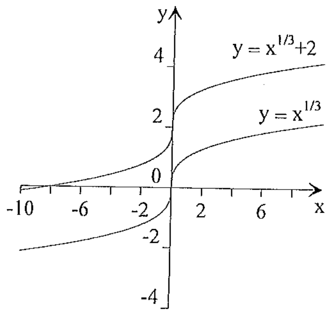
Topic 7. Graphs.
Level 3.
Problem GRA3_01.
Sketch (showing critical points) the graphs of: a) $y = x^{1/3}$; b) $y = x^{1/3} + 2$.

Solution:



a) $y = x^{1/3}$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$.

$\frac{dy}{dx}$ is not defined at x = 0

$\Rightarrow$ (0, 0) is a critical point.

$\frac{dy}{dx} \to \infty$ as $x \to 0$

$\Rightarrow$ the tangent line at (0, 0) is vertical.

b) $y = x^{1/3} + 2$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$.

$\frac{dy}{dx}$ is not defined at x = 0

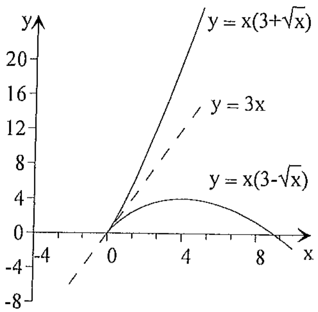
$\Rightarrow$ (0,2) is a critical point.

$\frac{dy}{dx} \to \infty$ as $x \to 0$

$\Rightarrow$ the tangent line at (0, 2) is vertical.

Problem GRA3_02.
Sketch (showing critical points) the graphs of: a) $y = x(3+\sqrt{x})$; b) $y = x(3+\sqrt{x})$.

Solution:



a) $y = 3x + x^{3/2}$

Domain $\{x : x \geq 0\}$ $\quad \frac{dy}{dx} = 3 + \frac{3}{2}x^{1/2}$, $x > 0$.

$\frac{dy}{dx} \to 3$ as $x \to 0^+ \Rightarrow y = 3x$ is the tangent line at the critical point (0, 0).

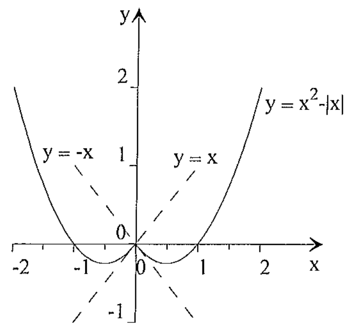
b) $y = 3x - x^{3/2}$

Domain $\{x : x \geq 0\}$ $\quad \frac{dy}{dx} = 3 - \frac{3}{2}x^{1/2}$, $x > 0$.

$\frac{dy}{dx} \to 3$ as $x \to 0^+ \Rightarrow y = 3x$ is the tangent line at the critical point (0, 0).

Problem GRA3_03.
Sketch (showing critical points) the graph of $y = x^2 - |x|$.

Solution:



$y = x^2 - |x|$

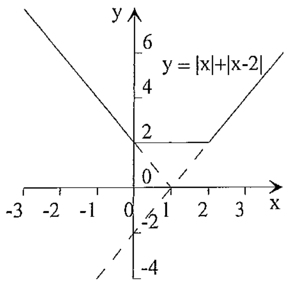
$$y = \begin{cases} x^2 - x, & x \geq 0 \\ x^2 + x, & x < 0 \end{cases} \qquad \frac{dy}{dx} = \begin{cases} 2x - 1, & x \geq 0 \\ 2x + 1, & x < 0 \end{cases}$$

$\frac{dy}{dx} \to -1$ as $x \to 0^+$ $\quad \frac{dy}{dx} \to 1$ as $x \to 0^-$

$\Rightarrow \frac{dy}{dx}$ is not defined at $x = 0$, and (0, 0) is a critical point.

Problem GRA3_04.
Sketch (showing critical points) the graph of $y = |x| + |x - 2|$.

Solution:



$y = |x| + |x-2|$

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$, $|x-2| = \begin{cases} x-2, & x \ge 2 \\ -x+2, & x < 2 \end{cases}$

Note that,

if $x \ge 2$, $y = x + x - 2 = 2x - 2$

if $2 > x \ge 0$, $y = x - x + 2 = 2$

if $x < 0$, $y = -x - x + 2 = -2x + 2$

$\Rightarrow y = \begin{cases} 2x-2, & x \ge 2 \\ 2, & 2 > x \ge 0 \\ -2x+2, & x < 0 \end{cases}$ $\quad \dfrac{dy}{dx} = \begin{cases} 2, & x > 2 \\ 0, & 0 < x < 2 \\ -2, & x < 0 \end{cases}$

$\dfrac{dy}{dx} \to 2$ as $x \to 2^+$ $\quad \dfrac{dy}{dx} \to 0$ as $x \to 2^-$

$\Rightarrow \dfrac{dy}{dx}$ is not defined at $x = 2$, and (2, 2) is a critical point.

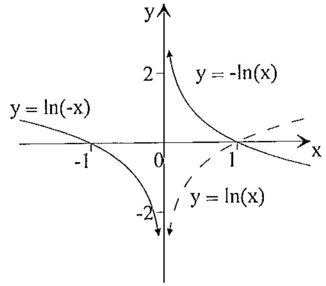
$\dfrac{dy}{dx} \to 0$ as $x \to 0^+$ $\quad \dfrac{dy}{dx} \to -2$ as $x \to 0^-$

$\Rightarrow \dfrac{dy}{dx}$ is not defined at $x = 0$, and (0, 2) is a critical point.

Problem GRA3_05.

Use the graph of $y = \ln x$ to sketch the graphs of: a) $y = \ln(-x)$, b) $y = -\ln x$.

Solution:



a) $y = \ln x$, domain $\{x: x > 0\}$.

$y = \ln(-x)$, domain $\{x: x < 0\}$.

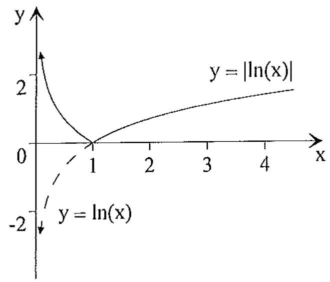
The graph of $y = \ln(-x)$ is a reflection of $y = \ln x$ in the y-axis.

b) The graph of $y = -\ln x$ is a reflection of $y = \ln x$ in the x-axis.

Problem GRA3_06.

Use the graph of $y = \ln x$ to sketch the graph of $y = |\ln x|$.

Solution:



$y = \ln x$, domain $\{x: x > 0\}$.

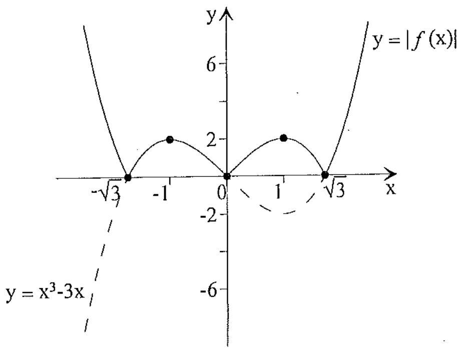
$y = |\ln x| = \begin{cases} \ln x, & \ln x \ge 0 \\ -\ln x, & \ln x < 0 \end{cases} \Rightarrow y = \ln x = \begin{cases} \ln x, & x \ge 1 \\ -\ln x, & 0 < x < 1 \end{cases}$

Hence the section of $y = \ln x$ which lies below the x-axis is reflected in the x-axis.

Problem GRA3_07.

Use the graph of $f(x) = x^3 - 3x$ (an odd function) to sketch (showing critical points) the graph of $y = |f(x)|$. Is this the graph of an even function?

Solution:



Let $g(x)=|x^3-3x| \Rightarrow g(-x)=|(-x)^3-3(-x)|=|-(x^3-3x)|=|(x^3-3x)| \Rightarrow g(-x)=g(x) \Rightarrow$ the graph of $y=g(x)$ is symmetric in the y-axis.

