_16.

Sketch the graph of $y = \frac{1}{2}\left(e^x - e^{-x}\right)$.

Solution:





The graph of $y = e^x - e^{-x}$ is obtained by subtraction of ordinates of the graphs $y = e^x$ and $y = e^{-x}$.

The graph $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ is obtained by enlarging $y = e^x - e^{-x}$ by a factor $\frac{1}{2}$ in the direction parallel to the y-axis.

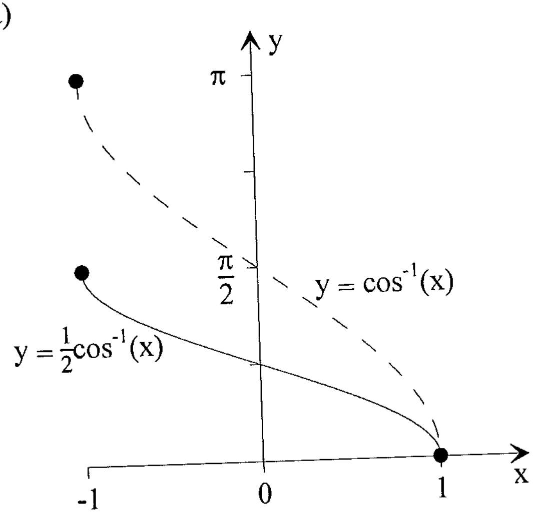
Problem GRA2_17.

Use the graph of $y=\cos^{-1}x$ to sketch the graphs of:

a) $y=\dfrac{1}{2}\cos^{-1}x$, b) $y=\cos^{-1}\left(\dfrac{x}{2}\right)$.

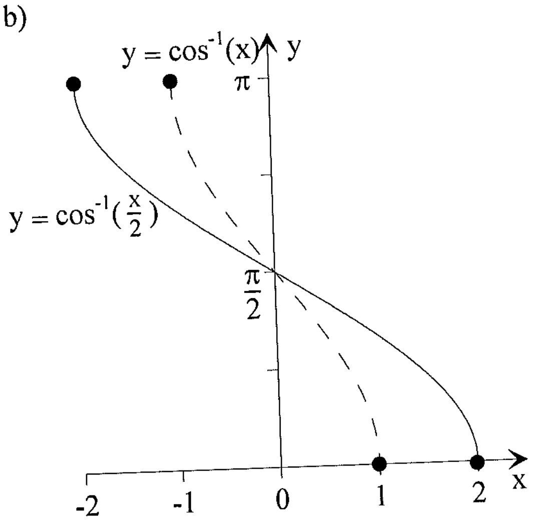
Solution:

a)



The graph $y=\dfrac{1}{2}\cos^{-1}x$ is obtained by enlarging $y=\cos^{-1}x$ by a factor $\dfrac{1}{2}$ in the direction parallel to the y-axis.

b)

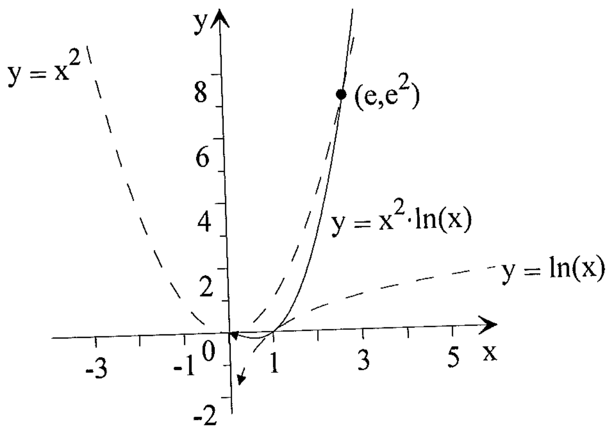


The graph $y=\cos^{-1}\left(\dfrac{x}{2}\right)$ is obtained by enlarging $y=\cos^{-1}x$ by a factor 2 in the direction parallel to the x-axis.

Problem GRA2_18.

Sketch the graph of $y=x^2\ln x$.

Solution:



The graph $y = x^2 \ln x$ is obtained by multiplication of ordinates $y = x^2$ and $y = \ln x$.

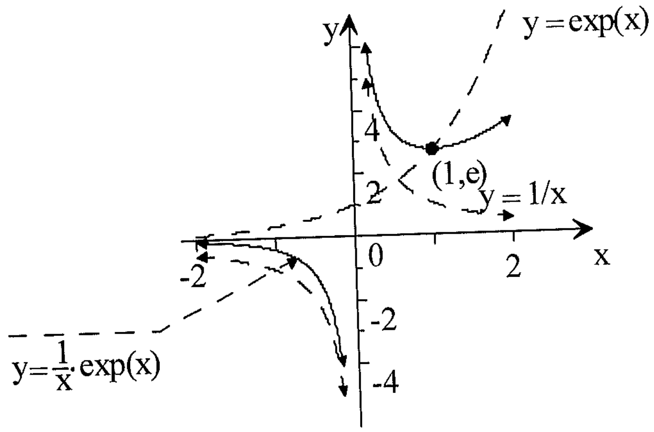
Features:

- Domain $\{x : x > 0\}$
- $y = 0$ when $x = 1$
- $y = x^2 \ln x$ lies above $y = x^2$ only for $x > e$ (where $\ln x > 1$).
- As $x \to 0^+$, $|\ln x| \to \infty$ more slowly than $\frac{1}{x}$ and hence $x \ln x \to 0^-$.

Problem GRA2_19.

Use the graphs of $y = x$ and $y = e^x$ to sketch the graph of $y = \frac{e^x}{x}$.

Solution:



The graph $y = \frac{e^x}{x}$ is obtained by multiplication of ordinates of $y = \frac{1}{x}$ and $y = e^x$.

