SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 3

JUNE 2007

MATHEMATICS
EXTENSION 1

Time Allowed: 70 minutes

Name ____________________

Teacher ____________________

Instructions:

- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start each question on a new page.
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	**Total**
/8	/10	/7	/8	/10	/9	/52

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) Solve $2\cos^2 x = \cos x$ for $0 \le x \le 2\pi$ 3

b) Simplify $\dfrac{\log_m \sqrt{a}}{\log_m (a^2)}$ 2

c) Solve $\log_e(x+1) - \log_e x = 2$. Leave your answer in exact form. 2

d) Find $\int 3xe^{4x^2+7}dx$ 1

Question 2

a) Find $\int \dfrac{6x^2}{x^3+4}dx$ 1

b) Differentiate $\tan^3 x$ and hence find $\int \sec^2 x \tan^2 x \, dx$ 2

c) (i) Sketch the curve $y = \log_e 2x$. Show the x intercept 1

(ii) The area between the curve above, $y = 0$ and $y = 1$ is rotated about the y-axis. Find the generated volume in exact form. 3

d) (i) Use a change of base to express $\log_2 5x$ in base e 1

(ii) Hence or otherwise, find $\dfrac{d}{dx}(\log_2 5x)$ 2

Question 3

a) (i) Show that $\sin x - \cos^2 x \sin x = \sin^3 x$ 1

(ii) Hence, and using the substitution $u = \cos x$, or otherwise, find $\int \sin^3 x \, dx$ 2

b) Given the curve represented by $y = \sin^2 x$,

(i) Sketch the curve for $-\pi \le x \le \pi$ 1

(ii) Find the total area between the x-axis and the curve above 3

Question 4

a) The function f is defined as $y = x(x-2)$

(i) Sketch f and state the largest positive domain for which an inverse f^{-1} exists. 2

(ii) Sketch f^{-1}. Show two key points 1

(iii) Find the coordinates of the point where f and f^{-1} intersect 1

b) Explain, without evaluating, why $\sin^{-1}(\sin \frac{3\pi}{4}) \neq \frac{3\pi}{4}$ 1

c) (i) Write the expansion of $\tan(\theta - \alpha)$ 1

(ii) Hence or otherwise, express $\tan\left[\cos^{-1}(-x)\right]$ in terms of x only 2

Question 5

a) Differentiate $y = \tan^{-1}(\sin 2x)$ 2

b) Consider the function $f(x) = \cos^{-1}(x^2)$

(i) Write the domain and range of $y = f(x)$ 2

(ii) Find the slope of the tangent where the curve crosses the y axis. 2

(iii) Sketch the curve $y = f(x)$ 1

c) Use the expansion of $\sin(A+B)$ to express $\sin^{-1}(\frac{4}{5}) + \sin^{-1}(\frac{12}{13})$ in the form $\sin^{-1}$ M. 3

Question 6

a) Find $\int \dfrac{dx}{\sqrt{9-4x^2}}$ 2

b) (i) Find $\dfrac{d}{dx}(x\tan^{-1} x)$ 1

(ii) Hence, and using a suitable rearrangement, evaluate $\int_0^1 \tan^{-1} x \, dx$ 3

c) Using a diagram, or otherwise, evaluate $\int_0^1 \sin^{-1} x \, dx$. Give your answer in exact form. 3

EXT1: SOLUTIONS – S.T.H.S

1) a) $2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$

$\cos x = 0$ or $\frac{1}{2}$ ← (1)

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$ (1) (1)

b) $\dfrac{\frac{1}{2}\log_m a}{2\log_m a} = \dfrac{1}{4}$ ← (1)

D

c) $\log_e\left(\frac{x+1}{x}\right) = 2$ (1)

$\therefore \frac{x+1}{x} = e^2$

$\therefore x + 1 = xe^2$

$\therefore x(1-e^2) = -1$

$\therefore x = \frac{-1}{1-e^2}$ or $\frac{1}{e^2-1}$ (1)

d) $\frac{3}{8}\int 8x e^{4x^2+7}\,dx$

$= \frac{3}{8} e^{4x^2+7} + c$ (1)

2) a) $2\int \frac{3x^2}{x^3+4}\,dx$

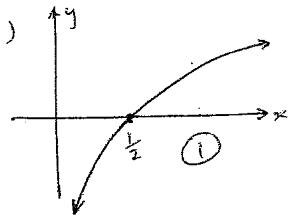
$= 2\log(x^3+4) + c$ (1)

b) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x \sec^2 x$ (1)

$\therefore \int \sec^2 x \tan^2 x\,dx$

$= \frac{1}{3}\tan^3 x + c$ (1)

c) (i)



(ii) $y = \log_e 2x \Rightarrow 2x = e^y$

$x = \frac{1}{2}e^y$

$\therefore Vol = \pi\int_0^1 \left(\frac{1}{2}e^y\right)^2 dy$ (1)

$= \frac{\pi}{4}\int_0^1 e^{2y}\,dy$

$= \frac{\pi}{4}\left[\frac{1}{2}e^{2y}\right]_0^1$ (1)

$= \frac{\pi}{8}\left(e^2 - e^0\right)$

$= \frac{\pi}{8}(e^2-1)\ u^3$ (1)

d) (i) $\log_2 5x = \dfrac{\log_e 5x}{\log_e 2}$ (1)

(ii) deriv. $= \dfrac{\frac{5}{5x}}{\log_e 2}$ (1)

$= \dfrac{1}{x\log_e 2}$ (1)

3) a) i) $\sin x(1-\cos^2 x) = \sin x \sin^2 x$ (1)