Those sections of $y=x^3-3x$ which lie below the x-axis are reflected in the x-axis.

$$y=|x^3-3x|=\begin{cases}3x-x^3, & x<-\sqrt{3}\\ x^3-3x, & -\sqrt{3}\le x<0\\ 3x-x^3, & 0\le x<\sqrt{3}\\ x^3-3x, & x\ge\sqrt{3}\end{cases} \qquad \frac{dy}{dx}=\begin{cases}3-3x^2, & x<-\sqrt{3}\\ 3x^2-3, & -\sqrt{3}<x<0\\ 3-3x^2, & 0<x<\sqrt{3}\\ 3x^2-3, & x>\sqrt{3}\end{cases}$$

$\frac{dy}{dx}\to 3$ as $x\to 0^+$, $\frac{dy}{dx}\to -3$ as $x\to 0^-$ $\Rightarrow \frac{dy}{dx}$ is not defined at $x=0$, and (0; 0) is a critical point.

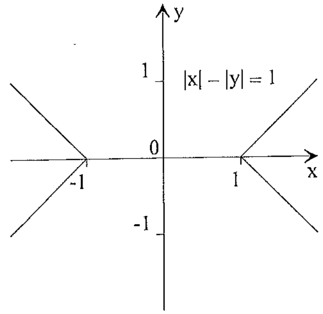
$\frac{dy}{dx}\to 6$ as $x\to -\sqrt{3}^+$, $\frac{dy}{dx}\to -6$ as $x\to -\sqrt{3}^-$ $\Rightarrow \frac{dy}{dx}$ is not defined at $x=-\sqrt{3}$, and $(-\sqrt{3};0)$ is a critical point.

Hence the symmetric about the y-axis point $(\sqrt{3},0)$ is also a critical point.

Problem GRA3_08.

Sketch the graph of $|x|-|y|=1$.

Solution:



$|x|-|y|=1 \Rightarrow |x|=1+|y|$

Clearly $y\ge 0 \Rightarrow$ domain $\{x:|x|\ge 1\}$.

If $x\ge 1$, then $y=x-1$ or $y=1-x$. Hence $\frac{dy}{dx}=1, x>1$, or $\frac{dy}{dx}=-1, x>-1$.

As $y\to 0^+, x\to 1^+ \Rightarrow \frac{dy}{dx}\to 1$, and as $y\to 0^-, x\to 1^+ \Rightarrow \frac{dy}{dx}\to -1$.

Hence $\frac{dy}{dx}$ is not defined at $x=1$, and (1, 0) is a critical point.

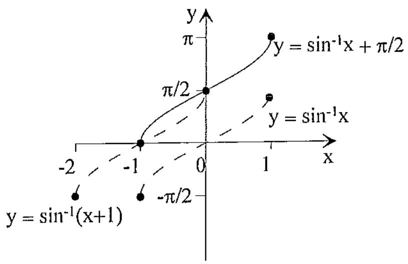
The curve is symmetric about $x=0$, since the transformation $x\to -x$ leaves the Cartesian equation of the curve unchanged. Hence, if $x\le -1$, then $y=-x-1$ or $y=1+x$. And hence the symmetric point (-1, 0) is also critical.

Problem GRA3_09.

Use the graph of $y=\sin^{-1}x$ to sketch the graphs of:

a) $y=\sin^{-1}x+\frac{\pi}{2}$ b) $y=\sin^{-1}(x+1)$.

Solution:



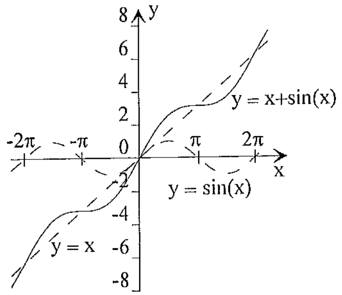
a) The graph $y=\sin^{-1}x+\frac{\pi}{2}$ is obtained by translating the graph $y=\sin^{-1}x$ through $\frac{\pi}{2}$ units up.

b) The graph $y = \sin^{-1}(x+1)$ is obtained by translating the graph $y = \sin^{-1} x$ through one unit to the left.

Problem GRA3_10.

Use the graphs of $y = x$ and $y = \sin x$ (both odd functions) to sketch the graph of $y = x + \sin x$. Is this the graph of an odd function?

Solution:

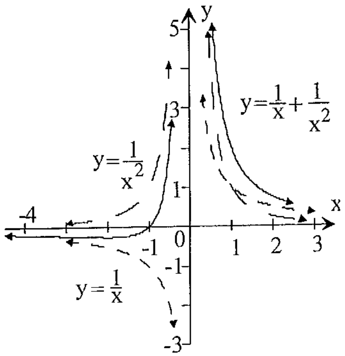


Let $f(x) = x + \sin x \Rightarrow f(-x) = -x + \sin(-x) = -x - \sin x \Rightarrow f(-x) = -f(x)$, i.e., the function $x + \sin x$ is an odd function. The ordinates of the graph $y = x + \sin x$ are obtained by summing the ordinates of the graphs $y = x$ and $y = \sin x$.

Problem GRA3_11.

Sketch the graph of $y = \frac{1}{x} + \frac{1}{x^2}$.

Solution:

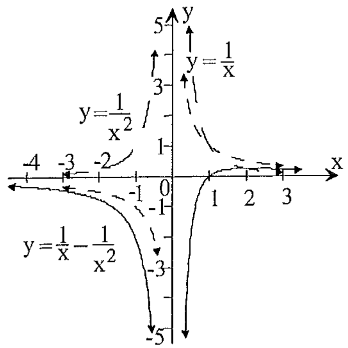


The ordinates of the graph $y = \frac{1}{x} + \frac{1}{x^2}$ are obtained by summing the ordinates of the graphs $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

Problem GRA3_12.

Sketch the graph of $y = \frac{1}{x} - \frac{1}{x^2}$.

Solution:

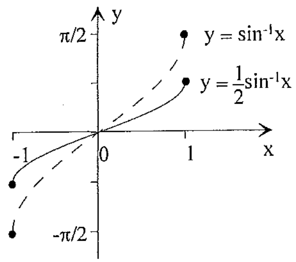


The ordinates of the graph $y = \frac{1}{x} - \frac{1}{x^2}$ are obtained by applying the procedure of subtraction of ordinates of the graphs $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

Problem GRA3_13.

Use the graph of $y = \sin^{-1} x$ to sketch the graph of: $y = \frac{1}{2}\sin^{-1} x$.

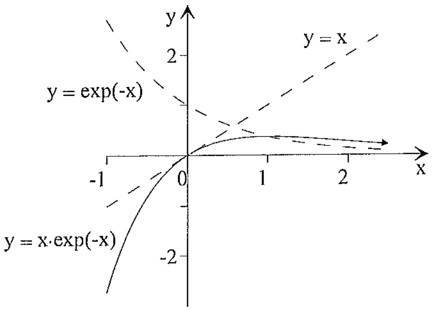
Solution:



The graph $y = \frac{1}{2}\sin^{-1} x$ is obtained by enlarging the graph $y = \sin^{-1} x$ by a factor $\frac{1}{2}$ in the direction parallel to the y-axis.

Problem GRA3_14.
Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = xe^{-x}$.

Solution:

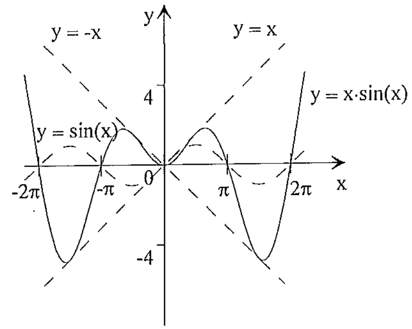


The graph of $y = xe^{-x}$ is obtained by multiplication of ordinates of $y = x$ and $y = e^{-x}$.
Features of $y = xe^{-x}$:

- $y = 0$ when $x = 0$.
- $y = xe^{-x}$ lies below $y = x$ for all real x as $xe^{-x} < x$ for $x > 0$ and $xe^{-x} < x$ for $x < 0$.
- $y = xe^{-x}$ lies below $y = e^{-x}$ for $0 < x < 1$.
- As $x \to +\infty$, $e^{-x} \to 0$ more quickly than any power of $\frac{1}{x}$ and hence $xe^{-x} \to 0$.
- As $x \to -\infty$, $xe^{-x} \to -\infty$ more quickly than e^{-x}.