Features of $y = \frac{e^x}{x}$:

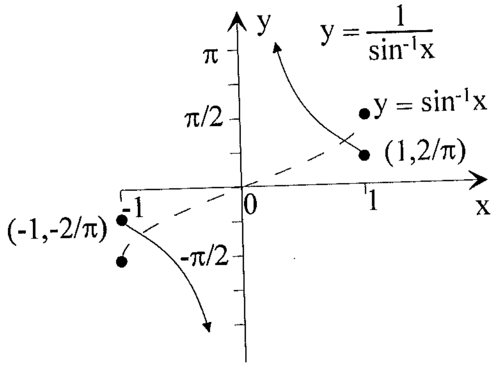
- Domain $\{x : \ x \neq 0\}$

- $y = e$ as $x = 1$
- $y = \frac{e^x}{x}$ lies above $y = \frac{1}{x}$ for $x > 0$ and for $x < 0$
- As $x \to 0^+$, $e^x \to 1 \Rightarrow y = \frac{e^x}{x} \to +\infty$
- As $x \to 0^-$, $e^x \to 1 \Rightarrow y = \frac{e^x}{x} \to -\infty$

Problem GRA2_20.

Sketch the graph of $y = \frac{1}{\sin^{-1} x}$.

Solution:



The graph $y = \frac{1}{\sin^{-1} x}$ are constructed by considering the features of $y = \sin^{-1} x$.

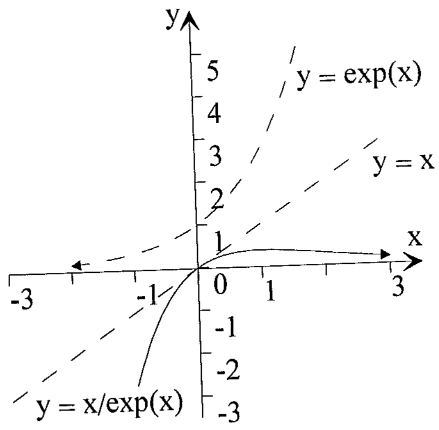
Features of $y = \frac{1}{\sin^{-1} x}$:

- $\sin^{-1} x$, $\frac{1}{\sin^{-1} x}$ have the same sign.
- $\sin^{-1} x$ increases $\Rightarrow \frac{1}{\sin^{-1} x}$ decreases.
- $\sin^{-1} x = 0$ when $x = o \Rightarrow$ the line $x = 0$ is the vertical asymptote of $y = \frac{1}{\sin^{-1} x}$.

Problem GRA2_21.

Use the graphs of $y = x$ and $y = e^x$ to sketch the graph of $y = \frac{x}{e^x}$.

Solution:



The graph of $y = \frac{x}{e^x}$ is obtained by division of ordinates of the graphs $y = x$ and $y = e^x$.

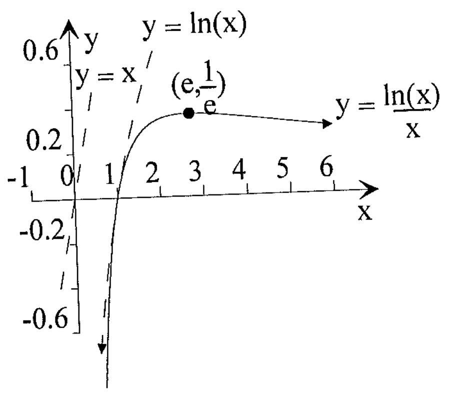
Features:

- $y = 0$ when $x = 0$.
- For all real x, $xe^x < x \Rightarrow$ the graph $y = \frac{x}{e^x}$ lies below the line $y = x$.
- $y = \frac{x}{e^x} > 0$ only for $x > 0$, and $y = \frac{x}{e^x} < 0$ only for $x < 0$.
- As $x \to +\infty$, $e^x \to +\infty$ more quickly than any power of x and hence $\frac{x}{e^x} \to 0^+$.
- As $x \to -\infty$, $\frac{x}{e^x} = xe^{-x} \to -\infty$ more quickly than e^{-x}.

Problem GRA2_22.

Use the graphs of $y = x$ and $y = \ln x$ to sketch the graph of $y = \frac{\ln x}{x}$.

Solution:



The graph of $y = \frac{\ln x}{x}$, domain $\{x : x > 0\}$, is obtained by division of ordinates of the graphs $y = \ln x$ and $y = x$.

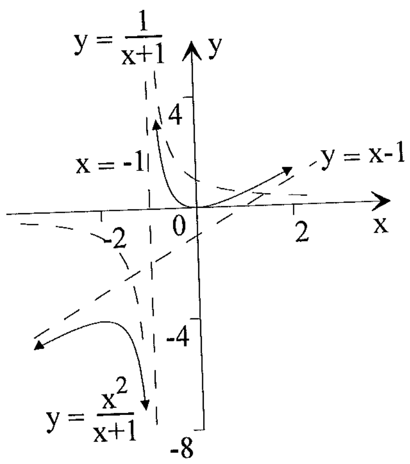
Features:

- $y = 0$ when $x = 1$.
- As $x \to +\infty$, $\ln x \to +\infty$ at a much slower rate than any power of x and hence $\frac{\ln x}{x} \to 0^+$.
- As $x \to 0^+$, $\frac{\ln x}{x} \to -\infty$.
- For all $x > 0$, $\frac{\ln x}{x} < \ln x \Rightarrow$ the graph of $y = \frac{\ln x}{x}$ lies below $y = \ln x$.

Problem GRA2_23.

Show that $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Hence sketch the graph of $y = \frac{x^2}{x+1}$.

