J.M.J.CH.

MARCELLIN COLLEGE RANDWICK

HSC ASSESSMENT TASK # 1

MATHEMATICS EXTENSION I

2009

Weighting: 30% (HSC Assessment Mark)

NAME:

MARK: / 60

PERCENTAGE: %

RANK ON THIS TASK:

Time Allowed: 90 minutes

Directions:

- There are **FIVE** questions on this paper
- Marks have been allocated for each question
- Answer each question on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work
- The *Standard Integrals* have been attached as the final page

Question One *Start a new page* 9/12

a. A is the point (-4, 2) and B is the point (3, -1). Find the coordinates of the point P which divides the interval AB externally in the ratio 2:1. 2

b. Find the acute angle between the lines $3x - y - 2 = 0$ and $x + 2y - 3 = 0$. Give the answer correct to the nearest degree. 2 (-1)

c. Solve the equation $1 + \cos 2x = \sin 2x$ for $-2\pi \le x \le 2\pi$. 4 (-2)

d. Differentiate $x\tan^{-1}\frac{x}{2}$. 2

e. It is known that two of the roots of the equation $2x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other. Find the value of k. 2

Question Two *Start a new page* 10/12

a. Use the substitution $u = x^2$ to find $\int \frac{x}{\sqrt{1-x^2}}\,dx$. 3 (-2)

b. Consider the function $f(x) = 3\tan^{-1}x$. 4

i. State the range of the function $y = f(x)$.

ii. Sketch the graph of $y = f(x)$.

iii. Find the gradient of the tangent to the curve $y = f(x)$ at $x = \frac{1}{\sqrt{3}}$.

c. Evaluate $\sin^{-1}(\cos\frac{2\pi}{3})$. 2

d. i. If $t = \tan\frac{\theta}{2}$, show that $3\cos x + 4\sin x + 5 = \frac{2t^2 + 8t + 8}{1 + t^2}$. 1

ii. Hence, or otherwise, solve the equation $3\cos x + 4\sin x + 5 = 0$ for $0° \le x \le 360°$ correct to the nearest degree. 2

Question Three *Start a new page* 11/12

a. Use the method of Mathematical Induction to show $9^n - 1$ is divisible by eight for all positive integer values of n. **3**

b. Differentiate $\cos^{-1}(\sin x)$. **2** (-1)

c. Find:

i. $\int \frac{2}{\sqrt{1-9x^2 dx}}$ **2**

ii. $\int \frac{3}{4+2x^2} dx$ **2**

d. Evaluate $\sin\left[\cos^{-1}\left(\frac{2}{3}\right)+\tan^{-1}\left(-\frac{3}{4}\right)\right]$. **3**

Question Four *Start a new page* 12/12

a. At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = 1+\cos^2 x$. If $y = \pi$ when $x = \pi$, find the value of y when $x = 2\pi$. **3**

b. The polynomial $P(x) = x^3 - 6x^2 + kx + 14$ has a zero at $x = -2$.

i. Find the value of k. **1**

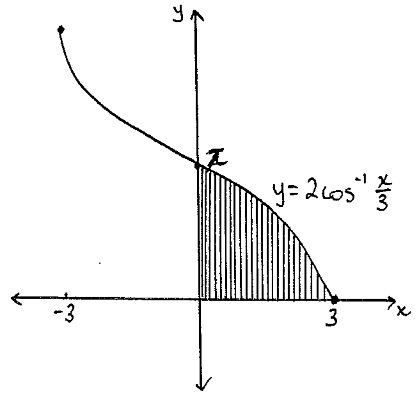
ii. Express $P(x)$ as a product of linear factors. **2**

iii. By sketching the graph of $y = P(x)$, hence, or otherwise, solve $P(x) < 0$. **2**

c. The portion of the curve $y = \sin x + \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. Show that the volume of the solid of revolution generated is $\frac{\pi}{2}(\pi+2)u^3$. **4**

Question Five *Start a new page* 12/12

a.