Problem GRA3_15.
Use the graphs of $y = x$ and $y = \sin x$ (both odd functions) to sketch the graph of $y = x\sin x$.
Is this the graph of an even function?

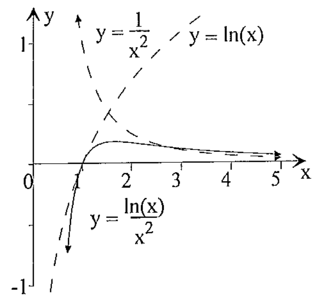
Solution:



The graph of $y = x\sin x$ is obtained by multiplication of ordinates of $y = x$ and $y = \sin x$.
Let $f(x) = x\sin x \Rightarrow f(-x) = (-x)\sin(-x) = x\sin x \Rightarrow y = x\sin x$ is an even function and hence its graph has axis symmetry about the y-axis.
For $x \geq 0$, $-x \leq x\sin x \leq x$ and hence the graph $y = x\sin x$ lies between the lines $y = \pm x$, touching these lines when $\sin x = \pm 1$.

Problem GRA3_16.
Sketch the graph of $y = \frac{\ln x}{x^2}$.

Solution:



The graph $y = \frac{\ln x}{x^2}$ is obtained by multiplication of ordinates $y = \ln x$ and $y = \frac{1}{x^2}$.

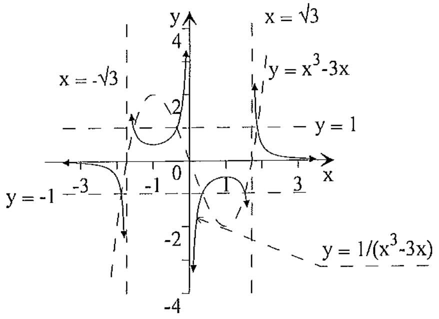
Features:

- Domain $\{x : x > 0\}$
- $y = 0$ when $x = 1$
- $y = \frac{\ln x}{x^2}$ lies above $y = \frac{1}{x^2}$ only for $x > e$ (where $\ln x > 1$).
- As $x \to +\infty$, $x^2 \to +\infty$ more quickly than $\ln x$ and hence $\frac{\ln x}{x^2} \to 0^+$.

Problem GRA3_17.

Use the graph of $f(x)=x^3-3x$ (an odd function) to sketch the graph of $y=\frac{1}{f(x)}$. Is this the graph of an odd function?

Solution:



$\frac{1}{f(-x)}=\frac{1}{(-x)^3-3(-x)}=\frac{-1}{x^3-3x}=\frac{-1}{f(x)}\Rightarrow y=\frac{1}{x^3-3x}$ is an odd function.

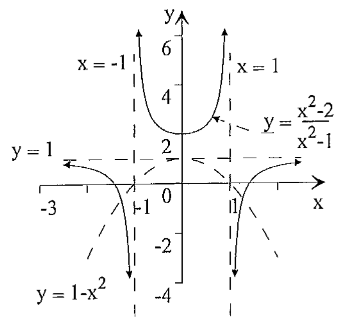
Features:

- $f(x), \frac{1}{f(x)}$ have the same sign.
- $f(x)=0$ when $x=\pm\sqrt{3}$ or $x=0\Rightarrow$ the lines $x=-\sqrt{3}$, $x=+\sqrt{3}$ and $x=0$ are the vertical asymptotes of $y=\frac{1}{f(x)}$.
- As $x\to\infty$, $f(x)\to\infty$ $\Rightarrow\frac{1}{f(x)}\to 0$.
- $(-1, 2)$ and $(1, -2)$ are maximum and minimum turning points of $y=f(x)$ respectively $\Rightarrow(-1,\frac{1}{2})$ and $(1,-\frac{1}{2})$ are minimum and maximum turning points of $y=\frac{1}{f(x)}$ respectively.

Problem GRA3_18.

Show that $\frac{x^2-2}{x^2-1}=1-\frac{1}{x^2-1}$. Hence sketch the graph of $y=\frac{x^2-2}{x^2-1}$.

Solution:



$\frac{x^2-2}{x^2-1}=\frac{(x^2-1)-1}{x^2-1}=1-\frac{1}{x^2-1}$

The graph $y=-\frac{1}{x^2-1}$ has been translated one unit upward. $y=1$ is an asymptote of $y=\frac{x^2-2}{x^2-1}$ as $x\to\infty$. The graph $y=-\frac{1}{x^2-1}$ is a reflection in the x-axis of $y=\frac{1}{x^2-1}$.

The graph $y=\frac{1}{x^2-1}$ is a reciprocal of $y=x^2-1$.

Consider the graphs $y=f(x)$ and $y=\frac{1}{f(x)}$, where $f(x)=x^2-1$.

Features:

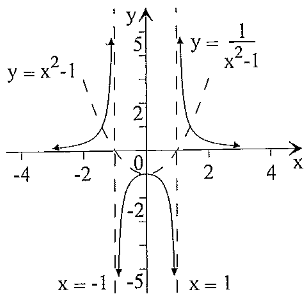
- $f(x), \frac{1}{f(x)}$ have the same sign.
- $f(x)=0$ when $x=\pm 1$ $\Rightarrow$ the lines $x=-1$ and $x=1$ are vertical asymptotes of $y=\frac{1}{f(x)}$.
- As $x\to\infty$, $f(x)\to+\infty$ $\Rightarrow\frac{1}{f(x)}\to 0^+$.
- Minimum turning point of $y=f(x)$ is $(0,-1)$ $\Rightarrow$ maximum turning point of $y=\frac{1}{f(x)}$ is $(0,-1)$.

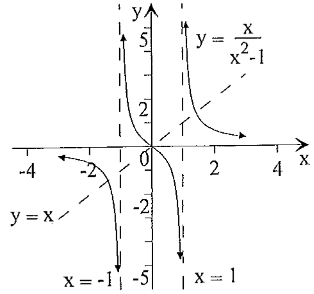
Problem GRA3_19.

Sketch the graphs of a) $y=\frac{x}{x^2-1}$, b) $y=\frac{x^2}{x^2-1}$.

Solution:

a) The graph $y=\frac{x}{x^2-1}$ is obtained by multiplication of ordinates $y=\frac{1}{x^2-1}$ and $y=x$.





Features of the graph $y=\frac{1}{x^2-1}$

The graph $y=\frac{1}{x^2-1}$ is a reciprocal of $y=x^2-1$:

- $y=x^2-1$ and $y=\frac{1}{x^2-1}$ have the same sign
- $x^2-1=0$ when $x=\pm1 \Rightarrow$ the lines $x=-1$ and $x=1$ correspond to vertical asymptotes of $y=\frac{1}{x^2-1}$
- As $x\to\infty$, $x^2-1\to+\infty \Rightarrow \frac{1}{x^2-1}\to 0^+$.
- Minimum turning point of $y=x^2-1$ (0, -1) $\Rightarrow$ maximum turning point of $y=\frac{1}{x^2-1}$ is (0, -1).