$= \sin^3 x$

ii) $\int \sin^3 x\,dx = \int(\sin x - \cos^2 x \sin x)\,dx$

$= \int \sin x(1-\cos^2 x)\,dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

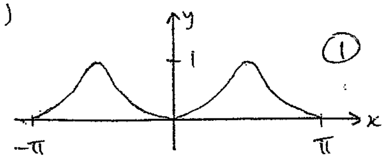
$= \int \cancel{\sin x}(1-u^2)\frac{du}{-\cancel{\sin x}}$ (1)

$= \int (u^2 - 1)\,du$

$= \frac{u^3}{3} - u + c$

$= \frac{\cos^3 x}{3} - \cos x + c$ (1)

b) (i)



(ii) $A = 2\int_0^{\pi} \sin^2 x\,dx$

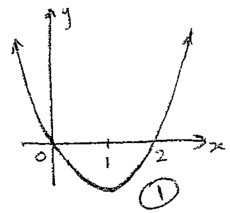
$= 2\int_0^{\pi} \frac{1}{2}(1-\cos 2x)\,dx$ (1)

$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$ (1)

$= (\pi - 0) - (0 - 0)$

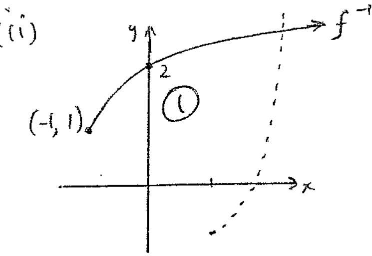
$= \pi\ u^2$ (1)

4) a) (i)



$x \geq 1$ (1)

(ii)



(iii) intersect on $y = x$

$\therefore x(x-2) = x$

$\therefore x^2 - 2x - x = 0$

$\therefore x(x-3) = 0$

$\therefore x = \cancel{0}$ or 3

$\therefore$ intersect at $(3, 3)$ (1)

b) Range of $\sin^{-1} m$ is

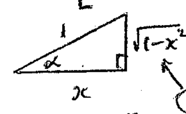
$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (1)

c) (i) $\tan(\theta - \alpha) = \dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$ (1)

(ii)

$\tan[\cos^{-1}(-x)] = \tan(\pi - \cos^{-1}x)$

$= \tan(\pi - \alpha)$



(1)

$\cos\alpha = \frac{x}{1}$

$= \dfrac{\tan\pi - \tan\alpha}{1 + \tan\pi\tan\alpha}$

$= \dfrac{0 - \frac{\sqrt{1-x^2}}{x}}{1 + 0}$

$= -\dfrac{\sqrt{1-x^2}}{x}$ (1)

5

a) $\frac{dy}{dx} = \frac{1}{1+(\sin 2x)^2} \times \cos 2x \times 2$

$= \frac{2\cos 2x}{1+\sin^2 2x}$ ← 1 ← 1

b) (i) $-1 \le x^2 \le 1$

$\therefore 0 \le x^2 \le 1$

$\therefore$ D: $-1 \le x \le 1$ 1

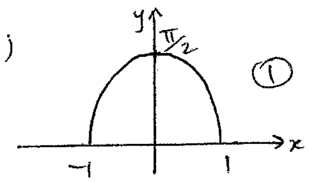
R: $0 \le y \le \frac{\pi}{2}$ 1

(ii) $\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}} \times 2x$ 1

$= \frac{-2x}{\sqrt{1-x^4}}$ 1

When $x=0$, slope of tangent $=0$

(iii)

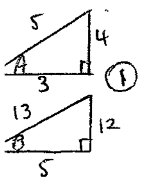


1

c) Let $A = \sin^{-1}\frac{4}{5}$, $B = \sin^{-1}\frac{12}{13}$

$\therefore \sin A = \frac{4}{5}$, $\sin B = \frac{12}{13}$

$\sin(A+B) = \sin A\cos B + \sin B\cos A$



1

$= \frac{4}{5} \times \frac{5}{13} + \frac{12}{13} \times \frac{3}{5}$ 1

$= \frac{20}{65} + \frac{36}{65}$

$= \frac{56}{65}$

$\therefore A+B = \sin^{-1}\left(\frac{56}{65}\right)$ 1

$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

6

a) $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{4}\sqrt{9/4-x^2}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{(\frac{3}{2})^2 - x^2}}$

1 method

1 answer

$= \frac{1}{2}\sin^{-1}\left(\frac{x}{3/2}\right)$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

b) (i)

$\frac{d}{dx}(x\tan^{-1}x) = 1 \times \tan^{-1}x + \frac{1}{1+x^2} \times x$

$= \tan^{-1}x + \frac{x}{1+x^2}$ 1

(ii)

$\int_0^1 \tan^{-1}x = \int_0^1 \frac{d}{dx}(x\tan^{-1}x) - \int_0^1 \frac{x}{1+x^2}$ 1

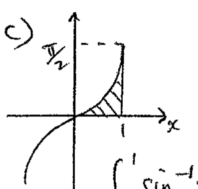
$= \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$ 1

$= \tan^{-1}1 - 0 - \frac{1}{2}(\log 2 - \log 1)$

$= \frac{\pi}{4} - \frac{1}{2}\log 2$ 1

(approx. 0·4)

c)



1

$\int_0^1 \sin^{-1}x\,dx = \left(\frac{\pi}{2} \times 1\right) - \int_0^{\pi/2} (\sin y)\,dy$

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\pi/2}$ 1

$= \frac{\pi}{2} + (\cos\frac{\pi}{2} - \cos 0)$

$= \frac{\pi}{2} + 0 - 1$

$= \frac{\pi}{2} - 1$ 1