Marcellin College Randwick

MATHEMATICS EXTENSION II - HSC

ASSESSMENT EVENT #2 - COMPLEX NUMBERS & POLYNOMIALS

TIME ALLOWED: 90 minutes

TOTAL POSSIBLE SCORE: 65 marks

INSTRUCTIONS:

- There are **THREE** questions on this paper
- Attempt all questions
- All working must be shown
- The marks have been allocated for each question
- Marks **may** be deducted for poorly set out and / or untidy solutions

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Question 1 **Marks**

(a) Let $z = 4-3i$ and $w = 2+i$. Find the following in the form $x+iy$.

(i) $z+3w$ 1

(ii) $\dfrac{\overline{z}}{w}$ 1

(b) Let $z = 2+2i$

(i) Express z in the form $r(\cos\vartheta + i\sin\vartheta)$. 2

(ii) Hence express z^{16} in the form $a+ib$ where a and b are real. 2

(c) Solve $z^2 = 5-12i$, giving your answer in the form $x+iy$, where x and y are real. 4

(d) Let $\alpha = -\sqrt{3}+i$ and $\beta = 1-i$.

(i) Express $\overline{\alpha}$ and β in mod-arg form. 2

(ii) Find $\overline{\alpha}\beta$ in mod-arg form. 2

→ (iii) Hence, or otherwise, find the exact value of $\tan\dfrac{11\pi}{12}$. Express your answer in its simplest form. 2

(e) Sketch separately the following loci in an Argand plane and state the Cartesian equations in each case given that:

(i) $|z-3i| = |z-4|$ 2

(ii) $\arg(z+2) = \dfrac{-\pi}{6}$ 2

Question 2 **Marks**

(a) The points O, A, Z and C on the Argand diagram represent the complex numbers 0, 1, z and $z+1$ respectively, where $z = \cos\vartheta + i\sin\vartheta$ is any complex number of modulus 1, with $0 < \vartheta < \pi$.

(i) Explain why $OACZ$ is a rhombus. 2

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary. 2

(iii) Find the modulus and argument of $z+1$ 2

↳ arg.

(b) The complex number z is such that $z \neq i$ and $|z| = 1$, and $w = \dfrac{2+z}{i-z}$.

(i) Find an expression for z in terms of w. 1

(ii) Explain why $|w+1| = |w+2i|$. 1

(iii) Hence, or otherwise, find the locus of w and describe it geometrically. 2

(c) (i) Sketch the locus of z on the Argand diagram where the inequalities $|z-1| \le 3$ and $\operatorname{Im}(z) \ge 3$ hold simultaneously. 3

(ii) Sketch the region defined by $1 \le |z-2+3i| \le 3$ and $\dfrac{\pi}{4} < \arg(z-2+3i) < \dfrac{2\pi}{3}$. 3

concentric

(d) Let α and 1 be zeros of the polynomial $z^5 - 1 = 0$.

(i) Prove that $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$. 2

(ii) The roots of the quadratic equation $x^2 + px + q = 0$ are $\alpha^2 + \alpha^3$ and $\alpha + \alpha^4$. Find the values of p and q given that they are real numbers. 2

(iii) If $\alpha = cis\dfrac{2\pi}{5}$ show that $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$. 3

Question 3 **Marks**

(a) Find all the roots of $P(x) = x^4 - 8x^3 + 39x^2 - 122x + 170$ given that $3-i$ is one of the roots. 4

(b) Let α, β and γ be the non-zero roots of the equation $x^3 + rx + s = 0$.