The sketch shows the graph of the curve $y = f(x)$ where $f(x) = 2\cos^{-1}\frac{x}{3}$.
The area under the curve for $0 \le x \le 3$ is shaded.

i. Find the y intercept. **1**

ii. Determine the inverse function $y = f^{-1}(x)$, and write down the domain D of this inverse function. **2**

iii. Calculate the area of the shaded region. **2**

b. $P(2p, p^2)$ is a point on the parabola $x^2 = 4y$. At P a tangent PT is drawn. Parallel to the tangent PT a straight line is drawn through the origin O, cutting the parabola again at B.

i. Draw a diagram, marking on it the above information. **1**

ii. By first finding the equation of the line OB, show that B has coordinates $(4p, 4p^2)$. **2**

iii. Determine the coordinates of the point M, the midpoint of PB. **1**

iv. Find the equation of the locus of M as P moves on the parabola. $x^2 = 4y$. **3**

Kenneth Schokrele 1/2

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Question One

a) ~~A(−4, 2)~~ B(3, −1)

$(-4, 2)$ $(3, -1)$

$2 : -1$

$x = \frac{2(3)+4}{1}$ $y = \frac{-2-2}{1}$

$= 6+4$ $= -4$

$= 10$

$\therefore (10, -4)$

$x = \frac{2(3)+4}{1}$ $y = \frac{-2-2}{1}$

$= 10$ $= -4$

$\therefore (10, -4)$

b) ~~$-y = -3x+2$, $y = 3x-2$, $m_1 = 3$; $2y = -x+3$, $m = -\frac{1}{2}$~~

~~$\tan\theta = \left|\frac{m_1+m_2}{1}\right|$~~

~~$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{3+\frac{1}{2}}{1-\frac{3}{2}}\right|$~~

~~$\theta = 54°29'$~~

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{3+\frac{1}{2}}{1-\frac{3}{2}}\right|$

$= \left|\frac{7}{2} \div -\frac{1}{2}\right| = |7| = 7.$

c) $1 + \cos^2 x - \sin^2 x = 2\sin x\cos x$

$1 = \sin^2 x + \cos^2 x$

$1 - \sin^2 x = \cos^2 x$

$\cancel{2}\cos^2 x = \cancel{2}\sin x\cos x$

~~$\cos^2 x - \sin x\cos x = 0$~~

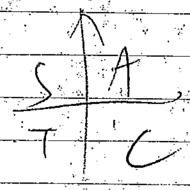
~~$\cos x(\cos x - \sin x) = 0$~~

$\cos x = \sin x$

$\therefore \tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{7\pi}{4}$

and



2

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

m_1: $-y = -3x+2$

$y = 3x+2$

$\therefore m = 3$

m_2: $2y = -x+3$

$y = -\frac{1}{2}x + 3$

$m_2 = -\frac{1}{2}$

$\therefore \left|\frac{3+\frac{1}{2}}{1-\frac{3}{2}}\right| = \left|\frac{\frac{7}{2}}{-\frac{1}{2}}\right| = |-7| = 7$

$\therefore \tan\theta = 7$

$\theta = 81°52'$

1

2/2

d) $y = x\tan^{-1}\frac{x}{2}$

$\frac{dy}{dx} = \tan^{-1}\frac{x}{2} + x \cdot \frac{1}{1+\frac{x^2}{4}} \cdot \frac{1}{2}$

$= \tan^{-1}\frac{x}{2} + \frac{x}{2\left(\frac{4+x^2}{4}\right)}$

$= \tan^{-1}\frac{x}{2} + \frac{2x}{4+x^2}$

e) ~~$\alpha \mid \beta \mid \gamma$~~

$\alpha \mid \beta \mid \gamma$

$\alpha \mid \beta \mid \frac{1}{\beta}$

$\Sigma\alpha = -\frac{1}{2} = \alpha + \beta + \frac{1}{\beta}$

$\Sigma\alpha\beta = -\frac{k}{2}$

$\alpha\beta\gamma = \alpha = -3$

$-\frac{1}{2} = -3 + \frac{\beta^2+1}{\beta}$

$\frac{5}{2} = \frac{\beta^2+1}{\beta}$

$5\beta = 2\beta^2 + 2$

$2\beta^2 - 5\beta + 2 = 0$

$\therefore \beta = 2, \frac{1}{2}$

$\therefore$ Roots: $2, \frac{1}{2}, -3$

$\alpha\beta + \beta\gamma + \alpha\gamma = -\frac{k}{2}$

$1 - 6 - \frac{3}{2} = -\frac{k}{2}$

$-5 - \frac{3}{2} = -\frac{k}{2}$

$\frac{k}{2} = \frac{3}{2} + 5$

$k = 3 + 10$

$= 13$

Keirck Jehobvai 1/2

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Question 2

a) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ $u = x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