Solution:



$$\frac{x^2}{x+1} = \frac{x^2 - 1 + 1}{x+1} = \frac{x^2 - 1}{x+1} + \frac{1}{x+1} = x - 1 + \frac{1}{x+1}.$$

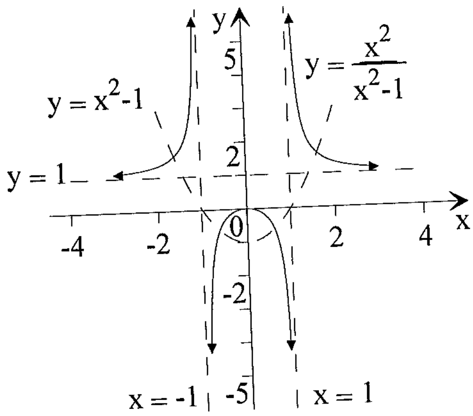
The graph has been constructed by addition of the ordinates of $y = x - 1$ and $y = \frac{1}{x+1}$.

$y = x - 1$ is an asymptote as $x \to \infty$.

Problem GRA2_24.

Sketch the graph of $y = \frac{x^2}{x^2 - 1}$.

Solution:



$$y = \frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1} \quad \Rightarrow y = 1 + \frac{1}{x^2-1}$$

The graph $y = \frac{1}{x^2-1}$ has been translated one unit upward.

$y = 1$ is an asymptote as $x \to \infty$. The graph $y = \frac{1}{x^2-1}$ is a reciprocal of $y = x^2 - 1$.

Consider the graphs $y = f(x)$ and $y = \frac{1}{f(x)}$, where $f(x) = x^2 - 1$.

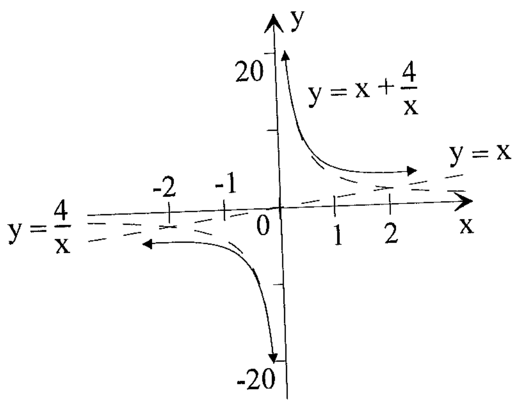
Features:

- $f(x), \frac{1}{f(x)}$ have the same sign.
- $f(x) = 0$ when $x = \pm 1$ $\Rightarrow$ the lines $x = -1$ and $x = 1$ are vertical asymptotes of $y = \frac{1}{f(x)}$.
- As $x \to \infty$, $f(x) \to +\infty$ $\Rightarrow \frac{1}{f(x)} \to 0^+$.
- Minimum turning point of $y = f(x)$ is $(0, -1)$ $\Rightarrow$ maximum turning point of $y = \frac{1}{f(x)}$ is $(0, -1)$.

Problem GRA2_25.

Sketch the graph of $y = \frac{x^2+4}{x}$.

Solution:



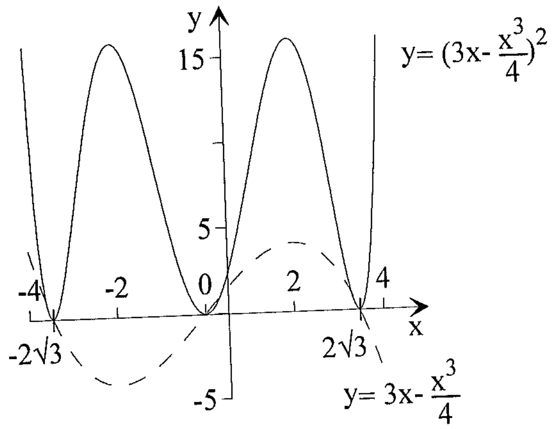
$y = \frac{x^2+4}{x} = x + \frac{4}{x}$. Domain $\{x: \quad x \neq 0\}$.

The graph has been constructed by addition of the ordinates of $y = x$ and $y = \frac{4}{x}$. $y = x$ is an asymptote as $x \to \infty$.

Problem GRA2_26.

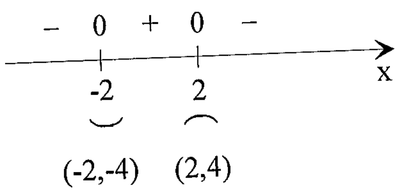
Use the graph of $y = 3x - \frac{x^3}{4}$ to sketch the graph of. $y = \left(3x - \frac{x^3}{4}\right)^2$

Solution:



$f(x) = 3x - \frac{x^3}{4}$ $\quad f'(x) = 3 - \frac{3}{4}x^2$

Sign of $f'(x)$



$y = [f(x)]^2 \quad \frac{dy}{dx} = 2f(x)f'(x) \quad \frac{dy}{dx} = 2x(3 - \frac{x^2}{4})(3 - \frac{3}{4}x^2)$

$\frac{dy}{dx} = \frac{3}{8}x(2\sqrt{3} - x)(2\sqrt{3} + x)(2 - x)(2 + x)$

Sign of $\frac{dy}{dx}$

$- \; 0 \; + \; 0 \; - \; 0 \; + \; 0 \; - \; 0 \; +$

$-2\sqrt{3} \quad -2 \quad 0 \quad 2 \quad 2\sqrt{3} \quad x$

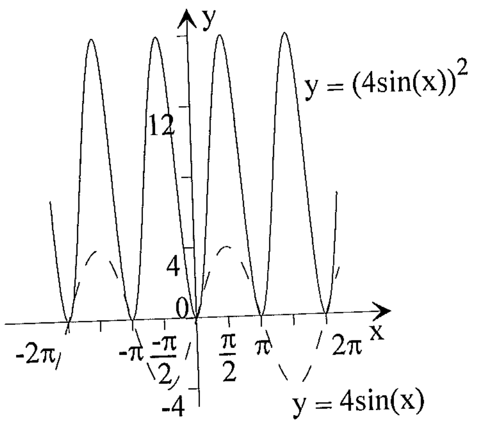
$\smile \quad \frown \quad \smile \quad \frown \quad \smile$

(-2√3,0) (-2,16) (0,0) (2,16) (2√3,0)

Problem GRA2_27.