(i) Find in terms of r, the simplified value of $\alpha^2 + \beta^2 + \gamma^2$. 2

(ii) Find in terms of r and s, the simplified value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. 2

(iii) Find in terms of r and s, the cubic equations (in general form) whose roots are

(A) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. 3

(B) $\alpha+\beta-\gamma, \beta+\gamma-\alpha$ and $\gamma+\alpha-\beta$. 3

(c) Suppose $x^3 + rx + s = 0$ (with r and s being non-zero and real) has a double root. Show that $x = -\dfrac{3s}{2r}$. 4

(d) The polynomial equation $x^5 - ax^2 + b = 0$ has a multiple root. Show that $108a^5 = 3125b^3$. 4

16 Good.

Name:

Teacher's initials: AB

Question: 1

Page 1 of 3

a) i. $4-3i+6+3i$

$= 10.$ ✓

ii. $\frac{4+3i}{2+i} = \frac{(4+3i)(2-i)}{4+1} = \frac{8-4i+6i+3}{5} = \frac{11}{5}+\frac{2}{5}i$ ✓

b). i. $r=\sqrt{4+4}=\sqrt{8}.$

$= 2\sqrt{2}.$ $\qquad \therefore 2\sqrt{2}\ \text{cis}\frac{\pi}{4}.$ ✓

$\arg(z) = \tan^{-1}1$

$= \frac{\pi}{4}.$

ii. $\left(2\sqrt{2}\ \text{cis}\frac{\pi}{4}\right)^{16}.$

$= 65536 \times 256\ \text{cis}\,4\pi.$

$= 16777216\ \text{cis}\,4\pi$ ✓

$= 16777216\ \text{cis}\,0.$

$= 16777216$

c). $x^2-y^2+2xyi = 5-12i.$

$x^2-y^2=5$ $\qquad 2xy=-12$

$y=-\frac{6}{x}.$

$x^2-\frac{36}{x^2}=5.$ $\qquad {}^{1}_{1}X^{-9}_{4}$

$x^4-5x^2-36=0$

$x^2=9,\ -4.$ ✓

$x^2\neq-4\ (x\in R).$

$\therefore x^2=9.$

$x=\pm3.$ $\longrightarrow$ When $x=3,\ y=-2.$

$x=-3,\ y=2.$

$\therefore z=\pm(3-2i)$ good.

Name:

Teacher's initials: AB

Question: 1

Page 2 of 3

d) i. $\bar{\alpha} = -\sqrt{3}-i.$ $\qquad$ Silly

$\therefore r=\sqrt{3+1}$

$= 2.$ $\qquad$ ~~α~~$= 2\ \text{cis}\left(-\frac{5\pi}{6}\right).$

$\arg(z) = \tan^{-1}\frac{1}{\sqrt{3}}.$

$= \frac{\pi}{6}.$

3^{rd} quad. $\therefore \arg(z) = -\frac{5\pi}{6}$ ✓ $\qquad -1$

$\beta = 1-i.$

$r=1.$

$\arg(z) = \tan^{-1}1$ $\qquad \therefore \beta = \text{cis}\left(-\frac{\pi}{4}\right).$

$= \frac{\pi}{4}.$

But, 4^{th} quad $\therefore \arg(z) = -\frac{\pi}{4}$

ii. $\bar{\alpha}\beta = 2.1\left(\text{cis}\left(-\frac{5\pi}{6}+\left(-\frac{\pi}{4}\right)\right)\right).$

$= 2\ \text{cis}\left(\frac{-10\pi-3\pi}{12}\right)$ $\qquad$ M

$= 2\ \text{cis}\left(-\frac{13\pi}{12}\right). = 2\ \text{cis}\left(\frac{11\pi}{12}\right).$

~~$= 2\ \text{cis}\left(-\frac{\pi}{12}\right).$~~

iii. $\bar{\alpha}\beta = (-\sqrt{3}-i)(1-i).$

$= -\sqrt{3}+\sqrt{3}i-i+i^2.$

$= (-\sqrt{3}-1)+(2i).$ $\qquad = 2\left[\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right].$

$= \frac{-\sqrt{3}-1}{2}+i$ $\qquad = \cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}.$

