J.M.J.Ch.

MARCELLIN COLLEGE RANDWICK

YEAR 11 ACCELERATED

MATHEMATICS

HSC ASSESSMENT TASK # 2

2006

Weighting: 20% of HSC Assessment Mark.

STUDENT NAME: ________________ **MARK:** / 30

RANK ON THIS TASK: /

Time Allowed: **50 minutes.**

Directions:
- Answer all questions on separate lined paper.
- Show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Begin your answers to each new question on a new answer page.

Structure: 2 questions each worth 15 marks – Total 30 marks.

OUTCOMES TO BE ASSESSED:

- H1 – seeks to apply mathematical techniques to problems in a wide range of practical contexts
- H4 – expresses practical problems in mathematical terms based on simple given models
- H5 – applies appropriate techniques from the study of calculus and trigonometry to solve problems
- H6 – uses the derivative to determine the features of the graph of a function
- H7 – uses the features of a graph to deduce information about the derivative
- H8 – uses techniques of integration to calculate areas and volumes

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1

a. If $y = e^x \cos x$, find $\frac{dy}{dx}$. **2**

b.

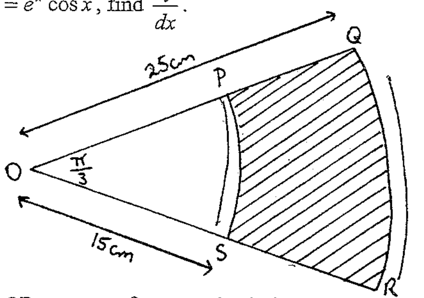


Figure not to scale

PS and QR are arcs of concentric circles with O as the centre. Find the perimeter of the shaded region. Leave your answer in exact form. **2**

c. i. Differentiate $\cos^3 x$ with respect to x. **1**

ii. Hence or otherwise, find $\int \cos^2 x \sin x \; dx$. **2**

d. Solve the equation $2\sin 2x + 1 = 0$ in the domain $0 \le x \le \pi$. **2**

e. If $\frac{dy}{dx} = 3\sin x + \sec^2 x$, find y in terms of x, if $y = 4$ when $x = 0$. **2**

f. i. Sketch the graph of $y = \sin x$ and $y = \cos x$ on the same diagram in the domain $0 \le x \le \frac{\pi}{2}$. **1**

ii. Find the coordinates of their point/s of intersection in this interval. **1**

iii. Determine the area bounded by the curves and the x-axis for $0 \le x \le \frac{\pi}{2}$. **2**

Question 2

a. The rate of decrease of water in a leaking container after t minutes is given by $\frac{dV}{dt} = -40 + 0 \cdot 3t$. Initially, there was 2000 cm^3 in the container. Find how much water remained in the container after 10 minutes. **2**

b. An isotope of carbon, C_{14} decays at a rate proportional to the mass present. The rate of change is given by $\frac{dM}{dt} = -kM$ where k is a positive constant and M is the mass present.

i. Show $M = M_0 e^{-kt}$ is a solution to this equation. **1**

ii. The half-life of this isotope of C_{14} is 5600 years. This means it takes 5600 years for 100 grams of C_{14} to decay to 50 grams. Find the value of k correct to 3 significant figures. **2**

iii. Archaeologists use radiocarbon dating to establish the age of discoveries. Calculate the age of an item in which only one-eighth of the original carbon remains. **2**

c. A particle is moving in a straight line. It starts 2 metres to the right of a fixed point O on the line and at time t seconds its velocity v ms^{-1} is given by $v = 2 - 4e^{-t}$.

i. Find when the particle comes to rest. **1**

ii. Find the distance travelled by the particle before it comes to rest. **2**

d. A defective rocket rises vertically upwards and then crashes back to the ground. The rocket's height above the ground at time t seconds after take-off, is h metres, where $h = 12t^2 - 2t^3$.