Features of the graph $y=\frac{x}{x^2-1}$:

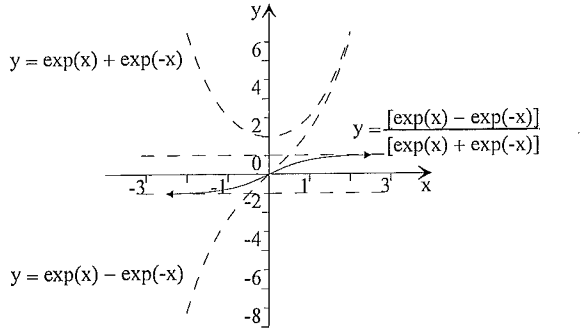
- $y=0$ when $x=0$
- As $x\to\infty$, $y=\frac{x}{x^2-1}\to 0 \Rightarrow$ the line $x=0$ is a horizontal asymptote.

b) Hence the graph $y=1+\frac{1}{x^2-1}$ is obtained from the graph $y=\frac{1}{x^2-1}$ by translating one unite upward.

Problem GRA3_20.

Sketch the graph of $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Solution:



The graph of $y=e^x-e^{-x}$ is obtained by subtraction of ordinates of the graphs $y=e^x$ and $y=e^{-x}$.

The graph of $y=e^x+e^{-x}$ is obtained by summing the ordinates of the graphs $y=e^x$ and $y=e^{-x}$.

The graph of $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is obtained by division of ordinates of the graphs $y=e^x-e^{-x}$ and $y=e^x+e^{-x}$.

Features of $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$:

- $y=0$ when $x=0$
- Let $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$, then $f(-x)=\frac{e^{-x}-e^{+x}}{e^{-x}+e^{+x}}=-f(x) \Rightarrow$ the graph $y=f(x)$ is the graph of an odd function and hence it is symmetric about origin.
- As $x\to+\infty$, $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{1-e^{-2x}}{1+e^{-2x}}\to 1 \Rightarrow$ the line $y=1$ is a horizontal asymptote of $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Problem GRA3_21.

Sketch the graph of $y=\frac{\cos x-\sin x}{\cos x+\sin x}$.

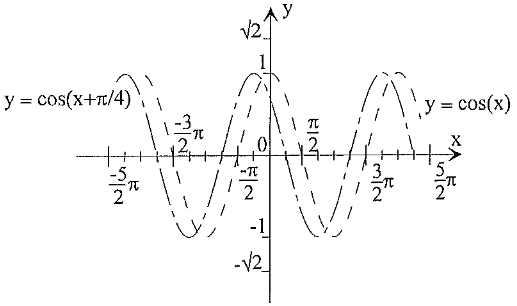
Solution:

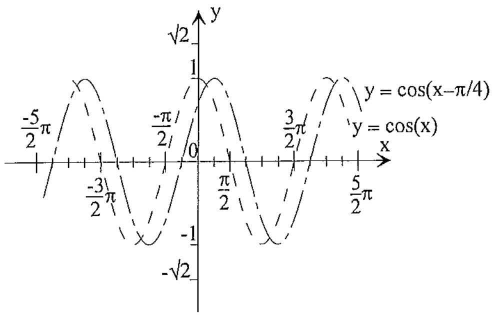
$$y=\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x}=\frac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)} \Rightarrow y=\frac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)}..$$

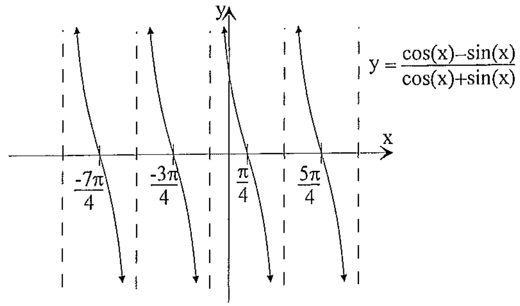
The graph $y=\cos\left(x+\frac{\pi}{4}\right)$ is obtained by translating the graph $y=\cos x$ through $\frac{\pi}{4}$ units to the left.

The graph $y=\cos\left(x-\frac{\pi}{4}\right)$ is obtained by translating the graph $y=\cos x$ through $\frac{\pi}{4}$ units to the right.

The graph $y=\dfrac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)}$ is obtained by division of ordinates of the graphs $y=\cos\left(x+\frac{\pi}{4}\right)$ and $y=\cos\left(x-\frac{\pi}{4}\right)$.







Features of the graph $y=\dfrac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)}$:

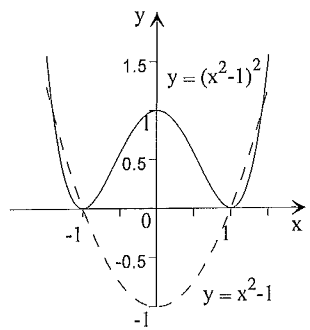
- $y=0$ when $\cos\left(x+\frac{\pi}{4}\right)=0$, i.e., $x=\frac{\pi}{4}+\pi n$, n integral.
- $\cos\left(x-\frac{\pi}{4}\right)=0$, as $x=\frac{3\pi}{4}+\pi n$, n integral

$\Rightarrow$ as $x\to\frac{3\pi}{4}+\pi n$, $y=\dfrac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)}\to\infty$, and hence the lines $x=\frac{3\pi}{4}+\pi n$, n integral, are the vertical asymptotes.

Problem GRA3_22.

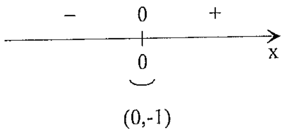
Use the graph of $y=x^2-1$ to sketch the graph of $y=\left(x^2-1\right)^2$.

Solution:



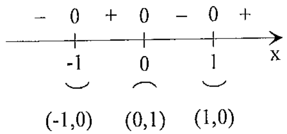
$f(x) = x^2 - 1 \quad f'(x) = 2x$

Sign of $f'(x)$



$y = [f(x)]^2 \quad \frac{dy}{dx} = 2f(x)f'(x) \quad \frac{dy}{dx} = 4x(x-1)(x+1)$

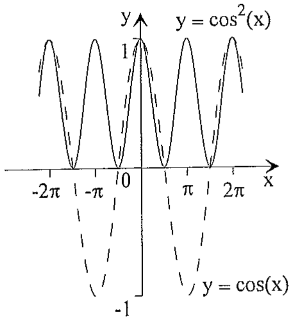
Sign of $\frac{dy}{dx}$



Problem GRA3_23.

Use the graph of $y = \cos x$ to sketch the graph of $y = (\cos x)^2$.

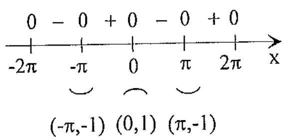
Solution:



$f(x) = \cos x \quad f'(x) = -\sin x$

Critical points are $n\pi$, n - integral

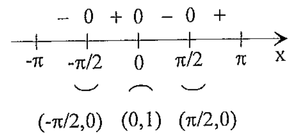
Sign of $f'(x)$



$y = [f(x)]^2 \quad \frac{dy}{dx} = 2f(x)f'(x) \quad \frac{dy}{dx} = -2\cos x \sin x = -\sin 2x$

Critical points are $n\frac{\pi}{2}$, n - integral

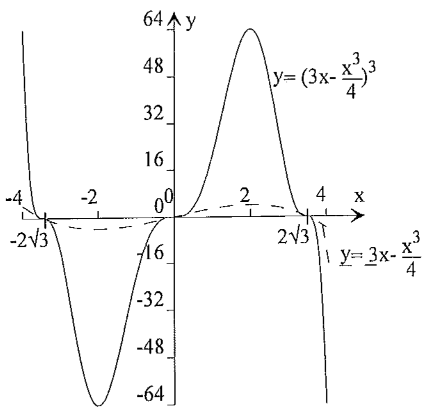
Sign of $\frac{dy}{dx}$



Problem GRA3_24.