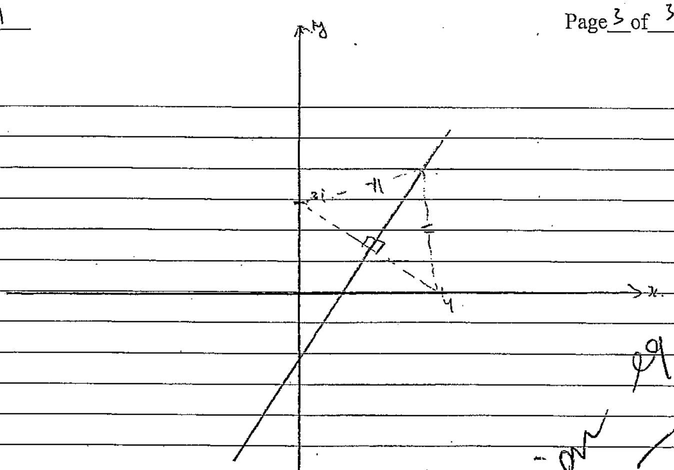
$\therefore \frac{\sin\frac{11\pi}{12}}{\cos\frac{11\pi}{12}} = \frac{1}{\frac{-\sqrt{3}-1}{2}}$ $\qquad$ M

$= \frac{2}{-\sqrt{3}-1}$ $\qquad -1$

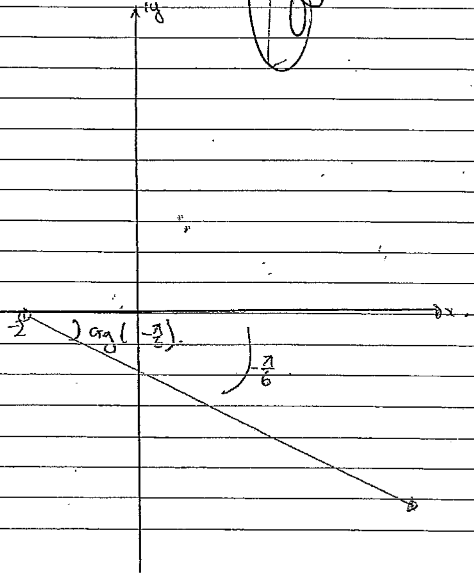
Name: __ Teacher's initials: AB

Question: 1 Page 3 of 3

e) i.



ii)



Question on gradient.

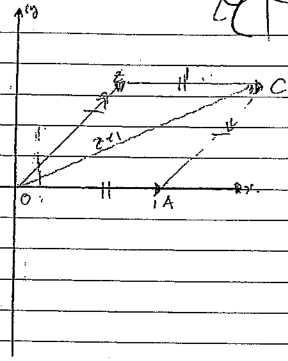
18

Excellent

Name: _ Teacher's initials: AB

Question: 2 Page 1 of 5

a) i.



From the diagram,
$|OZ| = |AC|$.
$|OA| = |ZC|$.
$\angle ZOA = \angle ACZ$.

∴ This is a rhombus.

M

ii. ~~$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 1) + i\sin\theta}$~~

~~$= \frac{(\cos\theta - 1)(\cos\theta + 1) - i^2\sin^2\theta}{(\cos\theta + 1)^2 + \sin^2\theta}$.~~

~~$= \frac{\cos^2\theta - 1 + \sin^2\theta}{(\cos\theta + 1)^2 + \sin^2\theta}$.~~

(Answer on attached sheet).

iii. $x + iy + 1$
$= (x + 1) + iy$.

~~$2 + 2\cos\theta$~~

$|z| = \sqrt{(x+1)^2 + y^2}$.

$\arg(z) = \cancel{\tan^{-1}} \frac{y}{x+1}$.