Use the graph of $y = 4\sin x$ to sketch the graph of $y = (4\sin x)^2$.

Solution:



$f(x) = 4\sin x \quad f'(x) = 4\cos x$

Critical points are $\frac{\pi}{2} + n\pi$, n - integral

Sign of $f'(x)$

$0 \; - \; 0 \; + \; 0 \; - \; 0$

$\ldots \frac{-3}{2}\pi \quad \frac{-\pi}{2} \quad \frac{\pi}{2} \quad \frac{3}{2}\pi \ldots \quad x$

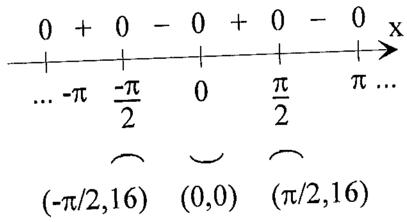
$\smile \quad \frown$

(-π/2,−4) (π/2,4)

$y = [f(x)]^2 \quad \frac{dy}{dx} = 2f(x)f'(x) \quad \frac{dy}{dx} = 2(4\sin x)(4\cos x) = 16\sin 2x$

Critical points are $n\frac{\pi}{2}$, n - integral

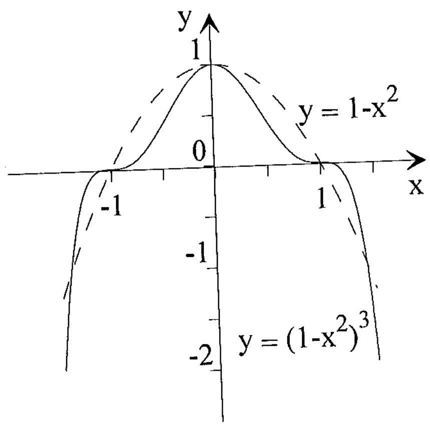
Sign of $\frac{dy}{dx}$



Problem GRA2_28.

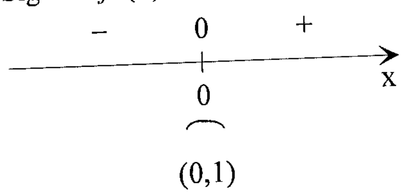
Use the graph of $y = 1 - x^2$ to sketch the graph of $y = (1 - x^2)^3$.

Solution:



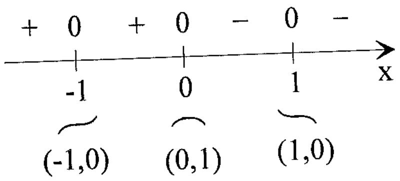
$f(x) = 1 - x^2 \quad f'(x) = -2x$

Sign of $f'(x)$



$y = [f(x)]^3 \quad \frac{dy}{dx} = 3[f(x)]^2 f'(x) \quad \frac{dy}{dx} = -6x(1 - x^2)^2$

Sign of $\frac{dy}{dx}$

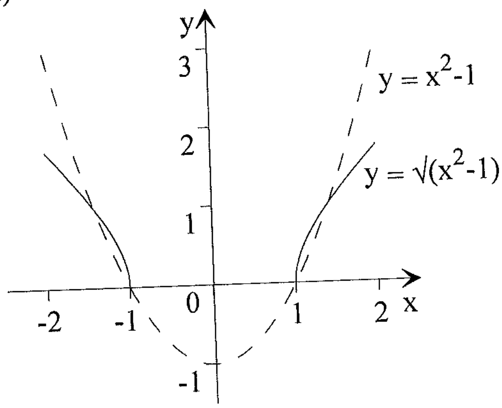


Problem GRA2_29.

For the function $f(x) = x^2 - 1$ use the graph of $y = f(x)$ to sketch the graphs of a) $y = \sqrt{f(x)}$, b) $y^2 = f(x)$.

Solution:

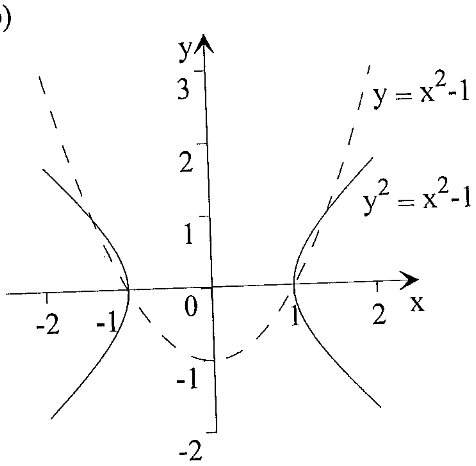
a)



Features:

- $y = \sqrt{f(x)}$ is defined only where $f(x) \geq 0$.
- $f(x) = 0$ where $x = \pm 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$ is not defined at $x = \pm 1 \Rightarrow$ (-1,0) and (1,0) are critical points.
- $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{f(x)}} \to \infty$ as $x \to -1^-$ or $x \to 1^{+1} \Rightarrow$ the tangent lines at $(\pm 1,0)$ are vertical.
- The graph $y = \sqrt{f(x)}$ lies above the graph $y = f(x)$ where $f(x) < 1$. The graph $y = \sqrt{f(x)}$ lies below the graph $y = f(x)$ where $f(x) > 1$.
- $y = f(x)$, $y = \sqrt{f(x)}$ intersect where $f(x) = 1$ or $f(x) = 0$.

b)



$y=\sqrt{f(x)} \Rightarrow y^2 = f(x) \Rightarrow (-y)^2 = f(x)$. Hence the graph $y^2 = f(x)$ is obtained by reflecting $y=\sqrt{f(x)}$ in the x-axis. The graph $y^2 = f(x)$ has vertical tangent lines at the critical points $(\pm 1, 0)$.