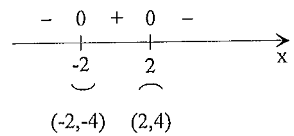
Use the graph of $y = 3x - \frac{x^3}{4}$ to sketch the graph of. $y = \left(3x - \frac{x^3}{4}\right)^3$

Solution:



$f(x) = 3x - \frac{x^3}{4} \quad f'(x) = 3 - \frac{3}{4}x^2$

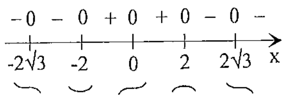
Sign of $f'(x)$



$y=[f(x)]^3 \quad \frac{dy}{dx}=3[f(x)]^2 f'(x) \quad \frac{dy}{dx}=3(3x-\frac{x^3}{4})^2(3-\frac{3}{4}x^2)$

$\frac{dy}{dx}=\frac{9}{64}x^2(2\sqrt{3}-x)^2(2\sqrt{3}+x)^2(2-x)(2+x)$

Sign of $\frac{dy}{dx}$

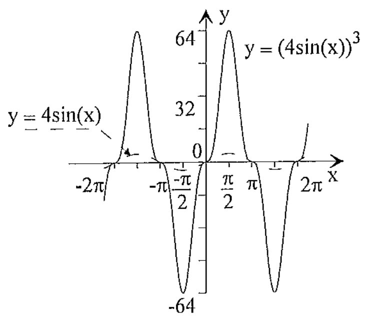


(-2√3,0) (-2,64) (0,0) (2,64) (2√3,0)

Problem GRA3_25.

Use the graph of $y=4\sin x$ to sketch the graph of $y=(4\sin x)^3$.

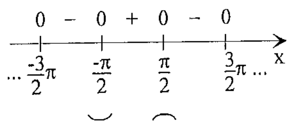
Solution:



$f(x)=4\sin x \quad f'(x)=4\cos x$

Critical points are $\frac{\pi}{2}+n\pi$, n - integral

Sign of $f'(x)$

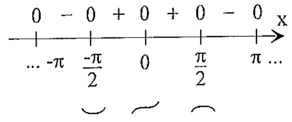


(-π/2,−4) (π/2,4)

$y=[f(x)]^3 \quad \frac{dy}{dx}=3[f(x)]^2 f'(x) \quad \frac{dy}{dx}=3(4\sin x)^2(4\cos x)=96\sin x\sin 2x$

Critical points are $n\frac{\pi}{2}$, n - integral

Sign of $\frac{dy}{dx}$



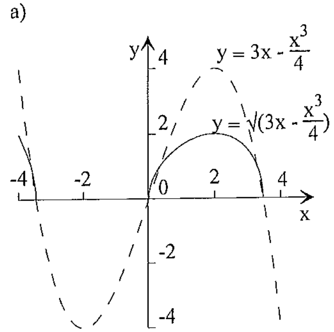
(-π/2,-64) (0,0) (π/2,64)

Problem GRA3_26.

For the function $f(x)=3x-\frac{x^3}{4}$ use the graph of $y=f(x)$ to sketch the graphs of a) $y=\sqrt{f(x)}$, b) $y^2=f(x)$.

Solution:

a)



Features:

- $y=\sqrt{f(x)}$ is defined only where $f(x)\ge 0$.
- $f(x)=0$ where $x=\pm 2\sqrt{3}$ or $x=0 \Rightarrow \frac{dy}{dx}=\frac{f'(x)}{2\sqrt{f(x)}}$ is not defined at

$x=\pm 2\sqrt{3}$ and $x=0 \Rightarrow (\pm 2\sqrt{3},0)$ and $(0,0)$ are critical points.

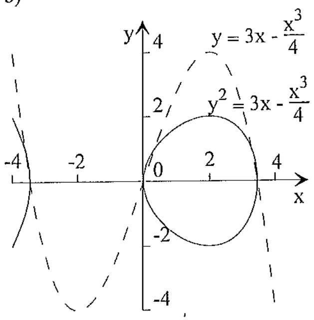
- $\frac{dy}{dx}=\frac{3}{8}\frac{4-x^2}{\sqrt{f(x)}} \to \infty$ as $x\to \pm 2\sqrt{3}^-$ or $x\to 0^+ \Rightarrow$ the tangent lines at $(\pm 2\sqrt{3},0)$ and $(0,0)$ are vertical.
- (2,4) is a maximum turning point of $y=f(x) \Rightarrow$ (2,2) is a maximum turning point of $y=\sqrt{f(x)}$.

- $y = \sqrt{f(x)}$ lies below $y = f(x)$ where $f(x) > 1$.

$y = \sqrt{f(x)}$ lies above $y = f(x)$ where $f(x) < 1$.

$y = \sqrt{f(x)}$, $y = f(x)$ intersect where $f(x) = 1$ or $f(x) = 0$.

b)



$y = \sqrt{f(x)} \Rightarrow y^2 = f(x) \Rightarrow (-y)^2 = f(x)$. Hence the graph $y^2 = f(x)$ is obtained by reflecting $y = \sqrt{f(x)}$ in the x-axis.

The graph $y^2 = f(x)$ has vertical tangent lines at the critical points $(\pm 2\sqrt{3}, 0)$ and $(0,0)$.

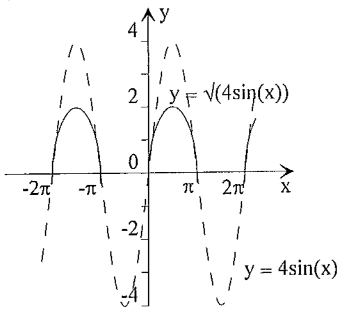
Problem GRA3_27.

For the function $f(x) = 4\sin x$ use the graph $y = f(x)$ to sketch the graphs of

a) $y = \sqrt{f(x)}$, b) $y^2 = f(x)$.

Solution:

a)



Features:

- $y = \sqrt{f(x)}$ is defined only where $f(x) \ge 0$.
- $f(x) = 0$ where $x = n\pi$, n integral $\Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$ is not defined at $x = n\pi$, n integral

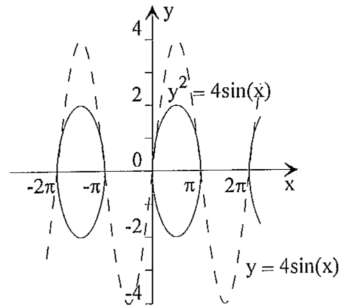
$\Rightarrow (n\pi, 0)$, n integral, are critical points.

- $\dfrac{dy}{dx} = \dfrac{2\cos x}{\sqrt{f(x)}} \to \infty$ as $x \to (2\pi n)^+$, n integral, or $x \to (n\pi)^-$, n odd $\Rightarrow$ the tangent lines at $(n\pi, 0)$, n integral, are vertical.
- $\left(\dfrac{\pi}{2} + 2\pi n, 4\right)$, n integral, are maximum turning of $y = f(x) \Rightarrow \left(\dfrac{\pi}{2} + 2\pi n, 2\right)$, n integral, are maximum turning points of $y = \sqrt{f(x)}$.
- $y = \sqrt{f(x)}$ lies below $y = f(x)$ where $f(x) < 1$.

$y = \sqrt{f(x)}$ lies above $y = f(x)$ where $f(x) > 1$.

$y = \sqrt{f(x)}$, $y = f(x)$ intersect where $f(x) = 1$ or $f(x) = 0$.

b)

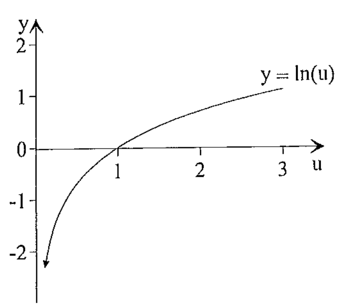


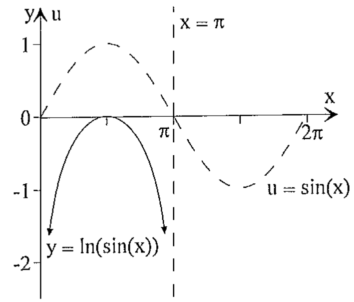
$y = \sqrt{f(x)} \Rightarrow y^2 = f(x) \Rightarrow (-y)^2 = f(x)$. Hence the graph $y^2 = f(x)$ is obtained by reflecting $y = \sqrt{f(x)}$ in the x-axis. The graph $y^2 = f(x)$ has vertical tangent lines at the critical points $(\pi n, 0)$, n integral.