Question 2.

i. $\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 1) + i\sin\theta}$

$= \frac{\quad}{(\cos\theta + 1)^2 - i^2\sin^2\theta}$

$= \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 1) + i\sin\theta}$

$= \frac{\{(\cos\theta - 1) + i\sin\theta\}\{(\cos\theta + 1) - i\sin\theta)}{(\cos\theta + 1)^2 - i^2\sin^2\theta}$

$= \frac{\cos^2\theta - 1 - i\cos\theta\sin\theta + i\sin\theta\cos\theta + i\sin\theta + \sin\theta}{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$

$= \frac{1 - 1 + i\sin\theta}{2 + 2\cos\theta}$

$= \frac{i\sin\theta}{2 + 2\cos\theta}$ ∴ This is purely imaginery

✓ 2

use property of rhombus

Name: _ ______ Teacher's initials: AB

Question: 2

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b). i. $w = \frac{(2+z)(-z-i)}{(i-z)(-z-i)}$. $z = \ldots$

Because $|z| = 1$, $\sqrt{x^2+y^2} = 1$.

$x^2+y^2 = 1$.

$iw - wz = 2+z$. $\therefore z = \frac{iw-2}{w+1}$ ✓

$iw - 2 = wz+z$

$iw - 2 = z(w+1)$

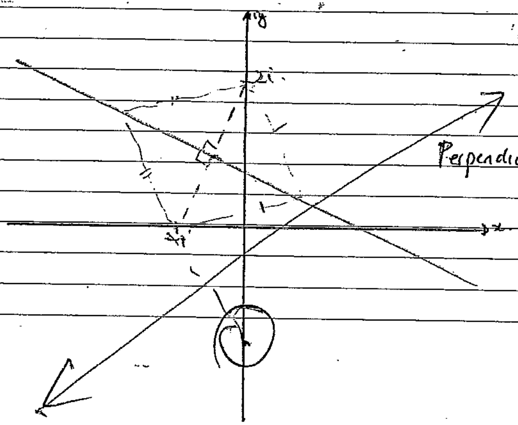
ii. $|z| = 1$.

$\therefore |iw-2| = 1$. (or α)

$|w+1| = 1$ (or α).

★ $|iw-2| = |w+2i|$. ✓ 2

iii. $|w+1| = |w+2i)$



Perpendicular Bisector.

M

be careful

Name: _

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Question: 2

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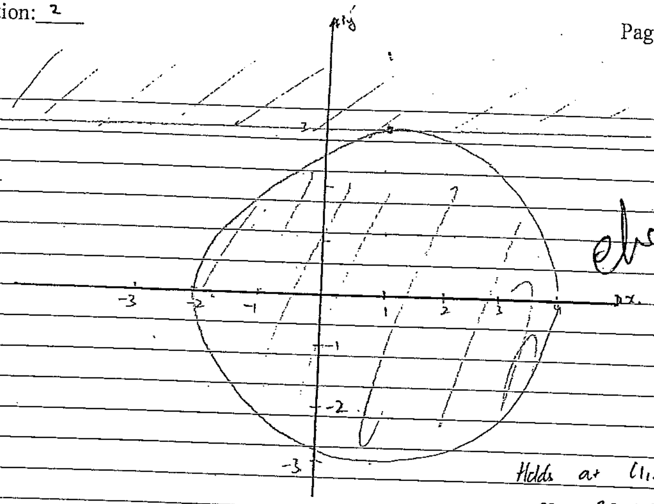
c) i.

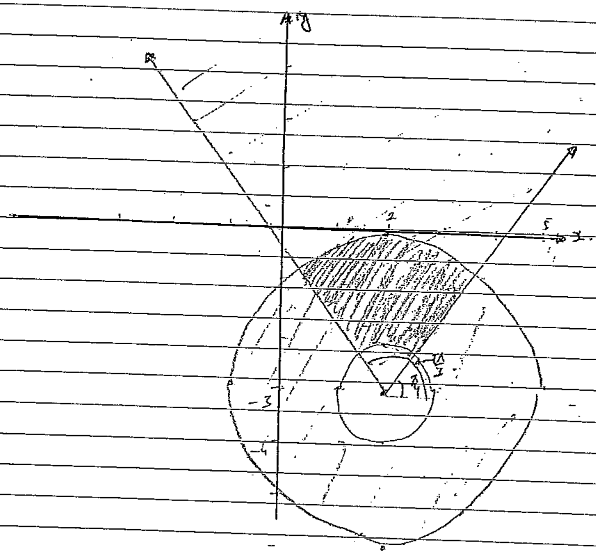


check regions

2

Holds at (1,2)
or $z = 1+2i$.

ii.



