Name: ____________________

Teacher: ____________________

MORIAH COLLEGE

Year 11

MATHEMATICS Extension 1

Date: Monday 25th November, 2002

Time Allowed: I Hour

Examiners: N Franks, D. Steel, J. Taylor

INSTRUCTIONS :

- **Do Questions 1 & 2 in one booklet**
- **Do Questions 3 & 4 in a second booklet**
- **Do Question 5 in the third booklet**
- **Show all necessary working.**

Question 3. (7 marks)

For the function $f(x) = \frac{1}{x}e^{-x}$,

i) prove that the graph $y = f(x)$ has a stationary point at (-1,-e).

ii) Determine the nature of this stationary point.

iii) State the values of $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

iv) Sketch the curve

Question 4. (10 marks)

a) Find $\frac{dy}{dx}$ given $y = \log \frac{\sqrt{2x+1}}{x^2}$ 2

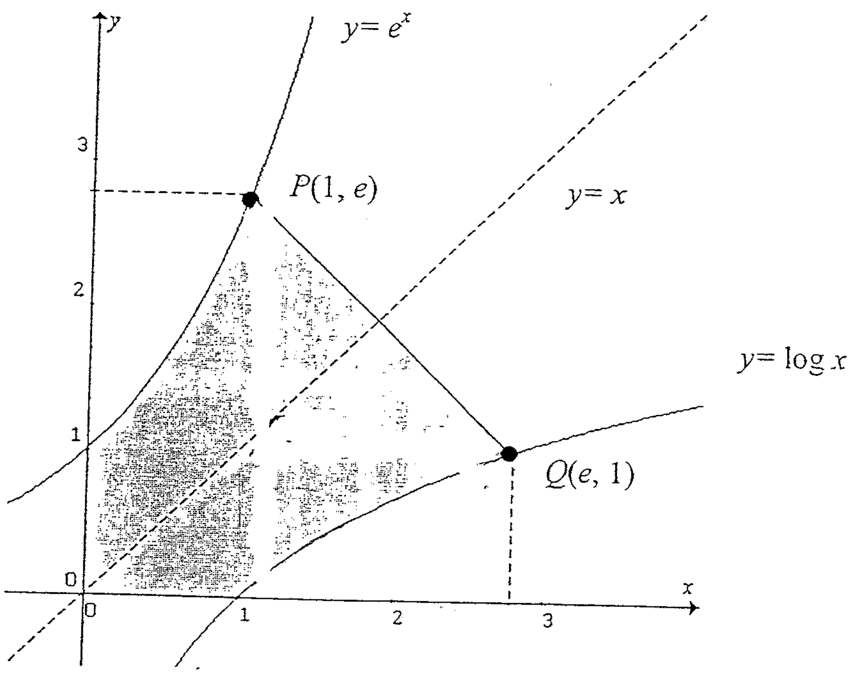
b) Evaluate $\int_1^4 \frac{1}{\sqrt{x}}\left(\sqrt{x}+1\right)^4 dx$ using the substitution $u = \sqrt{x}+1$. 4

c) Find the equation of the normal to the point where $x = \log 2$ on the curve $y = e^{3x}$ 4

Question 5. (10 marks)

a) Differentiate $\left(x^2+2x+2\right)e^{-x}$ and hence evaluate $\int_1^2 x^2 e^{-x}\,dx$ correct to 3 decimal places. 5

b) The graphs of $y = e^x$ and $y = \log x$ are drawn below. 5
The interval joining $P(1, e)$ and $Q(e, 1)$ is constructed.



Find the value of the shaded area.

Standard integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

1a) $\int \left(x+\frac{1}{x^2}\right)^2 dx$

$= \int x^2 + \frac{2}{x} + \frac{1}{x^4}\, dx$

$= \frac{x^3}{3} + 2\ln x - \frac{1}{3x^3} + c$

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b) $\int 5^x dx = \int e^{x\ln 5} dx$

$= \frac{5^x}{\ln 5} + c$

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c) $\int \frac{x}{x^2-4}\, dx$

$= \frac{1}{2}\ln(x^2-4) + c.$

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d) $\int 1 + e^{-x}\, dx$

$= x - e^{-x} + c$

$= x - \frac{1}{e^x} + c$

2(a) $\int_{-4}^{-1} \frac{3}{1-2x} = \frac{\ln 27}{2}.$

$LHS = \frac{-3}{2}\left[\ln(1-2x)\right]_{-4}^{-1}$

$= \frac{-3}{2}(\ln 3 - \ln 9)$

$= \frac{-3}{2}\ln\frac{1}{3}$

$= \frac{3}{2}\ln 3$

$= \frac{\ln 3^3}{2}$

$= \frac{\ln 27}{2}$

$= RHS.$

2b $V = \pi\int_0^2 \left(\frac{1}{\sqrt{2^x+1}}\right)^2 dx.$

x	0	1	2
y^2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$.
$\times$	1	2	1

$h=1$

$A = \frac{\pi}{2}\left\{\frac{1}{2} + \frac{2}{3} + \frac{1}{5}\right\}.$

$= 41\pi/60\ u^3$ or $2.15\,u^3$ (2dp)

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3. $f(x) = \frac{1}{x}e^{-x}$

i) $u = x^{-1}$ $v = e^{-x}$

$u' = -x^{-2}$ $v' = -e^{-x}$

$f'(x) = -\frac{1}{x}e^{-x} - \frac{1}{x^2}e^{-x}$

$= \frac{-1}{xe^x}\left[1 + \frac{1}{x}\right]$

For St. pts $f'(x) = 0$

ie. $\frac{-1}{xe^x}\left[1+\frac{1}{x}\right] = 0$, $x \neq 0$

$\therefore x = -1$

When $x=-1$, $f(x) = -1e^1$

$= -e$

$\therefore$ st pt at $(-1, -e)$

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ii)

x	-2	-1	$-\frac{1}{2}$
$f'(x)$	1.8...	0	-3.5...