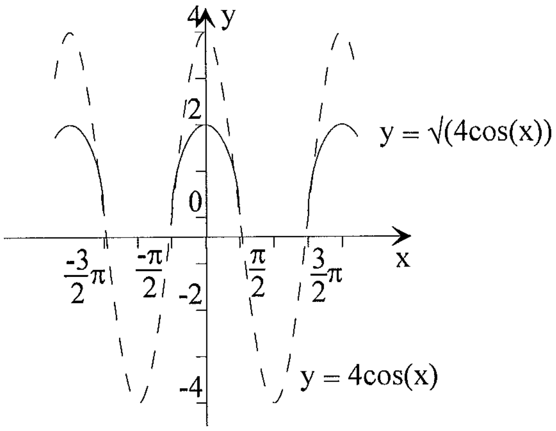
Problem GRA2_30.

Use the graph of $y = 4\cos x$ to sketch the graphs of:

a) $y=\sqrt{4\cos x}$ b) $y^2 = 4\cos x$

Solution:

a)



Let $f(x)=4\cos x$

Features of $y=\sqrt{f(x)}$:

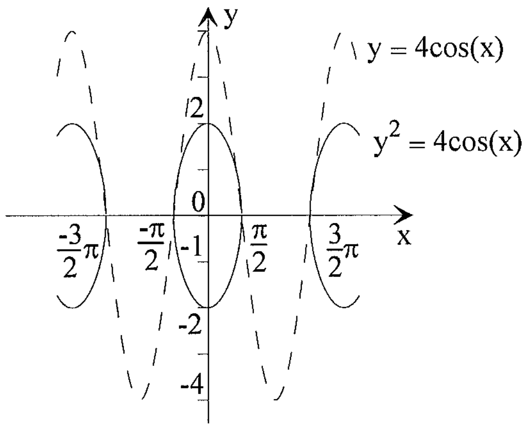
- $y=\sqrt{f(x)}$ is defined only where $f(x)\geq 0$.
- $f(x)=0$ where $x=\frac{\pi}{2}+\pi n$, n - integral $\Rightarrow \frac{dy}{dx}=\frac{f'(x)}{2\sqrt{f(x)}}$ is not defined at

$x=\frac{\pi}{2}+\pi n$, n - integral $\Rightarrow (\frac{\pi}{2}+n\pi, 0)$, n - integral, are critical points.

- $\frac{dy}{dx}=\frac{-2\sin x}{\sqrt{f(x)}} \to \infty$ as $x \to (\frac{\pi}{2}+2\pi n)^-$, n - integral, or $x \to (-\frac{\pi}{2}+2\pi n)^+$, n - integral, $\Rightarrow$

the tangent lines at $(\frac{\pi}{2}+\pi n, 0)$, n - integral, are vertical.

b)

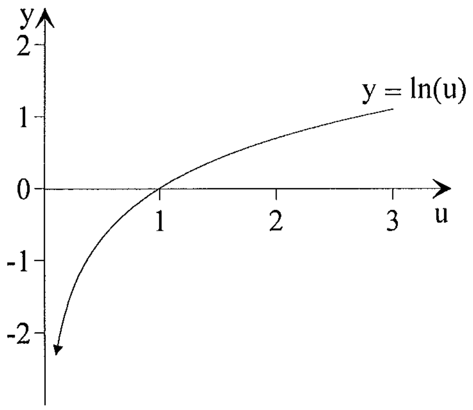


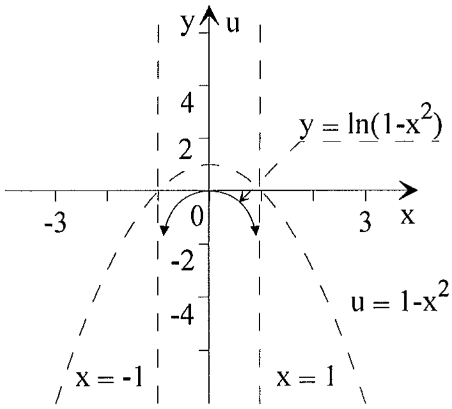
$y = \sqrt{f(x)} \Rightarrow y^2 = f(x) \quad \Rightarrow (-y)^2 = f(x)$ Hence the graph $y^2 = f(x)$ is obtained by reflecting $y = \sqrt{f(x)}$ in the x-axis. The graph $y^2 = f(x)$ has vertical tangent lines at the critical points $(\frac{\pi}{2} + n\pi, 0)$, n - integral.

Problem GRA2_31.

Sketch the graph of $y = \ln(1 - x^2)$.

Solution:





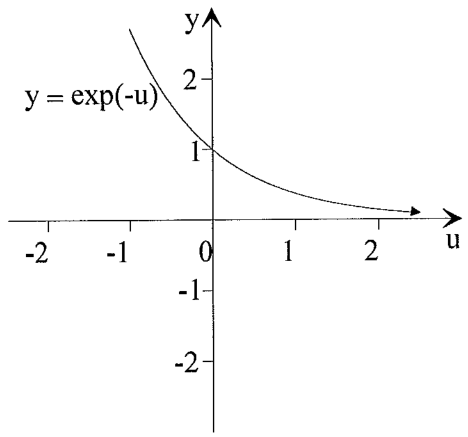
Features of the graph of the composite function $y = \ln(1 - x^2)$:

