LOCUS AND THE PARABOLA 2 — Ext. 1

1) Derive the equation of the tangent to the parabola $x^2 = 4y$ at the point $(2t,t^2)$.

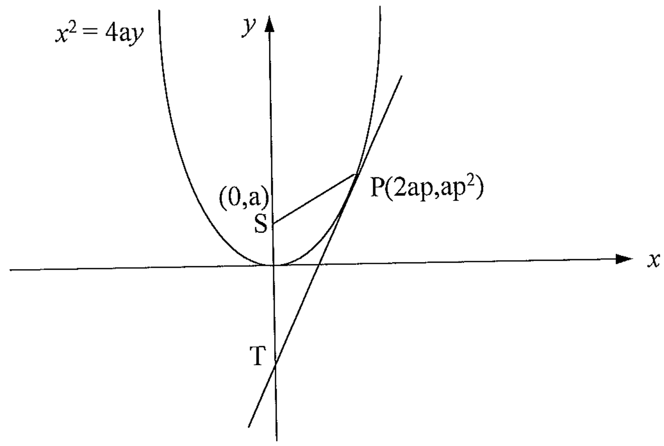
2) Derive the equation of the normal to the parabola $x^2 = 4ay$ at the point $(2ap,ap^2)$.

3) (a) Derive the equation of the chord PQ where $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ are points on the parabola $x^2 = 4ay$.
(b) Show that if PQ is a focal chord, then $pq = -1$.

4) Find the point of intersection of the tangents to the curve $x^2 = 8y$ at points (4,2) and (-8,8).

5) Tangents are drawn from an external point $\left(1\frac{1}{2},-1\frac{1}{2}\right)$ to the parabola $x^2 = 12y$.
(a) Find the equation of the chord of contact of the tangents.
(b) Find the coordinates of the points where the tangents meet the parabola.

6)



Point $P(2ap,ap^2)$ lies on the parabola $x^2 = 4ay$ and S is the focus. The tangent at P meets the y-axis at T.
(a) Find the coordinates of T.
(b) Prove that SP = ST.
(c) Hence show that <SPT = <STP.

7) (a) Find the coordinates of D, where the tangents to the parabola $x^2 = 4ay$ from $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ intersect.
(b) If F is the focus of the parabola, show that $FP = a(p^2 + 1)$.
(c) Find the locus of D if P and Q move so that FP + FQ = 6a.

8) If $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ are points on the parabola $x^2 = 4ay$ such that PQ is a focal chord, show that the tangents at P and Q intersect on the directrix.

9) (a) Write down the equations of normals to the parabola $x^2 = 4ay$ at the points $P(2ap,ap^2)$ and $Q(2aq,aq^2)$.
(b) Find the coordinates of N, the point of intersection of the normals at P and Q.
(c) Find the equation of the locus of N if chord PQ passes through the point (0,2a)

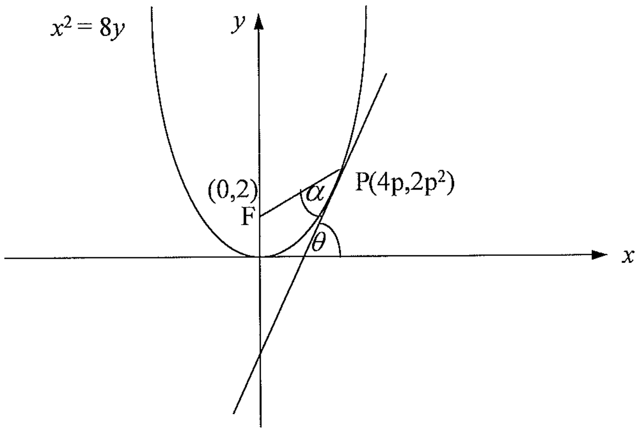
10) Find the locus of the midpoint of PQ where $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ are points on the parabola $x^2 = 4ay$ and PQ is a focal chord.

