J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION 2 (HALF-YEARLY)

MATHEMATICS

2012

Weighting: 20% (HSC Assessment Mark)

NAME: ____________________

MARK: / 60

Time Allowed: 90 minutes

Topics: Graphs, Complex Numbers, Circle Geometry & Polynomials.

Directions:

- There are two sections:
  - **Section 1** multiple choice 4 marks
  - **Section 2** three questions worth 56 marks
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \textit{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$

NOTE : $\ln x = \log_e x,\quad x > 0$

Section 1
Multiple Choice (4 marks) Use a SEPARATE writing booklet.

Marks

1. Find two square roots of $3+4i$ in the form $a+ib$, where a and b are real. 1

(a) $\pm(1+2i)$ (b) $\pm(2+i)$ (c) $\pm(1-2i)$ (d) $\pm(2-i)$

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2. Simplify i^{11}. 1

(a) 1 (b) -1 (c) i (d) $-i$

---

3. If α, β, and γ are the roots of the equation $x^3+px+q=0$, find the values of $(\alpha-1)(\beta-1)(\gamma-1)$. 1

(a) $-(q+p+1)$ (b) $-\dfrac{p}{q}$ (c) $q+p$ (d) $(q+p+1)$

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4. Express $z=4+4\sqrt{3}i$ in modulus-argument form. 1

(a) $4cis\dfrac{\pi}{6}$ (b) $8cis\dfrac{\pi}{6}$ (c) $4cis\dfrac{\pi}{3}$ (d) $8cis\dfrac{\pi}{3}$

Section II
Total marks – 54
Attempt Questions 1–3

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1*(24 marks)*

(a) Let $z=\dfrac{2-4i}{1+i}$.

(i) Find $\overline{z}$, giving your answer in the form $a+bi$, where a and b are real. 2

(ii) Find iz. 1

(b) By applying De Moivre's theorem and also expanding $(\cos\theta+i\sin\theta)^3$, express $\cos 3\theta$ & $\sin 3\theta$ in terms of $\cos\theta$ & $\sin\theta$. Hence, find $\tan 3\theta$ in terms of $\tan\theta$. 3

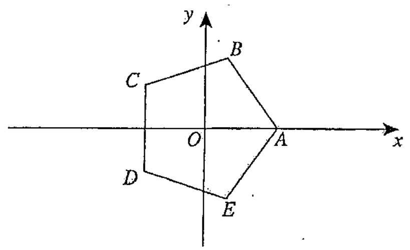
(c) (i) Express $\dfrac{1}{\overline{z}-i}$ in the form $x+iy$. 2

(ii) If $z=x+iy$, where x and y are real, and $\dfrac{1}{\overline{z}-i}=2$ is purely imaginary, show that the locus of z is a circle and find its centre and radius. 2

(e) Sketch the region on the Argand diagram where the inequalities $|z|\le 2$ and $\pi\ge\arg z\ge-\dfrac{\pi}{4}$ hold simultaneously. 3

Marks

(f)



In the diagram above, the complex numbers z_0, z_1, z_2, z_3, z_4, are represented by the vertices of a regular polygon with centre O and the vertices A, B, C, D, E respectively.

Given that $z_0 = 2$:

(i) Express z_2 in modulus-argument form. **2**

(ii) Find the value of $z_2^{\ 5}$. **2**

(iii) Show that the perimeter of the pentagon is $20\sin\frac{\pi}{5}$. **2**

(g) Consider the curve $x^2 - xy + y^2 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{2x-y}{x-2y}$. **2**

(ii) Hence find the two stationary points on the curve. **2**

(iv) Find the values of x where there are vertical tangents. **1**

Marks

Question 2*(12 marks)*

(a) Sketch the graph of $f(x) = (x-3)(1-x)$ and hence draw separate sketches of the following graphs: **1**