$= \int \frac{x}{\sqrt{1-u}} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int (1-u)^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u}}\,du$

$= \frac{1}{2}\int \frac{1}{\sqrt{(1)^2-(\sqrt{u})^2}}\,du$

$= \frac{1}{2}\sin\sqrt{u}\cdot\frac{1}{\frac{1}{2}u^{-\frac{1}{2}}} + C = \sqrt{u}\sin^{-1}\sqrt{u} + c$

But, $u = x^2$: $= x\sin^{-1}x + c$

~~But, $u = x^2$~~

~~$\therefore = \frac{1}{2}\sin^{-1}x^2 + C$~~

$\int \frac{x}{\sqrt{1-u}}\,dx$ $\frac{du}{dx} = 2x$, $x = \sqrt{u}$, $dx = \frac{du}{2x}$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-(\sqrt{u})^2}}\,du$

$= \frac{1}{2}\left[\frac{\sin^{-1}\sqrt{u}}{(\sqrt{u})'}\right] + C$

~~$= \frac{1}{2}\sin^{-1}x + C$~~

$= \frac{1}{2}\sin^{-1}\sqrt{u} \cdot \frac{1}{\frac{1}{2}u^{-\frac{1}{2}}} + C$

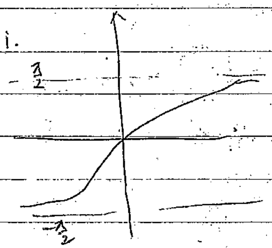
$= \frac{1}{2}\sin^{-1}\sqrt{u} \times 2\sqrt{u}$

$= \sqrt{u}\sin^{-1}\sqrt{u} + c$

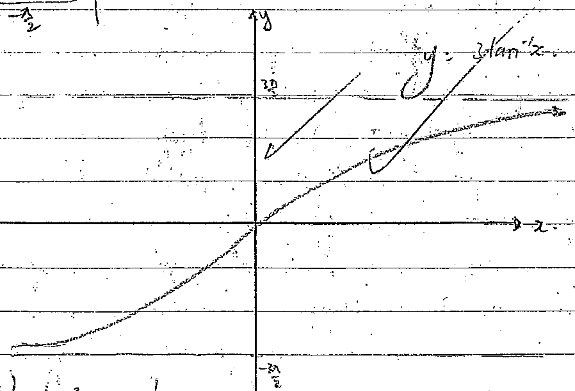
But, $u = x^2$

$\therefore = x\sin^{-1}x + c$

b) i. Range: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



ii.



$y = 3\tan^{-1}x$

iii. ~~$y = f$~~

$f(x) = 3\tan^{-1}(x)$

$f'(x) = 3 \cdot \frac{1}{1+x^2}$

$= \frac{3}{1+x^2}$

At $x = \frac{1}{\sqrt{3}}$,

$f'(\frac{1}{\sqrt{3}}) = \frac{3}{1+\frac{1}{3}}$

$= \frac{3}{\frac{4}{3}}$

$= 3 \times \frac{3}{4} = \frac{9}{4}$

$\therefore$ At $x = \frac{1}{\sqrt{3}}$, gradient is $\frac{9}{4}$.

$\frac{3}{1+x^2}$

$= \frac{3}{1+x^2}$

$f'(x) = \frac{3}{1+\frac{1}{3}} = 3 \times \frac{3}{4} = \frac{9}{4}$

Keirck Jehobvai 2/2

c) $\sin^{-1}(\cos 120°)$

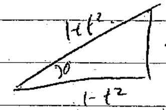
$= \sin^{-1}(-\cos 60°)$

$= \sin^{-1}\left(-\frac{1}{2}\right)$

$= -\frac{\pi}{6}$

d) i. If $\tan\frac{\theta}{2} = t$,

then $\tan\theta = \frac{2t}{1-t^2}$



$4t^2+$

$\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$= 3\left(\frac{1-t^2}{1+t^2}\right) + \frac{8t}{1+t^2} + 5$

$= \frac{3 - 3t^2 + 8t + 5 + 5t^2}{1+t^2}$

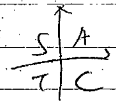
$= \frac{2t^2 + 8t + 8}{1+t^2}$

ii. $\frac{2t^2 + 8t + 8}{1+t^2} = 0$.   But, $t = \tan\frac{\theta}{2}$.