11) The parabola $x^2 = 4y$ has tangents drawn from points R(-4,4) and S(2,1) on the parabola.

(a) Find the point of intersection T of these tangents.

(b) Point Q($2q,q^2$) lies between R and S on the parabola. If the tangent at Q meets line ST at its midpoint, show that the tangent at Q is parallel to RS.

12)



The point P($4p,2p^2$) lies on the parabola $x^2 = 8y$, $p > 0$. F is the focus of the parabola. The tangent at P makes an angle of θ with the x-axis.

(a) Show that $\tan\theta = p$.

(b) Show that the gradient of FP is $\dfrac{p^2-1}{2p}$.

(c) Show that $\tan\alpha = \dfrac{1}{p}$ where α is the angle between FP and the tangent at P, and $p > 0$.

(d) Find the value of $\theta + \alpha$

(e) Find the coordinates of P if $\theta = \alpha$.

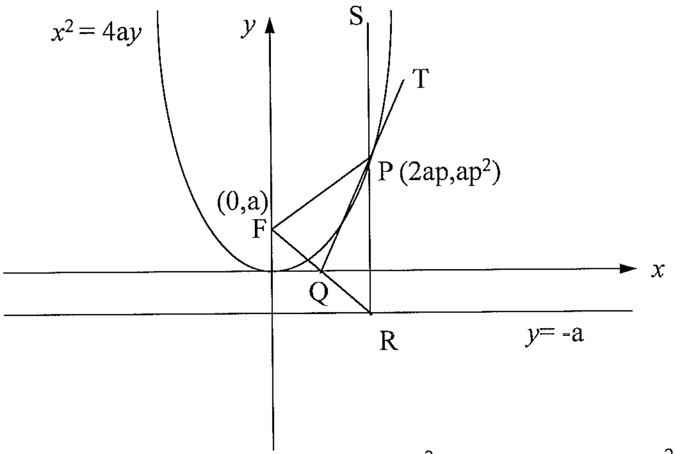
13) (a) Find the equation of the tangent to the parabola $x^2 = 4ay$ at point P($2ap,ap^2$).

(b) Find S, the point of intersection of the tangents at P and Q($2aq,aq^2$).

(c) The tangents meet at S at an angle of 45^0. Show that $p - q = 1 + pq$.

(d) Find the equation of the locus of S.

14)



A tangent is drawn from $P(2ap,ap^2)$ on the parabola $x^2 = 4ay$ with x-intercept Q. RS is drawn parallel to the y-axis such that R lies on the directrix. F is the focus of the parabola.

(a) Find the coordinates of Q.

(b) Show that PQ and FQ are perpendicular.

(c) Show that FP = PR.

(d) Prove that <SPT = <FPQ.

ANSWERS

1) $tx - y - t^2 = 0$

2) $x + py - 2ap - ap^3 = 0$

3) (a) $(p + q)x - 2y - 2apq = 0$

(b) Substitute (0,a):

$$(p + q)0 - 2a - 2apq = 0$$
$$-2apq = 2a$$
$$pq = -1$$

4) (-2,-4)

5) (a) $x - 4y + 6 = 0$ (b) (6,3) and (-3,0.75)

6) (a) $(0,-ap^2)$

(b) $SP = \sqrt{(2ap-0)^2 + (ap^2 - a)^2}$

$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$$
$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$
$$= \sqrt{a^2(p^4 + 2p^2 + 1)}$$
$$= \sqrt{a^2(p^2+1)^2}$$
$$= a(p^2 + 1)$$

$ST = a - (-ap^2)$

$= a + ap^2$

$= a(1 + p^2)$

$= SP$

(c) ST = SP, so SPT is an isosceles triangle.