Problem GRA3_28.

Use the graphs of $y = \ln u$ and $u = \sin x$ $(0 \le x \le 2\pi)$ to sketch the graph of $y = \ln(\sin x)$ $(0 \le x \le 2\pi)$.

Solution:





Features of the graph $y = \ln(\sin x)$:

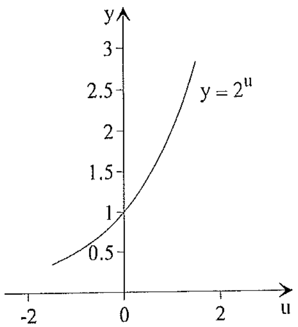
- $y = \ln u$ is defined where $u = \sin x > 0 \Rightarrow$ domain $\{x : 0 < x < \pi\}$.
- Vertical asymptote of $y = \ln u$ at $u = 0$.

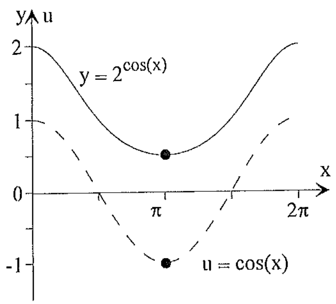
But $u = \sin x$ and $\sin x = 0$ at $x = 0$ or $x = \pi$ $\Rightarrow x = 0$ and $x = \pi$ are vertical asymptotes of $y = \ln(\sin x)$.

- $u = \sin x \le 1$ $\Rightarrow y = \ln u \le 0$.
- $y = \ln u$ is an increasing function $\Rightarrow y = \ln(\sin x)$ increases as $\sin x$ increases and decreases as $\sin x$ decreases.
- The maximum turning point $(\frac{\pi}{2}, 1)$ of $u = \sin x$ corresponds to the maximum turning point $(\frac{\pi}{2}, 0)$ of $y = \ln(\sin x)$.

Problem GRA3_29.
Use the graphs of $y = 2^u$ and $u = \cos x$ $(0 \le x \le 2\pi)$ to sketch the graph of $y = 2^{\cos x}$ $(0 \le x \le 2\pi)$.

Solution:



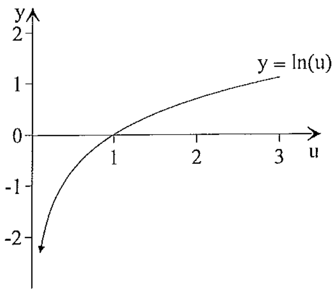


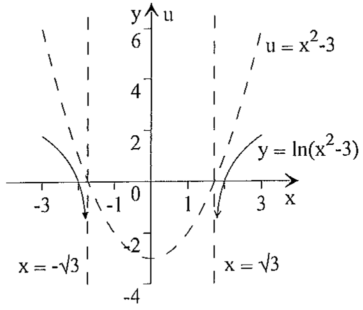
Features of the graph $y = 2^{\cos x}$:

- $y = 2^{\cos x}$, domain $\{x : 0 \le x \le 2\pi\}$.
- $y = 2^x$ is an increasing function $\Rightarrow y = 2^{\cos x}$ increases as $\cos x$ increases and decreases as $\cos x$ decreases.
- $(\pi, -1)$ is a minimum turning point of $u = \cos x$ $\Rightarrow (\pi, 2^{-1})$ is a minimum turning point of $y = 2^{\cos x}$.
- $(0, 1)$ and $(2\pi, 1)$ are maximum turning points of $u = \cos x$ $\Rightarrow (0, 2)$ and $(2\pi, 2)$ are maximum turning points of $y = 2^{\cos x}$.

Problem GRA3_30.
Use the graphs of $y = \ln u$ and $u = x^2 - 3$ (an even function) to sketch the graph of $y = \ln(x^2 - 3)$.

Solution:





Features of the graph $y = \ln(x^2 - 3)$:

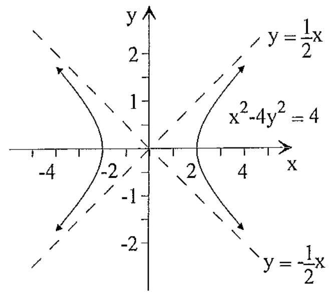
- $y = \ln(x^2 - 3)$ is defined where $u = x^2 - 3 > 0$.
- $\ln((-x)^2 - 3) = \ln(x^2 - 3) \Rightarrow$ the graph $y = \ln(x^2 - 3)$ is the graph of an even function.
- Vertical asymptote of $y = \ln u$ at $u = 0$.

But $u = x^2 - 3$ and $x^2 - 3 = 0$ at $x = \pm\sqrt{3}$ $\Rightarrow x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes of $y = \ln(x^2 - 3)$.

- $y = \ln u$ is an increasing function $\Rightarrow y = \ln(x^2 - 3)$ increases as $x^2 - 3$ increases and decreases as $x^2 - 3$ decreases.

Problem GRA3_31.
Sketch (showing critical points) the graph of $x^2 - 4y^2 = 4$.

Solution:



$x^2 - 4y^2 = 4$. Clearly $x^2, y^2 \ge 0 \Rightarrow$ domain $\{x : |x| \ge 2\}$. Take the derivative of both sides with respect to x. Consider y as a function of x and use the chain rule. Then we have

$$2x - 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(\frac{x}{y}\right).$$

As $y \to 0$, $x \to \pm 2 \Rightarrow \frac{dy}{dx} \to \infty$ and the curve has vertical tangent at the critical points (-2,0) and (2,0).

As $x = 0$, $y = \pm 1 \Rightarrow \frac{dy}{dx} = 0$ and the curve has horizontal tangent at (0,-1) and (0,1).

Clearly the curve is symmetric about the lines $y = 0$ and $x = 0$ as the transformation $y \to -y$ and $x \to -x$ leave the Cartesian equation of the curve unchanged.

$x^2 - 4y^2 = 4 \Rightarrow y = \pm\frac{|x|}{2}\left(1 - \frac{4}{x^2}\right)^{1/2}$. By expansion for the large values of x we have

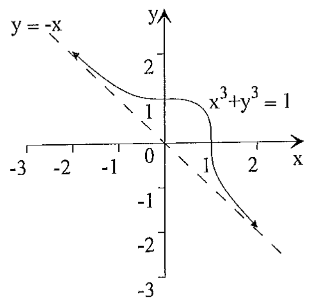
$y = \pm\frac{|x|}{2}\left(1 - \frac{2}{x^2} + \ldots\right) \Rightarrow y = \pm\frac{x}{2} + 0\left(\frac{1}{x}\right)$. Hence the curve has an oblique asymptotes

$y = \pm\frac{x}{2}$ as $x \to \pm\infty$.

Problem GRA3_32.

Sketch (showing critical points and stationary points) the graph of $x^3 + y^3 = 1$.

Solution:



$x^3 + y^3 = 1$. Take the derivative of both sides with respect to x. Consider y as a function of x and use the chain rule. Then we have $3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^2$.

As $y \to 0$, $x \to 1 \Rightarrow \frac{dy}{dx} \to -\infty$ and the curve has a vertical tangent at (1,0).

As $x = 0$, $y = 1 \Rightarrow \frac{dy}{dx} = 0$ and the curve has a horizontal tangent at (0,1),

Clearly the curve is symmetric about $y = x$, since the transformation $y \leftrightarrow x$ leaves the Cartesian equation of the curve unchanged.