$\frac{t^2 + 4t + 4}{1+t^2} = 0$ $\therefore \tan\frac{\theta}{2} = -2$,

$\therefore (t+2)^2 = 0$. $\frac{\theta}{2} = 116°34'$.

$\therefore t = -2$. $\theta = 233°8'$.



Question 3

a). Prove true for $n=1$.

$9^1 - 1 = 8 \rightarrow$ divisible by 8.

$\therefore$ It is true $n=1$.

Assume true for $n=k$,

$9^k - 1 = 8M$,

Prove true for $n=k+1$.

$9^{k+1} - 1 = 8M$ $\qquad 9^k = 8M+1$.

$9 \cdot 9^k - 1 = 8M$

LHS $= 9(8M+1) - 1 =$

$= 72M + 9 - 1$.

$= 72M + 8$.

$= 8(9M+1) \rightarrow$ divisible by 8.

Since $n=1$, and $n=1+1=2$ is true, by the principle of mathematical induction the statement is true for all integers of n.

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3 b). $\frac{d}{dx} \cos^{-1} u$. Let $u = \sin x$ $\qquad \cos^{-1}(\sin x)$

$= \frac{1}{\sqrt{1-u^2}} \cdot (u)'$. $\qquad \frac{d}{dx} = \frac{1}{\sqrt{1-\sin^2 x}} \cos x$

$= -\frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x$. $\qquad = -\frac{\cos x}{\sqrt{1-\sin^2 x}}$

$= -\frac{\cos x}{\sqrt{1-\sin^2 x}}$.

c). i. $2\int \frac{1}{\sqrt{(1)^2 - (3x)^2}}\,dx$. $\qquad 2\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$

$= 2\left[\frac{\sin^{-1} 3x}{3}\right] + C$ $\qquad = \frac{2 \sin^{-1} 3x}{3} + C$

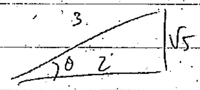
$= \frac{2}{3} \sin^{-1} 3x + C$.

ii. $3\int \frac{1}{2^2 + (\sqrt{2}x)^2}\,dx$, $\qquad 3\int \frac{1}{(2)^2 + (\sqrt{2}x)^2}$ $\qquad = 3 \cdot \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{2}$

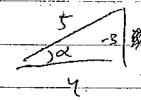
$= 3\left[\frac{1}{2\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{2}\right] + C$ $\qquad = \frac{3\sqrt{2}}{4} \tan^{-1} \frac{\sqrt{2}x}{2} + C$

$= \frac{3\sqrt{2}}{4} \tan^{-1} \frac{\sqrt{2}x}{2} + C$

d). ~~sin~~ $\cos^{-1}\frac{2}{3}$



$\tan^{-1}\left(-\frac{3}{4}\right)$



Let $\theta = \cos^{-1}\frac{2}{3}$. $\alpha = \tan^{-1}\left(-\frac{3}{4}\right)$.

(With the assumption that the range is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$)

$\sin[\theta + \alpha]$.

$= \sin\theta\cos\alpha + \cos\theta\sin\alpha$.

$= \frac{\sqrt{5}}{3} \cdot \frac{4}{5} + \frac{2}{3} \times -\frac{3}{5}$

$= \frac{4\sqrt{5}}{15} + \frac{-6}{15} = \frac{4\sqrt{5} - 6}{15}$

(12)

Question 4.

a). $\frac{dy}{dx} = 1 + \frac{1+\cos 2x}{2}$

$= \frac{3}{2} + \frac{\cos 2x}{2}$

$\int \frac{3}{2} + \frac{\cos 2x}{2} \, dx$

$= \frac{3}{2}x + \frac{\sin 2x}{4} + c$

~~When x = 2π~~

When $x = \pi$, $y = \pi$.

$\therefore \pi = \frac{3}{2}\pi + \frac{\sin 2\pi}{4} + c$

$\pi = \frac{3\pi}{2} + c$

$\therefore c = -\frac{\pi}{2}$

$\therefore y = \frac{3}{2}x + \frac{\sin 2x}{4} - \frac{\pi}{2}$

When $x = 2\pi$,

$y = 3\pi - \frac{\pi}{2}$

$= \frac{6\pi - \pi}{2} = \frac{5\pi}{2}$

3

~~$\int 1+\cos 2x$~~

~~$\int 1+\cos^2 x \, dx$~~

$\frac{dy}{dx} = 1 + \frac{1+\cos 2x}{2}$

$= \frac{3}{2} + \frac{\cos 2x}{2}$

$\therefore \int \frac{3}{2} + \frac{\cos 2x}{2} \, dx$

$y = \frac{3}{2}x + \frac{\sin 2x}{4} + c.$

At $x = \pi$, $y = \pi$.