So <SPT = <STP (base angles in isosceles triangle)

7) (a) (a(p + q),apq)

(b) $FP = \sqrt{(2ap-0)^2 + (ap^2 - a)^2}$

$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$$
$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$
$$= \sqrt{a^2(p^4 + 2p^2 + 1)}$$
$$= \sqrt{a^2(p^2+1)^2}$$
$$= a(p^2 + 1)$$

(c) $x^2 = 2a(y + 2a)$

8) Chord PQ has equation $(p + q)x - 2y - 2apq = 0$

Substitute (0,a):

$$(p + q)0 - 2a - 2apq = 0$$
$$-2apq = 2a$$
$$pq = -1$$

Intersection of tangents is (a(p + q), apq)

So $x = a(p + q)$ and $y = apq$

But pq = -1, so y = a(-1)

i.e. y = -a, so the tangents meet on the directrix.

9) (a) $x + py - 2ap - ap^3 = 0,\ x + qy - 2aq - aq^3 = 0$

(b) $(-apq(p + q),\ a(p^2 + pq + q^2 + 2))$

(c) $x^2 = 4a(y - 4a)$

10) $x^2 = 2a(y - a)$

11) (a) (-1,-2)

(b) RS has gradient $\dfrac{4-1}{-4-2} = -\dfrac{1}{2}$

Midpoint ST: $M = \left(\dfrac{-1+2}{2}, \dfrac{-2+1}{2}\right) = \left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$

QM has gradient $\dfrac{2q^2+1}{4q-1}$

Tangent at Q has gradient q.

$$\therefore \frac{2q^2+1}{4q-1} = q$$

$$2q^2 + 1 = 4q^2 - q$$

$$2q^2 - q - 1 = 0$$

$$(2q+1)(q-1) = 0$$

$$q = -\frac{1}{2}, 1$$

But $q = 1$ makes Q = (2,1) = S, so $q = 1$ is not a solution.

$$\therefore q = -\frac{1}{2}$$

So the tangent at Q is parallel to RS (equal gradients)

12) (a) Gradient of tangent at P is p.

So $\tan\theta = p$

(b) Gradient of SP $= \dfrac{2p^2-2}{4p-0}$

$$= \frac{p^2-1}{2p}$$

(c) $$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{p - \dfrac{p^2-1}{2p}}{1 + p\left(\dfrac{p^2-1}{2p}\right)}\right|$$

$$= \left|\frac{2p^2 - p^2 + 1}{2p + p^3 - p}\right|$$

$$= \left|\frac{p^2 + 1}{p(p^2 + 1)}\right|$$

$$= \left|\frac{1}{p}\right|$$

$$= \frac{1}{p}, p > 0$$

(d) 90^0 (e) (4,2)

13) (a) $px - y - p^2 = 0$ (b) $(a(p + q),apq)$

(c) Gradient of tangent at P is p, gradient of tangent at Q is q.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan 45^0 = \left|\frac{p - q}{1 + pq}\right|$$

$$1 = \left|\frac{p - q}{1 + pq}\right|$$

$p > q$, *so* $p - q > 0$

$$\therefore \quad 1 = \frac{p - q}{1 + pq}$$

$$\therefore 1 + pq = p - q$$

(d) $(y + a)^2 = x^2 - 4ay$

14) (a) (ap,0)

(b) PQ has gradient $\frac{ap^2 - 0}{2ap - ap} = \frac{ap^2}{ap} = p$

FQ has gradient $\frac{a - 0}{0 - ap} = \frac{a}{-ap} = -\frac{1}{p}$

$p \times -\frac{1}{p} = -1$ so PQ and FQ are perpendicular.

(c) $FP = \sqrt{(2ap - 0)^2 + (ap^2 - a)^2}$

$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$$

$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$

$$= \sqrt{a^2(p^4 + 2p^2 + 1)}$$

$$= \sqrt{a^2(p^2 + 1)^2}$$

$$= a(p^2 + 1)$$

$R = (2ap,-a)$

PR = $ap^2 - (-a) = a(p^2 + 1)$ = FP

(d) Triangle FPR is isosceles, so <PFR = <PRF

<FQP = <RQP = 90^0

So <FPQ = <QPR (< sum of triangle)

<QPR = <SPT (vertically opposite <s)

So <FPQ = <SPT