Note: $x \neq 0$.

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$\therefore$ local max turning pt at $(-1,-e)$

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iii) $x \to \infty$, $f(x) \to \frac{1}{\infty}e^{-\infty} \Rightarrow 0$

, $f(x) \to \frac{1}{-\infty}e^{\infty} \Rightarrow -\infty$

(as $\frac{1}{-\infty}$ is very small negative, but e^{∞} is extremely large).

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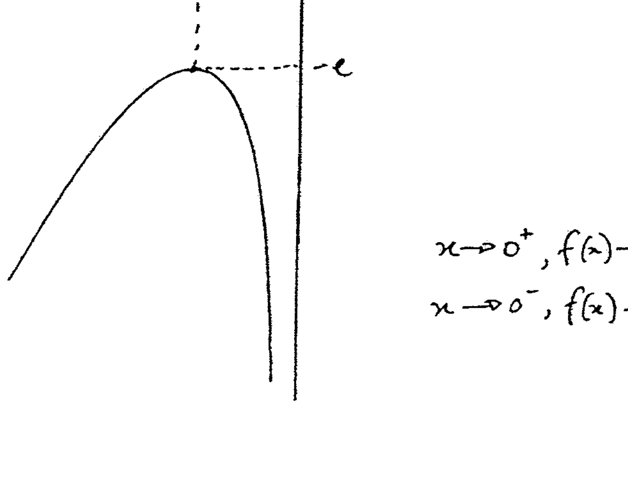
iv)



$x \to 0^+$, $f(x) \to \infty$

$x \to 0^-$, $f(x) \to -\infty$

$= \log(2x+1)^{1/2} - \log x^2$

$= \frac{1}{2}\log(2x+1) - 2\log x$

$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{2x+1}\cdot 2 - 2\cdot\frac{1}{x}$

$= \frac{1}{2x+1} - \frac{2}{x}$

) $\int_1^4 \frac{1}{\sqrt{x}}(\sqrt{x}+1)^4\,dx$

$u = \sqrt{x} + 1$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$x=1,\ u=2$

$x=4,\ u=3$

$2\int_2^3 u^4\,du$

$2\left[\frac{u^5}{5}\right]_2^3$

$\left[\frac{243}{5} - \frac{32}{5}\right]$

$\frac{422}{5}$ or 84.4.

$y = e^{3x}$

$y' = 3e^{3x}$

when $x = \log 2$

$y' = 24$.

$\therefore m$ of normal at $x = \log 2$ is $-\frac{1}{24}$

eqⁿ of normal at $(\log 2, 8)$ is

$y - 8 = -\frac{1}{24}(x - \log 2)$

ie. $x + 24y - 192 - \log 2 = 0$.

) $\frac{d}{dx}\left[(x^2+2x+2)e^{-x}\right]$

$= (2x+2)e^{-x} - (x^2+2x+2)e^{-x}$

$= -x^2e^{-x}$

$\int_1^2 x^2e^{-x}\,dx = -\left[(x^2+2x+2)e^{-x}\right]_1^2$

$= -\left[10e^{-2} - 5e^{-1}\right]$

$= \frac{5e-10}{e^2}$ or 0.486 (3dp)

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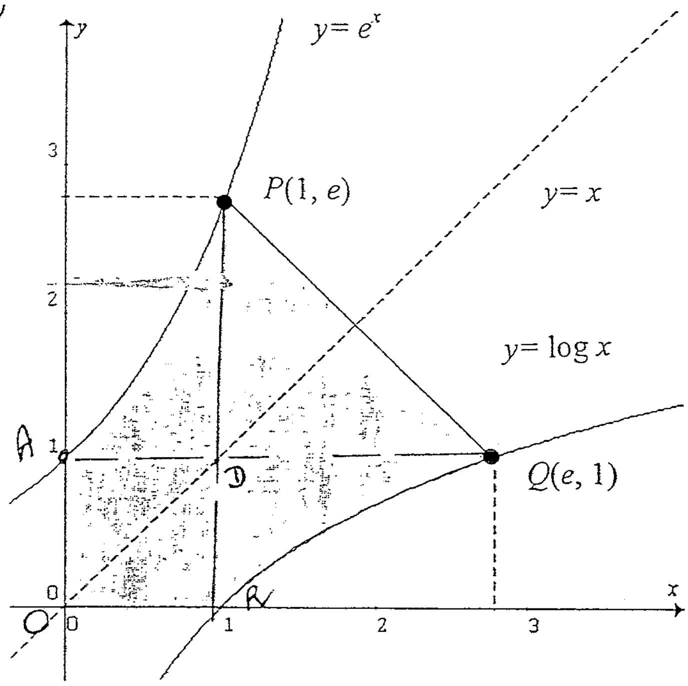
b)



Area OAPR $= \int_0^1 e^x\,dx = e - 1$ u²

Area ORDA $= 1$ u²

Area $\Delta PDQ = \frac{1}{2}(e-1)^2$

Shaded Area $= 2$(Area OAPR) $-$ Area(ORDA) $+$ Area ΔPDQ

$= 2(e-1) - 1 + \left[\frac{e^2-2e+1}{2}\right]$

$= \frac{4e - 4 - 2 + e^2 - 2e + 1}{2}$

$= \frac{e^2 + 2e - 5}{2}$