$\pi = \frac{3}{2}\pi + 0 + c.$

$c = -\frac{\pi}{2}.$

$\therefore y = \frac{3}{2}x + \frac{\sin 2x}{4} - \frac{\pi}{2}.$

At $x = 2\pi$,

$y = 3\pi - \frac{\pi}{2}$

$= \frac{5\pi}{2}.$

b). i. $P(-2) = -8 - 6(4) - 2k + 14 = 0.$

$= -8 - 24 - 2k + 14$

$= -32 + 14 - 2k$

$= -18 - 2k = 0.$

~~$2k = -18$~~

~~$k =$~~

$-2k = 18$

$k = -9.$

$P(-2) = -8 - 24 - 2k + 14$

$0 = -32 + 14 - 2k$

$-18 = -2k.$

$k = -9.$

1

ii. $P(x) = x^3 - 6x^2 - 9x + 14.$

$$
\begin{array}{r|l}
 & x^2 - 8x + 7 \\
x+2 & x^3 - 6x^2 - 9x + 14 \\
 & x^3 + 2x^2 \\
 & -8x^2 - 9x + 14 \\
 & -8x^2 - 16x \\
 & 7x + 14 \\
 & 7x + 14 \\
 & 0
\end{array}
$$

$\Rightarrow (x+2)(x-1)(x-7).$

$P(-2) = -8 - 24 + 18 + 14$

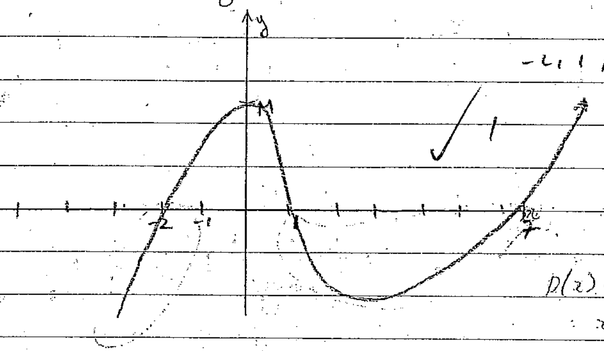
$= 32 - 32 = 0.$

$P(1) = -5 - 9 + 14 = 0.$

$P(7) = 7^3 - 6 \times 49 - 9 \times 7 + 14 = 0$

2

iii.



$-2, 1, 7.$

$P(x) < 0$

$\therefore x < -2$

$1 < x < 7.$

$P(x) < 0$

$\therefore x < -2 \quad 1 < x < 7.$

1

c). $V = \pi \int_a^b [f(x)]^2 dx.$

$= \pi \int_0^{\frac{\pi}{2}} [\sin x + \cos x]^2 dx$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$= \pi \int_0^{\frac{\pi}{2}} 1 + 2 \sin x \cos x \, dx$

$= \pi \int_0^{\frac{\pi}{2}} 1 + \sin 2x \, dx$

~~$= \pi [x - \cos$~~

$= \pi \left[x - \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}}$

$= \pi \left[\frac{\pi}{2} - \frac{\cos \pi}{2} + \frac{1}{2} \right].$

$= \frac{\pi}{2} [\pi + 1 + 1]$

$= \frac{\pi}{2} [\pi + 2] u^3$

4

~~$= \pi \int_0^{\frac{\pi}{2}} x$~~

$V = \pi \int_0^{\frac{\pi}{2}} \sin^2 x + \cos^2 x + 2 \sin x \cos x \, dx$

$= \pi \int_0^{\frac{\pi}{2}} 1 + \sin 2x \, dx = \pi \left[x - \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}}$

$= \pi \left[\frac{\pi}{2} - \frac{\cos \pi}{2} + \frac{1}{2} \right]$

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12

Question 5.