3

Name: _

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Question: 2

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d) i. $z^5 - 1 = 0$.

$(z-1)(z^4+z^3+z^2+z^1+1) = 0$.

$\therefore z = 1 \quad z^4+z^3+z^2+z+1 = 0$.

α o ~~one~~ of the other root.

$\therefore \alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0$.

2

ii. $\Sigma\alpha = \alpha^2+\alpha^3+\alpha+\alpha^4 = -p$. $\quad z^5 = 1$.

$\Sigma\alpha\beta = (\alpha^2+\alpha^3)(\alpha+\alpha^4) = q$.

$\alpha + \alpha^2+\alpha^3+\alpha^4 = -1$.

$\therefore -1 = -p$.

$\therefore p = 1$.

$\alpha^3+\alpha^6+\alpha^4+\alpha^7 = q$.

$\alpha^3+\alpha+\alpha^4+\alpha^2 = q$.

$\alpha+\alpha^2+\alpha^3+\alpha^4 = -1$.

$\therefore q = -1$.

$\therefore p = 1$
$q = -1$.

good

2

iii. $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

~~$\cos\frac{8\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} + 1 = 0$.~~

Back Page.

d). $\cos\frac{2\pi}{5} = 0.3091016994.$

$\frac{-1+\sqrt{5}}{4} = -\frac{1}{4}+\frac{\sqrt{5}}{4} = 0.3091016994. = \cos\frac{2\pi}{5}.$

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}.$

$(z-1)(z^4+z^3+z^2+z^1+1)=0.$

$\alpha+\alpha^2+\alpha^3+\alpha^4=-p \qquad (\alpha^2+\alpha^3)(\alpha+\alpha^4)=q.$

$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=-1.$

$=\cos(\pi+2k\pi).$

$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\left(\frac{-4\pi}{5}\right)+\cos\left(-\frac{2\pi}{5}\right).$

$=2\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\left(\frac{-4\pi}{5}\right)+i\sin\left(-\frac{4\pi}{5}\right)$

$=2\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$ ~~cos~~ $= \frac{-1+\sqrt{5}}{2}.$

$\therefore \cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}.$

✓ ✓ ✓

3

not a simple | elegant proof

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16 Excellent

Name: ___ Teacher's initials: AB

Question: 3. Page 1 of 4

a). $p(x) = x^4-8x^3+39x^2-122x+170.$

Real coefficient polys mean complex conjugate roots.

$\therefore (3+i)$ is the other one.

$\therefore 3-i,\ 3+i,\ \alpha,\ \beta.$

$(x-3+i)(x-3-i)$

$=(x-3)^2-(i)^2.$

$=x^2-6x+9+1.$

$=x^2-6x+10.$

$\therefore x^2-6x+10 \,\big)\, x^4-8x^3+39x^2-122x+170$ quotient $x^2-2x+17$

$x^4-6x^3+10x^2$ −

$-2x^3+29x^2-122x+170$

$-2x^3+12x^2-20x$ −

$17x^2-102x+170.$

$17x^2-102x+170$ −

$0.$

$\therefore (x^2-6x+10)(x^2-2x+17).$

Find roots of $x^2-2x+17$:

$x=\frac{1\pm\sqrt{1-17}}{1}$

$=1\pm\sqrt{-16}$

$=1\pm4i.$

$\therefore$ The roots are: $3-i,\ 3+i,\ 1+4i,\ 1-4i.$

✓ good ✓ 4

Name: __ ___ Teacher's initials: AB

Question: 3 Page 2 of 9

b) i. $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$.