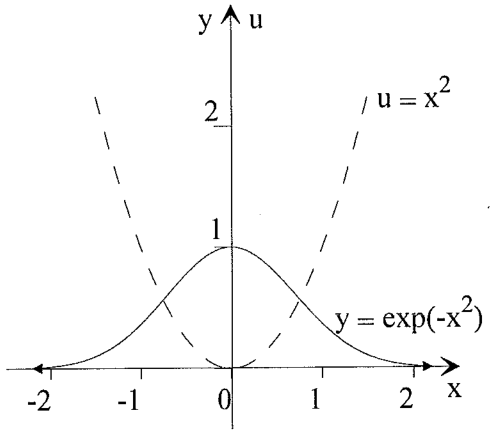
- $y = \ln u, \quad u = 1 - x^2 \quad \Rightarrow y = \ln(1 - x^2)$.
- $y = \ln u$ is defined where $u = 1 - x^2 > 0 \quad \Rightarrow$ domain $\{x : -1 < x < 1\}$
- Vertical asymptote of $y = \ln u$ at $u = 0$. But $u = 1 - x^2$ and $1 - x^2 = 0$ at $x = \pm 1 \quad \Rightarrow$ $x = -1$ and $x = 1$ are vertical asymptotes of $y = \ln(1 - x^2)$.
- $u = 1 - x^2 \le 1 \quad \Rightarrow y = \ln u \le 0$.
- $\ln(1 - (-x)^2) = \ln(1 - x^2) \quad \Rightarrow$ the function $y = \ln(1 - x^2)$ is even.
- $y = \ln u$ is an increasing function $\Rightarrow y = \ln(1 - x^2)$ increases as $1 - x^2$ increases and decreases as $1 - x^2$ decreases.

Problem GRA2_32.

Sketch the graph of $y = e^{-x^2}$.

Solution:





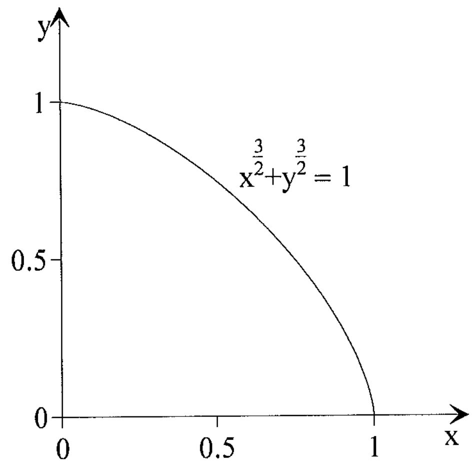
Features of the composite function $y = e^{-x^2}$:

- $y = e^{-u}, u = x^2 \quad \Rightarrow y = e^{-x^2}$.
- $e^{-(-x)^2} = e^{-x^2} \quad \Rightarrow y = e^{-x^2}$ is an even function.
- $y = e^{-u}$ is an decreasing function $\Rightarrow y = e^{-x^2}$ increases as x^2 decreases and decreases as x^2 increases.
- The minimum turning point $(0, 0)$ of $u = x^2$ corresponds to the maximum turning point $(0, 1)$ of $y = e^{-x^2}$.

Problem GRA2_33.

Sketch (showing critical points) the graph of $x^{3/2} + y^{3/2} = 1$.

Solution:



$x^{3/2} + y^{3/2} = 1$. Clearly $x, y \geq 0 \Rightarrow$ domain $\{x : 0 \leq x \leq 1\}$, range $\{y : 0 \leq y \leq 1\}$. Taking the derivative of both sides with respect to x, remembering that y is a function of x, we have

$$\frac{3}{2}x^{1/2} + \frac{3}{2}y^{1/2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{1/2}.$$

As $y \to 0^+, x \to 1^- \Rightarrow \frac{dy}{dx} \to -\infty$. Hence the curve has a vertical tangent line at the critical point (1,0).

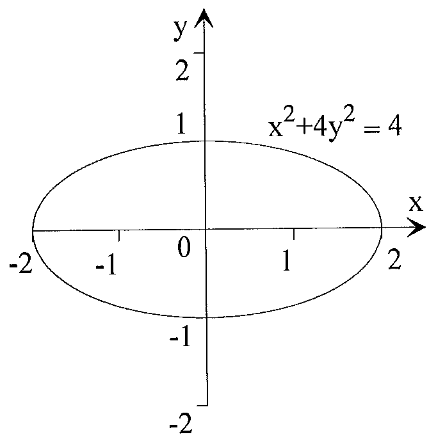
As $x \to 0^+, y \to 1^- \Rightarrow \frac{dy}{dx} \to 0^-$. Hence the curve has a horizontal tangent line at the critical point (0,1).

The curve is symmetric about $y = x$, since the transformation $y \leftrightarrow x$ leaves the Cartesian equation of the curve unchanged.

Problem GRA2_34.

Sketch (showing critical points and stationary points) the graph of $x^2 + 4y^2 = 4$.

Solution:



$x^2 + 4y^2 = 4$. Clearly $x^2, y^2 \geq 0$, hence domain $\{x : -2 \leq x \leq 2\}$, range $\{y : -1 \leq y \leq 1\}$. Take the derivative of both sides with respect to x. Consider y as a function of x and use the chain rule. Then we have $2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{4}\left(\frac{x}{y}\right)$.

As $y \to 0, x \to \pm 2 \Rightarrow \frac{dy}{dx} \to -\infty$ and the curve has vertical tangent at (-2,0) and (2,0).

As $x = 0, y = \pm 1 \Rightarrow \frac{dy}{dx} = 0$ and the curve has horizontal tangent at (0,-1) (0,1).