a) i. At $x=0$,

$f(0) = 2\cos^{-1}0$

$= 2 \cdot \frac{\pi}{2} = \pi.$ ✓ 1

$f(0) = 2\cos^{-1}0$

$= 2 \times \frac{\pi}{2} = \pi.$

ii. I: $x = 2\cos^{-1}\frac{y}{3}$

$\frac{x}{2} = \cos^{-1}\frac{y}{3}.$

$\cos\frac{x}{2} = \frac{y}{3}$

$y = 3\cos\frac{x}{2}.$ ✓ 1

Domain: $0 \le x \le 2\pi.$ ✓ 1

$x = 2\cos^{-1}\frac{y}{3}$

$\frac{x}{2} = \cos^{-1}\frac{y}{3}$

$\cos\frac{x}{2} = \frac{y}{3}$

$3\cos\frac{x}{2} = y.$

iii. $\int_0^{\pi} 3\cos\frac{x}{2}\,dx$

$= 3\int_0^{\pi} \cos\frac{x}{2}\,dx$ ✓ 1

$= 3\left[2\sin\frac{x}{2}\right]_0^{\pi}$

$= 3\left[2\sin\frac{\pi}{2} - 2\sin 0\right]$

$= 3\left[2\right]$ ✓ 1

$= 6$ (six)

$\int_0^{\pi} 3\cos\frac{x}{2}$

$= 3\int_0 \cos\frac{x}{2}\,dx$

$= 3\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi}$

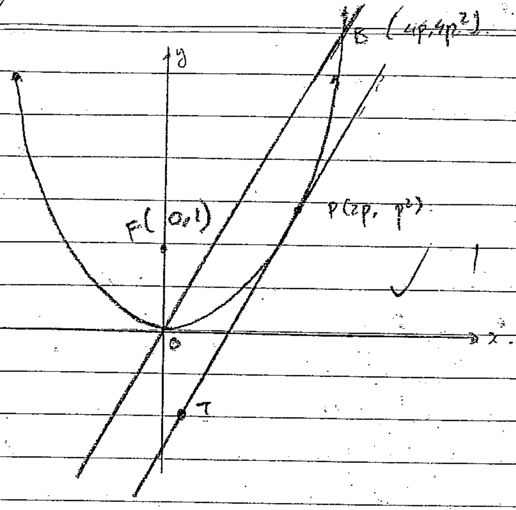
$= 3\left[2\sin\frac{\pi}{2} - \sin 0\right]$

$= 6$ 1

$= 6$. (six).

Kenrick J

b). i.



✓ 1

(What is T? I assumed that it is somewhere along the tangent)

(I also assumed B as the point on the right side of the arc and according to (ii))

ii. B's gradient is the same as PT.

$\therefore y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{2x}{4}$

$= \frac{x}{2}.$

At $x = 2p$,

$\frac{dy}{dx} = p.$

$\therefore y - p^2 = p(x - 2p).$

$y = px - p^2.$

Gradient of OB is the same!

$\therefore y - 0 = p(x - 0).$

$\therefore y = px.$ ✓ 1

$x^2 = 4px.$

$\therefore x = 4p.$

$\therefore 16p^2 = 4y.$

$\therefore y = 4p^2.$

$(4p, 4p^2)$ 1

iii. Midpt of PB

$P(2p, p^2)$ $B(4p, 4p^2).$

$M\left(\frac{2p+4p}{2}, \frac{4p^2+p^2}{2}\right)$

$M\left(3p, \frac{5p^2}{2}\right).$ ✓ 1

iv. $x = 3p$

$p = \frac{x}{3}$ ✓

$y = \frac{5p^2}{2}$

$y = \frac{5}{2} \times \frac{x^2}{9}.$ ✓ 1

$= \frac{5x^2}{18}.$ ✓ 1

$18y = 5x^2.$

Kenrick Johnson

Question 5

Walk out!

ii. $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$.

At $x = 2p$, $\frac{dy}{dx} = p$

$\therefore$ Tangent of OB is also p.

$\therefore y - 4p^2 = p(x - 4p)$

$y = px$

Sub this in.

$x^2 = 4px$.

$x = 4p$.

$(4p, 4p^2)$.

$y = px$

$= p(4p)$

$= 4p^2$

iii. Mid PB.

$M\left(\frac{2p + 4p}{2}, \frac{p^2 + 4p^2}{2}\right)$

$= M\left(\frac{6p}{2}, \frac{5p^2}{2}\right)$

$= M\left(3p, \frac{5p^2}{2}\right)$

iv. $x = 3p$. $y = \frac{5p^2}{2}$

$p = \frac{x}{3}$ $y = \frac{5\left(\frac{x^2}{9}\right)}{2}$

$= \frac{5}{2} \times \frac{x^2}{9}$

$18y = 5x^2$.