$\alpha+\beta+\gamma = -\frac{b}{a}$. $\therefore$ $-2r$.

$= 0$.

$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$.

$= \frac{r}{1} = r$. ✓ 2

ii. $\frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{(\alpha\beta\gamma)^2}$.

$\alpha\beta+\alpha\gamma+\beta\gamma$

$\alpha\beta+\alpha\gamma+\beta\gamma \,\big|\, \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

$\alpha^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma$

$\cancel{\alpha^2\gamma^2} + \beta^2\gamma^2 - \alpha^2\beta\gamma - \alpha\beta^2\gamma$

$\cancel{\alpha^2\gamma^2} + \alpha^2\gamma\beta + \alpha\beta\gamma^2$

$\beta^2\gamma^2 - 2\alpha^2\beta\gamma - \alpha\beta^2\gamma - \alpha\gamma^2\beta$

$\beta^2\gamma^2 + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$-2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\gamma^2\beta$. M

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$.

$\Sigma\alpha = -\frac{b}{a} = 0$. $-2(-s)r$.

$\Sigma\alpha\beta = \frac{c}{a} = r$. $= 2rs$.

$\alpha\beta\gamma = -\frac{d}{a} = -s$.

~~iii. A). $ax^3+bx^2+cx+d=0$.~~

~~$x^3+\frac{b}{a}x^2+\frac{c}{a}x+d=0$.~~

~~$\Sigma\alpha = -\frac{b}{a}$.~~

~~$\therefore \frac{b}{a} = -\Sigma\alpha$.~~

~~$= -$~~

BACK PAGE

A) Let α.

$\alpha = \frac{1}{x}$.

$\left(\frac{1}{x}\right)^3 + \frac{1}{x}r + s = 0$.

$\frac{1}{x^3} + \frac{r}{x} + s = 0$ $(\times)\ x^3$. 3

$1 + rx^2 + sx^3 = 0$. $\therefore$ $1+rx^2+sx^3=0$.

B). $\Sigma\alpha = 0$. $x^3 - \Sigma\alpha x + \Sigma\alpha\beta - \alpha\beta\gamma$.

Let $x = \alpha+\beta-\gamma$. $\Sigma\alpha = 0$. $\therefore$ $x^3 + \Sigma\alpha\beta x - \alpha\beta\gamma$.

~~$\alpha\beta$~~ $\gamma = \alpha+\beta+x$. ✗

$\alpha\beta+\alpha\gamma+\beta\gamma =$

Do not work on back of page!

Name: _ ___ Teacher's initials: AB

Question: 3 . Page 3 of 4.

c). $P(x) = P'(x) . = 0.$

$P(x) = x^3 + rx + s.$

$P'(x) = 3x^2 + r . = 0.$

$r = -3x^2.$

$3x^2 = -r.$

$x^2 = -\frac{r}{3}.$

$x^3 - 3x^3 + s = 0.$

$-2x^3 + s = 0.$

$s = 2x^3.$

$s = 2x . \left(-\frac{r}{3}\right).$

$-3s = 2rx.$

$x = \frac{-3s}{2r}.$

Excellent

4

$P\left(\frac{-3s}{2r}\right) = P'\left(-\frac{3s}{2r}\right) = 0.$

$\therefore -\frac{27s^3}{8r^3} - \frac{3s}{2} + s = 0.$ $-27s^3 - 12r^3s + s = 0.$