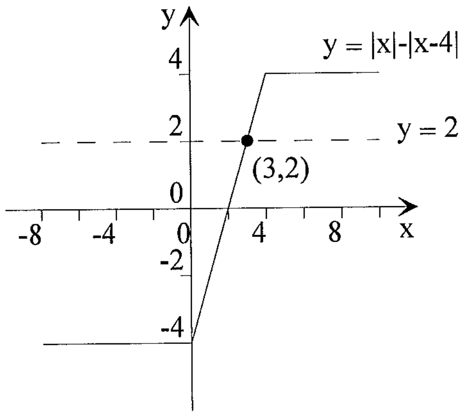
Clearly the curve is symmetric about the lines $y = 0$ and $x = 0$ as the transformation $y \to -y$ and $x \to -x$ leave the Cartesian equation of the curve unchanged.

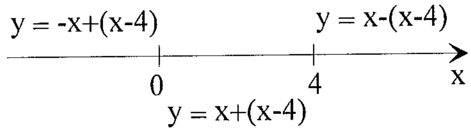
Problem GRA2_35.

Sketch the graph of $y = |x| - |x-4|$. Use this graph to solve the inequality $|x| - |x-4| > 2$.

Answer: $\{x : x > 3\}$.

Solution:





$$y = \begin{cases} -4, & x < 0 \\ 2x - 4, & 0 \le x < 4 \\ 4, & x > 4 \end{cases}$$

By inspection of the graph, $|x| - |x-4| > 2$ for $x > 3$.

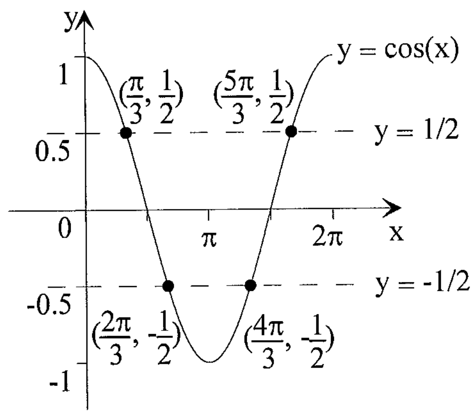
Problem GRA2_36.

Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$. Use this graph to solve the inequalities.

a) $\cos x \le \frac{1}{2}$, for $0 \le x \le 2\pi$; b) $|\cos x| \le \frac{1}{2}$, for $0 \le x \le 2\pi$.

Answer: a) $\{x : \frac{\pi}{3} \le x \le \frac{5}{3}\pi\}$; b) $\{x : \frac{\pi}{3} \le x \le \frac{2}{3}\pi \text{ or } \frac{4}{3}\pi \le x \le \frac{5}{3}\pi\}$.

Solution:



a) $\cos x = \frac{1}{2} \Leftrightarrow x = \pm\cos^{-1}\frac{1}{2} + 2\pi n$, n integral. But of these values of x only $\cos^{-1} = \frac{\pi}{3}$ and

$-\frac{\pi}{3} + 2\pi = \frac{5}{3}\pi$ are from the interval $[0; 2\pi]$. By inspection of the graph,

$\cos x \le \frac{1}{2}$, for $\frac{\pi}{3} \le x \le \frac{5}{3}\pi$.

b) $|\cos x| \le \frac{1}{2} \Leftrightarrow -\frac{1}{2} \le \cos x \le \frac{1}{2}$.

$\cos x = -\frac{1}{2} \Rightarrow x = \frac{2}{3}\pi$ or $x = \frac{4}{3}\pi$.

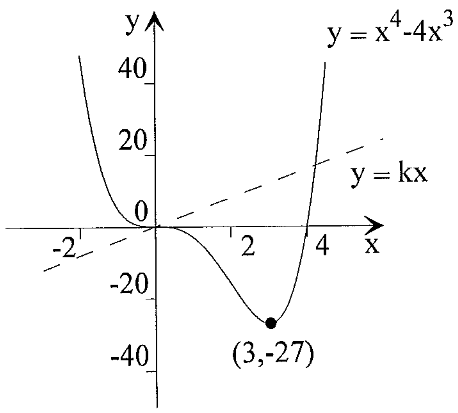
By inspecting the graph, $|\cos x| \le \frac{1}{2}$ for $\frac{\pi}{3} \le x \le \frac{2}{3}\pi$ or $\frac{4}{3}\pi \le x \le \frac{5}{3}\pi$.

Problem GRA2_37.

Sketch the graph of $y = x^4 - 4x^3$. Use this graph to find the number of real roots of the equation $x^4 - 4x^3 = kx$, where k is a positive real number.

Answer: 2 real roots.

Solution:

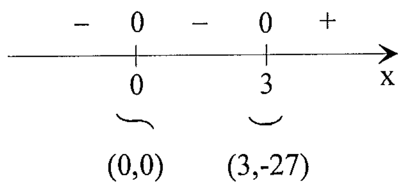


Features of the graph $y = x^4 - 4x^3$:

$y = 0$ when $x = 0$

$\frac{dy}{dx} = 4x^3 - 12x^2$

Sign of $\frac{dy}{dx} = 4x^2(x-3)$



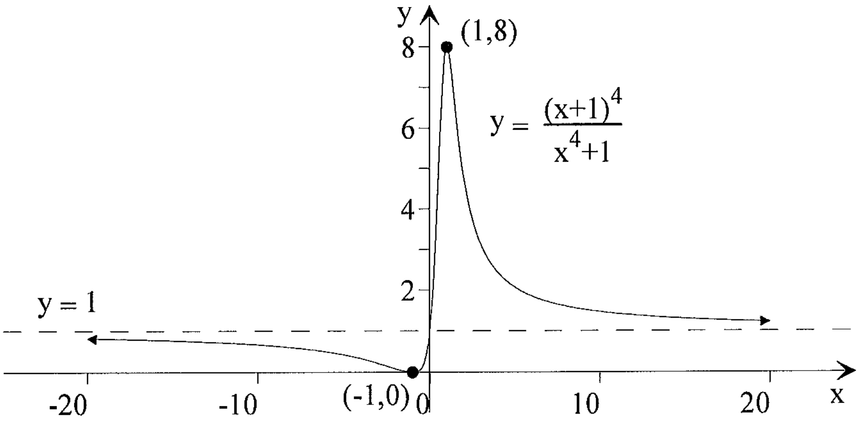
Real solution of the equation $x^4 - 4x^3 = kx$ are given by x-values where $y = x^4 - 4x^3$ and $y = kx$ intersect. $k > 0 \Rightarrow$ the equation has 2 real roots.