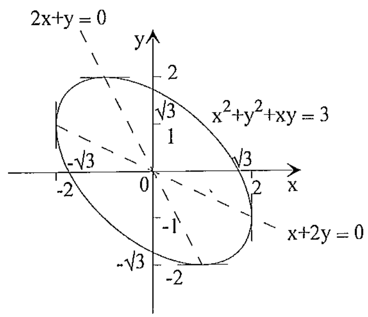
$x^3 + y^3 = 1 \Rightarrow y = -x\left(1 - \frac{1}{x^3}\right)^{1/3}$. By expansion for the large values of x we have

$y = -x\left(1 - \frac{1}{3x^3} + \ldots\right) \Rightarrow y = -x + 0\left(\frac{1}{x}\right)$. Hence the curve has an oblique asymptote $y = -x$ as $x \to \pm\infty$.

Problem GRA3_33.

Sketch (showing critical points and stationary points) the graph of $x^2 + y^2 + xy = 3$.

Solution:



$x^2 + y^2 + xy = 3$. Take the derivative of both sides with respect to x. Consider y as a function of x and use the chain and product rules. Then we have

$2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \Rightarrow (x + 2y)\frac{dy}{dx} = -(2x + y) \Rightarrow \frac{dy}{dx} = -\left(\frac{2x + y}{x + 2y}\right)$. Substituting of $x = -2y$ in the equation of the curve gives $x + 2y = 0 \Rightarrow$

$4y^2 + y^2 - 2y^2 = 3 \Rightarrow y^2 = 1$.

Hence $\begin{cases} y = -1 \\ x = 2 \end{cases}$ or $\begin{cases} y = 1 \\ x = -2 \end{cases}$.

In either case, $2x + y \neq 0$. Hence as $x + 2y \to 0$, $\frac{dy}{dx} \to \infty$ and the curve has vertical tangents at (2, -1) and (-2,1).

Similarly, $2x + y = 0 \Rightarrow x^2 = 1$.

Hence $\begin{cases} x = -1 \\ y = 2 \end{cases}$ or $\begin{cases} x = 1 \\ y = -2 \end{cases}$.

In either case, $x + 2y \neq 0$. Hence $2x + y = 0 \Rightarrow \frac{dy}{dx} = 0$ and the curve has horizontal tangents at (-1,2) or (1,-2).

Clearly the curve is symmetric about $y = x$ and $y = -x$, since the transformation $y \leftrightarrow x$ and $y \leftrightarrow x$ leave the Cartesian equation of the curve unchanged.

Problem GRA3_34.

Find the equation of the tangent to the curve $xy(x + y) + 16 = 0$ at the point on the curve where the gradient is -1.

Answer: $y + x + 4 = 0$.

Solution:

$x^2y + xy^2 + 16 = 0$

Consider y as a function of x and take the derivative of both sides with respect to x using the chain and product rules:

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

$$(x^2 + 2xy)\frac{dy}{dx} = -(y^2 + 2xy)$$

$$\frac{dy}{dx} = -1 \Rightarrow -(x^2 + 2xy) = -(y^2 + 2xy)$$

$\Rightarrow x^2 = y^2 \Rightarrow y = x$ or $y = -x$.

Substitution of $y = x$ in the equation of the curve gives

$x^2x + xx^2 + 16 = 0 \Rightarrow 2x^3 = -16 \Rightarrow x = -2$ and hence $y = -2$.

Substitution of $y = -x$ in the equation of the curve gives

$x^2(-x) + x(-x)^2 + 16 = 0 \Rightarrow 16 = 0$.

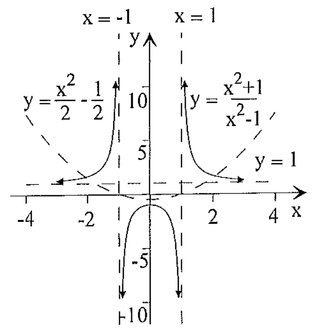
Hence the tangent touches the curve at the point $(x_0, y_0) = (-2, 2)$ where the gradient $k = -1$. So the equation of the tangent is $y - y_0 = k(x - x_0) \Rightarrow y + 2 = -(x + 2) \Rightarrow\ y + x + 4 = 0$.

Problem GRA3_35.

Sketch the graph of $y = \frac{x^2+1}{x^2-1}$. Use this graph to solve the inequality $\frac{x^2+1}{x^2-1} < 1$.

Answer: $\{x : -1 < x < 1\}$.

Solution:



$$\frac{x^2+1}{x^2-1} = \frac{x^2-1+2}{x^2-1} = 1 + \frac{2}{x^2-1}$$

The graph $y = \frac{2}{x^2-1}$ has been translated one unit upward. $y = 1$ is asymptote as $x \to \infty$. The graph $y = \frac{2}{x^2-1} = \frac{1}{\frac{x^2}{2} - \frac{1}{2}}$ is a reciprocal of $y = \frac{x^2}{2} - \frac{1}{2}$.

Consider the graph $y = f(x)$ and $y = \frac{1}{f(x)}$, where $f(x) = \frac{x^2}{2} - \frac{1}{2}$.

Features:

- $y = f(x)$, $y = \frac{1}{f(x)}$ have the same sign.
- $f(x) = 0$ when $x = \pm 1 \Rightarrow$ the lines $x = -1$ and $x = 1$ are vertical asymptotes of $y = \frac{1}{f(x)}$.
- As $x \to \infty$, $f(x) \to +\infty \Rightarrow \frac{1}{f(x)} \to 0^+$.
- Minimum turning point of $y = f(x)$ is $\left(0, -\frac{1}{2}\right) \Rightarrow$ maximum turning point of $y = \frac{1}{f(x)}$ is $(0, -2)$.

By inspection of the graph, $\frac{x^2+1}{x^2-1} < 1$ for $-1 < x < 1$.

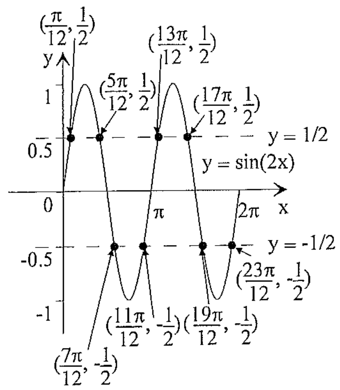
Problem GRA3_36.

Sketch the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$. Use this graph to solve the inequalities a) $\sin 2x \ge \frac{1}{2}$, for $0 \le x \le 2\pi$; b) $|\sin 2x| \ge \frac{1}{2}$, for $0 \le x \le 2\pi$.

Answer: a) $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$, $\frac{13\pi}{12} \le x \le \frac{17\pi}{12}$;

b) $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$, $\frac{13\pi}{12} \le x \le \frac{17\pi}{12}$, $\frac{7\pi}{12} \le x \le \frac{11\pi}{12}$, $\frac{19\pi}{12} \le x \le \frac{23\pi}{12}$.

Solution:



a) $\sin 2x = \frac{1}{2} \Leftrightarrow 2x = (-1)^n \sin^{-1}\frac{1}{2} + \pi n$, n integral

$\Rightarrow x = (-1)^n \frac{\pi}{12} + \frac{n}{2}\pi$, $n = 0,1,2,\ldots$ $(x \ge 0)$.

But $0 \le x \le 2\pi \Rightarrow$ there are exactly four values of x, namely $\frac{\pi}{12}, \frac{5}{12}\pi, \frac{13}{12}\pi, \frac{17}{12}\pi$.

By inspection of the graph, $\sin 2x \ge \frac{1}{2}$ for $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$ or $\frac{13\pi}{12} \le x \le \frac{17\pi}{12}$.

b) $|\sin 2x| \ge \frac{1}{2} \Leftrightarrow \sin 2x \ge \frac{1}{2}$ or $\sin 2x \le -\frac{1}{2}$

$\sin 2x = -\frac{1}{2} \Leftrightarrow 2x = (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) + \pi n$, n integral

$\Rightarrow x = (-1)^{n+1} \frac{\pi}{12} + \frac{n}{2}\pi$, $n = 1,2,\ldots$ $(x \ge 0)$.

But $0 \le x \le 2\pi \Rightarrow$ there are exactly four values of x, namely $\frac{7}{12}\pi, \frac{11}{12}\pi, \frac{19}{12}\pi, \frac{23}{12}\pi$.