d). $P(x) = x^5 - ax^2 + b = 0.$

$P'(x) = 5x^4 - 2ax$

$P''(x) = 20x^3 - 2a = 0.$

$20x^3 = 2a.$

$a = 10x^3.$ $x^3 = \frac{a}{10}.$

~~$P'(x)$~~

~~$P'(10x^2)$~~ $= 5x^4 - 20x^4$

$P(x) = x^5 - 10x^5 + b = 0.$

$-9x^5 + b = 0.$

$\therefore b = 9x^5.$

ON THE ATTACHED SHEET

$\therefore b = 9x^2 . \left(\frac{a}{10}\right).$

$10b = 9x^2 . a.$

Question 3 d). 4 of 4

$x^5 - ax^2 + b$

$P(x) = x^5 - ax^2 + b.$

$P'(x) = 5x^4 - 2ax = x(5x^3 - 2a) = 0.$ $5x^3 - 2a = 0.$

$P''(x) = 20x^3 - 2a = 0.$ $5x^3 = 2a.$

$x^3 = \frac{a}{10}.$ $x^3 = \frac{2a}{5}.$

$x = \sqrt[3]{\frac{a}{10}}.$

$\therefore$ ~~$\left(\frac{a}{10}\right)^{\frac{5}{3}}$~~ $x^2(x^3 - a) + b = 0.$

$x^2\left(\frac{a}{10} - a\right) + b = 0.$

$x^2\left(\frac{a - 10a}{10}\right) + b = 0.$

$x^2(-9a) + 10b = 0.$

$x^2 = \frac{10b}{9a}$

$\therefore 5(x^4) - 2ax = 0$ $x = \frac{\sqrt{10b}}{3\sqrt{a}}$

4

$\therefore 5\left(\frac{100b^2}{81a^2}\right) - 2ax = 0.$

$500b^2 - 162a^3x = 0.$

$500b^2 - 162a^3 . \frac{\sqrt{10b}}{3\sqrt{a}} = 0.$

$500b^2 - 54\frac{a^3}{a^{\frac{1}{2}}} . \sqrt{10b} = 0.$

Excellent

~~$500b^2$~~

~~$125b^2$~~

$250b^2 = 27a^{\frac{5}{2}}\sqrt{10b}.$

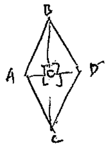
$62500b^4 = 7290a^5b.$

$62500b^3 = 7290a^5.$

$6250b^3 = 729a^5$

$\therefore 3125b^3 = 108a^5..$

- Diagonals are axes of symmetry
- Diagonals have ~~th~~ same midpoint and perpendicular



$|BO| = |OC|$
$|AO| = |OD|$

$\therefore$ Diagonals have the same midpoint of O.

$$\overline{\alpha^2\beta^2\gamma^2}$$

$$= \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{(\alpha\beta\gamma)^2}.$$

$$\begin{array}{r|l} & \alpha\beta + \alpha\gamma + \beta\gamma \\ \hline \alpha\beta + \alpha\gamma + \beta\gamma & \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2. \\ & \alpha^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma. \quad - \\ \hline & \alpha^2\gamma^2 + \beta^2\gamma^2 - \alpha^2\beta\gamma - \alpha\beta^2\gamma. \\ & \alpha\gamma^2 \quad + \alpha^2\beta\gamma \quad + \alpha^2\beta\gamma^2 \quad - \\ \hline & \beta^2\gamma^2 - 2\alpha^2\beta\gamma - \alpha\beta^2\gamma - \alpha\beta\gamma^2. \\ & \beta\gamma^2 \quad + \alpha\beta^2\gamma \quad + \alpha\beta\gamma^2 \quad - \\ \hline & -2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2 \end{array}$$

$$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma).$$

$$\frac{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha).}{(\alpha\beta\gamma)^2.}$$

~~[illegible] $- 2s($[illegible]$)$~~

$\Sigma\alpha = 0.$
$\Sigma\alpha\beta = r.$
$\alpha\beta\gamma = -s.$

$$\therefore \frac{r^2 + 2s(0)}{s^2} = \frac{r^2}{s^2}.$$