i. When does the rocket crash? **1**

ii. What is its velocity at this time? **1**

iii. When is the velocity of the rocket zero? **1**

iv. What is its maximum height? **2**

Question 1 — Peter Simpson

13

A $u = e^x$ $\qquad e^x\cos x + e^x(-\sin x)$

$\frac{du}{dx} = e^x$

$v = \cos x$ $\qquad = e^x(\cos x - \sin x)$ 1

$\frac{dv}{dx} = -\sin x$

B $l = r\theta$ $\qquad l = r\theta$

$= 15\frac{\pi}{3}$ $\qquad = 25\frac{\pi}{3}$

$= 5\pi$

$P = 5\pi + \frac{25\pi}{3} + 10 + 10$

$= \frac{40\pi}{3} + 20$ 2

C I $(\cos x)^3$

$3(\cos x)^2(-\sin x)$

$= -3\sin x(\cos x)^2$ 1

$\cos^3 x + c$ II $\int \cos^3 x$, $\qquad \int \cos^2 x \sin x$

$\cancel{(\cos x)^2}$ $\qquad -3\int \cos^2 x \sin x$

$\cancel{= (\cos x)^3 + c}$ $\qquad -3(\cos x)^3$ X

$-\frac{1}{3}\cos x^3 + c$ see next page *

D $\sin 2x = -\frac{1}{2}$ $\qquad x = 30$

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$, for $0 \le x \le 2\pi$

$x = \frac{7\pi}{12}, \frac{11\pi}{12}$ 2

E $y = 4$ $\quad x = 0$

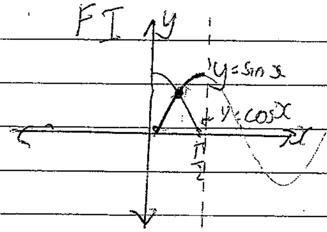
$y = -3\cos x + \tan x + c$

$4 = -3 + c$

$c = 7$

$\therefore y = -3\cos x + \tan x + 7$ 2

F I



1

$\cos x = \sin x$

$\tan x = 1$

$x = \frac{\pi}{4}$

when $x = \frac{\pi}{4}$ $\quad y = \sin\frac{\pi}{4}$

$y = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

$\therefore$ P.O.I $= \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ 1

Question 1 — Peter Simpson

F III $\int_0^{\pi/4} \sin x + \int_{\pi/4}^{\pi/2} \cos x$

$= \left[-\cos x\right]_0^{\pi/4} + \left[\sin x\right]_{\pi/4}^{\pi/2}$

$\left(-\frac{1}{\sqrt{2}} + 1\right) + \left[1 - \frac{1}{\sqrt{2}}\right]$

$= 1 - \frac{\sqrt{2}}{2} + 1 - \frac{\sqrt{2}}{2}$ 2

$= 2 - \frac{2\sqrt{2}}{2}$

$= 2 - \sqrt{2}\ u^2$

* (1) (c) (ii) Since $\frac{d}{dx}\left[\cos^3 x\right] = -3\sin x \cos^2 x$ from (i)

$\therefore \int \frac{d}{dx}\left[\cos^3 x\right] dx = -3\int \sin x \cos^2 x\, dx$

$\therefore -\frac{1}{3}\cos^3 x + c = \int \cos^2 x \cdot \sin x\, dx$.

Question 2 Peter Simpson

13

A $r = -40 + 0.3t$

$T = 0$ $V = 2000$

$V = -40t + \frac{0.3t^2}{2} + C$

$2000 = 0 + 0 + C$

$C = 2000$

$V = -40t + \frac{0.3t^2}{2} + 2000$ 2

$V = -40(10) + \frac{0.3(10)^2}{2} + 2000$

$= -400 + 15 + 2000$

~~1601 5 cm³~~ = ~~1615~~ 1615

B I $\frac{dm}{dt} = -k M_0 e^{-kt}$

as $m = M_0 e^{-kt}$

$\therefore = -kM$ 1

II $50 = 100e^{-k5600}$

$\ln 1/2 = -k \times 5600$

$k = -\frac{\ln(1/2)}{5600}$ 2

~~III~~ $k =$ ~~...~~ 0.000124

III $M = M_0 e^{-kt}$

$\frac{\ln(1/8)}{k} = t$ 2

$t = 16800$ years

C I $T = 0$ $x = 2$

$V = 2 - 4e^{-t}$

$V = 0$

$2 - 4e^{-t} = 0$

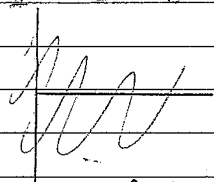
$4e^{-t} = 2$

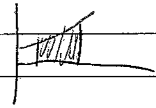
$e^{-t} = \frac{1}{2}$

$t = -\ln(1/2)$ 1

$t = 0.69$

II $V = 2 - 4e^{-t}$





Distance = Area under curve

$a = 4e^{t}$

$\int_0^{-\ln(1/2)}$ ~~4e^t~~ $2 - 4e^{-t}$ $2t + 4e^{t}$

$=$ ~~4e~~ $(4e^{t})_0^{-\ln(1/2)}$

~~4e~~ $40 .$ X

~~$\int^{4} 2 - 4e^{-t}$~~

~~$2t + 4e^{t} + C$~~ $2t + 4e^{t} + C$

$x = 2$ $t = 0$

$c = 2$

$\therefore x =$ ~~2t~~ $2t + 4e^{t} + 2$

area under V curve gives distance

$V = 2 - 4e^{-t}$

$\therefore (2t + 4e^{t}(12))_0^{-\ln(1/2)}$

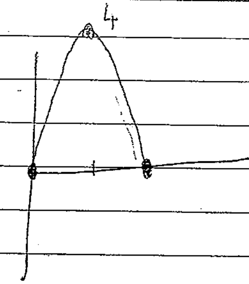
$= 11.386 -$

$= 2\ln 2 + 2 - [0 - 4]$

$= 2\ln 2 - 7$

Question 2 Peter Simpson

D I $h = 12t^2 - 2t^3$



$h = 0$

$12t^2 - 2t^3 = 0$

$t^2(12 - 2t) = 0$

$t = 0$ $t = 6$ 1

II $V = 24t - 6t^2$

$= 24(6) - 6(6)^2$

$= 144 - 216$ 1

$= -72 ms^{-1}$

III $V = 0$

$24t - 6t^2 = 0$

$t(24 - 6t) = 0$

$t = 0$ $t = 4$ 1

IV ~~V=~~ max ~~height~~ height - Turning pt

$v = 0$ 2

$v = 0$ at $t = 4$

when $t = 4$ $h = 64$

$\therefore$ Max height is $h = 64$ at $t = 4$