The equation $\sin 2x = \frac{1}{2}$ was solved in a).

By inspection of the graph, $|\sin 2x| \ge \frac{1}{2}$ for

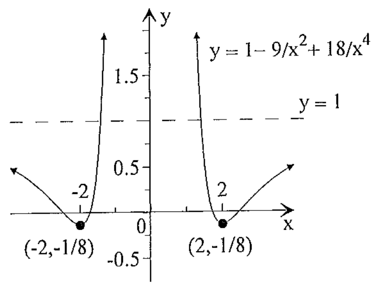
$\frac{\pi}{12} \le x \le \frac{5\pi}{12}, \frac{13\pi}{12} \le x \le \frac{17\pi}{12}, \frac{7\pi}{12} \le x \le \frac{11\pi}{12}, \frac{19\pi}{12} \le x \le \frac{23\pi}{12}$.

Problem GRA3_37.

Sketch the graph of $f(x) = 1 - \frac{9}{x^2} + \frac{18}{x^4}$. Use this graph to find the set of values of the real number k such that the equation $f(x) = k$ has four real distinct roots.

Answer: $-\frac{1}{8} < k < 1$.

Solution:



$y = 1 - \frac{9}{x^2} + \frac{18}{x^4}$

Domain $\{x : x \ne 0\}$

$\frac{dy}{dx} = \frac{18}{x^3} - \frac{72}{x^5}$

Sign of $\frac{dy}{dx}$

x		-2		0		2	
sign	−	0	+		−	0	+
		(-2,-1/8)				(2,1/8)	

As $x \to 0$, $y \to +\infty \Rightarrow$ the line $x = 0$ is a vertical asymptote.
As $x \to \infty$, $y \to 1^- \Rightarrow$ the line $y = 1$ is a horizontal asymptote.
Real solution of the equation $f(x) = k$ are given by x-values where $y = f(x)$ and $y = k$ intersect. Hence the equation has four real distinct roots for the following set of k

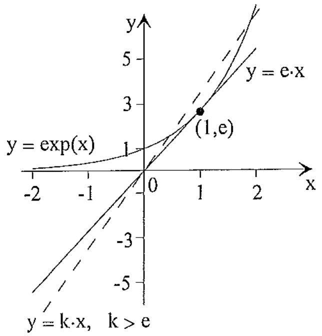
$\{k : -\frac{1}{8} < k < 1\}$.

Problem GRA3_38.

Find the gradient of the tangent to the curve $y = e^x$ which passes through the origin. Hence find the values of the real number k for which the equation $e^x = kx$ has exactly two real solutions.

Answer: e, $k > e$.

Solution:



Let the gradient of the tangent from the origin to the curve be equal to a. Then $a = \left(e^x\right)'$, i.e., $a = e^x$. In addition at the point (x, y) where the tangent touch the curve $y = e^x$ and simultaneously $y = ax$. Hence we have the simultaneous equations:

$$\begin{cases} a = e^x \\ y = e^x \\ y = ax \end{cases} \Leftrightarrow \begin{cases} a = e^x \\ ax = e^x \\ y = ax \end{cases} \Leftrightarrow \begin{cases} a = e^x \\ xe^x = e^x \\ y = ax \end{cases} \Leftrightarrow \begin{cases} a = e \\ x = 1 \\ y = e \end{cases}$$

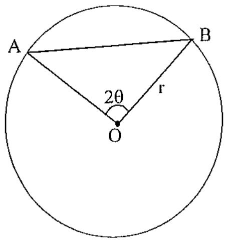
Real solutions of the equation $e^x = kx$ are given by x-values where $y = e^x$ and $y = kx$ intersect. Hence the equation has two real distinct roots for the following set of k $\{k : k > e\}$.

Problem GRA3_39.

The chord AB of a circle of radius r subtends an angle of 2θ radians at the centre O. The perimeter of the minor segment AB is k times the perimeter of the triangle OAB. Show that $k + (k-1)\sin\theta = \theta$. Use a graphical method to obtain an estimate of θ in the case when $k = \frac{1}{2}$.

Answer: 0.34.

Solution:



The perimeter of the triangle OAB is $2r + 2r\sin\theta$. The perimeter of the minor segment $2r\sin\theta + 2\theta r$.

Hence $2r\sin\theta + 2\theta r = k(2r + 2r\sin\theta)$

$\sin\theta + \theta = k(1 + \sin\theta)$

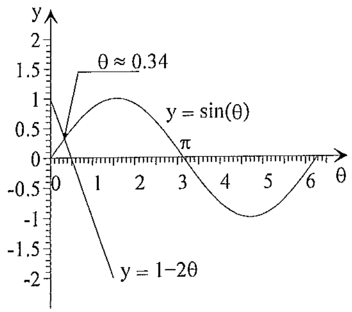
$k + (k-1)\sin\theta = \theta$

If $k = \frac{1}{2}$, then $\frac{1}{2} - \frac{1}{2}\sin\theta = \theta$

$\Rightarrow \sin\theta = 1 - 2\theta$.

Clearly solution of the equation $\sin\theta = 1 - 2\theta$ are given by θ-values where $y = \sin\theta$ and $y = 1 - 2\theta$ intersect.

Note that $0 < 2\theta < \pi \Rightarrow 0 < \theta < \frac{\pi}{2}$.



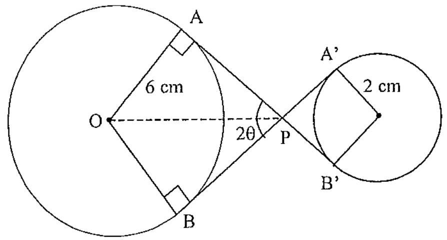
By inspection of the graph $\theta \approx 0.34$.

Problem GRA3_40.

A taut belt passes round two circular pulleys of radius 6 cm and 2 cm respectively. The straight portions of the belt are common tangents to the two pulleys and are inclined to each other at an angle of 2θ radians. The total length of the belt is 44 cm. Show that $\frac{\pi}{2} + \theta + \cot\theta = \frac{11}{4}$ and hence use a graphical method to obtain an estimate of θ.

Answer: $\theta \approx 2.48$

Solution:



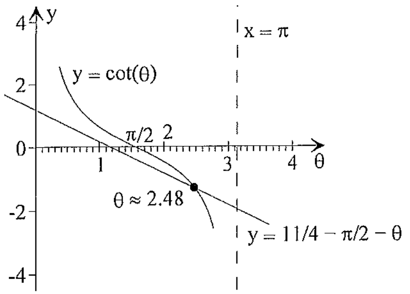
Consider the rectangular triangle OAP. $\angle OPA = \theta \Rightarrow AP = 6\cot\theta$. Analogously $BP = 6\cot\theta$.

In the quadrilateral $OAPB$ the sum of angles is $\angle AOB + \pi + 2\theta = 2\pi$

$\Rightarrow 2\pi - \angle AOB = \pi + 2\theta$.
Hence the length of the larger arc AB is $6(\pi + 2\theta)$.
So the length of the belt from the point P to the circular pulley of radius 6 cm and round it is $2 \cdot 6\cot\theta + 6(\pi + 2\theta) = 12\cot\theta + 6\pi + 12\theta$. The figure $PA'B'$ is similar to $PAB \Rightarrow$ the length of the second part of the belt is $\frac{2}{6}$ of the first part.

Hence the belt has the length $(12\cot\theta + 6\pi + 12\theta)(1 + \frac{1}{3}) = 44$

$\cot\theta + \frac{\pi}{2} + \theta = \frac{11}{4}$. Clearly solutions of the equations $\cot\theta = \frac{11}{4} - \frac{\pi}{2} - \theta$ are given by θ-values where $y = \cot\theta$ and $y = \frac{11}{4} - \frac{\pi}{2} - \theta$ intersect. Note that $0 < \theta < \pi$.



By inspection of the graph $\theta \approx 2.48$.