Problem GRA2_38.

Sketch the graph of $y = \frac{(x+1)^4}{x^4+1}$. Use this graph to find the set of values of the real number k for which the equation $(x+1)^4 = k(x^4+1)$ has two real distinct roots.

Answer: $\{k : 0 < k < 8,\ k \neq 1\}$.

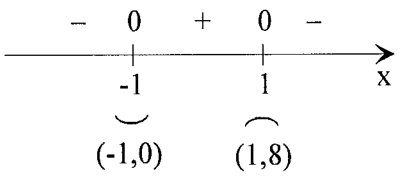
Solution:



Features of the graph $y = \frac{(x+1)^4}{x^4+1}$:

- $y = 0$ when $x = -1$
- As $x \to \infty$, $y \to 1$
- $y = 1$ when $x = 0$
- $\frac{dy}{dx} = \frac{4(x+1)^3(1-x^3)}{(x^4+1)^2}$

Sign of $\frac{dy}{dx}$



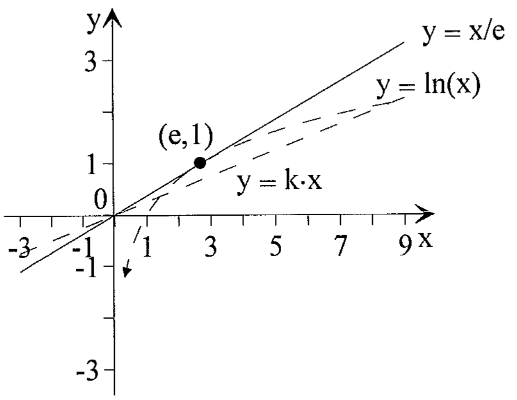
Real solution of the equation $(x+1)^4 = k(x^4+1)$ are given by x-values where $y = \frac{(x+1)^4}{x^4+1}$ and $y = k$ intersect. Hence the equation has two real distinct roots for the following set of k $\{k : 0 < k < 8,\ k \neq 1\}$.

Problem GRA2_39.

Find the gradient of the tangent from the origin to the curve $y = \ln x$. Hence find the set of values of the real number k such that the equation $\ln x = kx$ has two real distinct roots.

Answer: $\frac{1}{e}$, $0 < k < \frac{1}{e}$.

Solution:



Let the gradient of the tangent from the origin to the curve be equal to a. Then $a = (\ln x)'$, i.e., $a = \frac{1}{x}$. In addition at the point (x, y) where the tangent touch the curve $y = \ln x$ and simultaneously $y = ax$. Hence we have the simultaneous equations:

$$\begin{cases} a = \frac{1}{x} \\ y = \ln x \\ y = ax \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{x} \\ y = \ln x \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{x} \\ y = e \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{e} \\ y = e \\ y = 1 \end{cases}$$

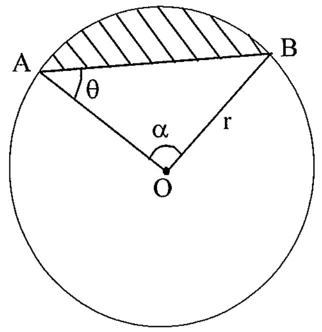
Real solution of the equation $\ln x = kx$ are given by x-values where $y = \ln x$ and $y = kx$ intersect. Hence the equation has two real distinct roots for the following set of k $\{k : 0 < k < \frac{1}{e}\}$.

Problem GRA2_40.

A chord AB of a circle makes on angle θ with the diameter passing through A. If the area of the minor segment is one-quarter the area of the circle, show that $\sin 2\theta = \frac{\pi}{2} - 2\theta$. Solve this equation graphically.

Answer: $\theta \approx 0.4$.

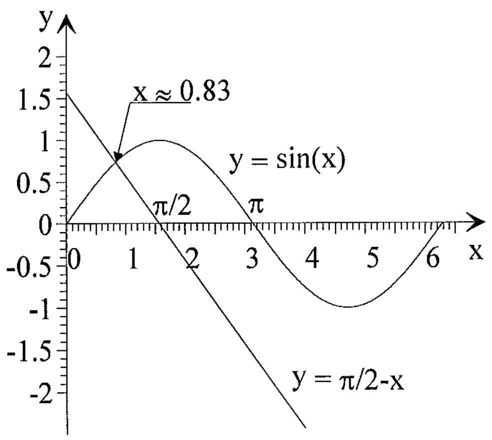
Solution:



Let the chord AB subtends an angle α at the centre of a circle. $|OA| = |OB| \Rightarrow$ the triangle OAB is isosceles $\Rightarrow \angle OAB = \angle OBA$. Hence $\alpha = \pi - 2\theta$.

Area of a segment $= \frac{1}{4}$ area of a circle $\Rightarrow \frac{1}{2}r^2(\alpha - \sin\alpha) = \frac{1}{4}\pi r^2$

$\pi - 2\theta - \sin(\pi - 2\theta) = \frac{\pi}{2} \Rightarrow \sin 2\theta = \frac{\pi}{2} - 2\theta$.



Solution of this equation is given by θ-values, where $y = \sin x$ and $y = \frac{\pi}{2} - x$ intersect, and $2\theta = x$. From the graph we have $2\theta \approx 0.83 \Rightarrow \theta \approx 0.4$.