(i) $y = \frac{1}{f(x)}$ **1**

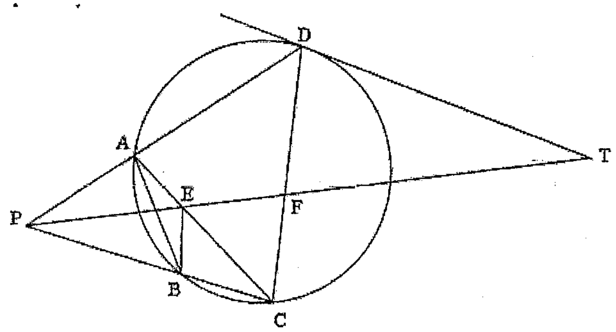
(ii) $y = \left|\frac{1}{f(x)}\right|$ **2**

(iii) $y = [f(x)]^2$ **1**

(iv) $y^2 = f(x)$ **2**

(v) $y = e^{f(x)}$ **1**

(b)



$ABCD$ is a cyclic quadrilateral. DA is produced and CB produced meet at P. T is a point on the tangent at D to the circle through A, B, C and D. PT cuts CA and CD at E and F respectively. $TF = TD$.

(i) Show that $AEFD$ is a cyclic quadrilateral. **2**

(ii) Show that $PBEA$ is a cyclic quadrilateral. **2**

Marks

Question 3 *(20 marks)*

(a) The zeros of $x^3 + px^2 + qx + r$ are α, β, and γ (where p, q and r are real numbers).

(i) Find $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Find a cubic polynomial with integer coefficients whose zeros 2α, 2β and 2γ. **2**

(b) (i) If $(x-r)^2$ is a factor of the polynomial $p(x)$, prove that $x-r$ is a factor of the polynomial $p'(x)$. **2**

(ii) The polynomial equation $x^4 + Bx^3 + Cx^2 - 24x + 36 = 0$ has a double zero at $x = 3$. Find the values of B and C, and hence write $P(x)$ as a product of four linear factors. **5**

(c) The polynomial equation $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a root of multiplicity 3. Find all the zeros of this polynomial. **2**

(d) Let α, β, and γ be the roots of the equation $x^3 - 2x^2 - 5x - 1 = 0$. Form an equation whose roots are $\dfrac{1}{\sqrt{\alpha}}$, $\dfrac{1}{\sqrt{\beta}}$ and $\dfrac{1}{\sqrt{\gamma}}$. **2**

(e) (i) Let $w = z + \dfrac{1}{z}$. Prove that **2**

$$w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$

(ii) Hence, or otherwise, find all solutions of **3**

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0,$$

in the range $0 \le \alpha \le 2\pi$.

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Multiple Choice

① $(a+ib)^2 = 3+4i$

$a^2 - b^2 + 2abi = 3+4i$

$2ab = 4$

$ab = 2$

$a^2 - b^2 = 3$

$\pm(2+i)$

(B)

② $i^{11} = i \times i^{10}$

$= i \times (i^2)^5$

$= i \times (-1)^5$

$= -i$

(d)

③ $(\alpha-1)(\beta-1)(\gamma-1)$

$(\alpha\beta - \alpha - \beta + 1)(\gamma - 1)$

$\alpha\beta\gamma - \alpha\gamma - \beta\gamma + \gamma - \alpha\beta + \alpha + \beta - 1$

$\alpha\beta\gamma - (\alpha\gamma + \beta\gamma + \alpha\beta) + (\alpha + \beta + \gamma) - 1$

$\left(\frac{-d}{a}\right) - \left(\frac{c}{a}\right) + \left(\frac{-b}{a}\right) - 1$

$-q - p + 0 - 1$

$-(q + p + 1)$

(a)

④

$z = 4 + 4\sqrt{3}i$

$|z| = \sqrt{4^2 + (4\sqrt{3})^2}$

$= 8$

$\arg(z) = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$

$= \frac{\pi}{3}$

$8 \operatorname{cis} \frac{\pi}{3}$

(d)

Question 1

(a) (i) $z = \frac{2-4i}{1+i} \times \frac{(1-i)}{(1-i)}$

$= \frac{2-6i-4}{2}$ ①

$= -1-3i$

$\therefore \bar{z} = -1+3i$ ①

(ii) $iz = i(-1-3i)$

$= -i+3$

$= 3-i$ ①

(b) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$LHS = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + i(3\cos^2\theta\sin\theta) - 3(\cos\theta\sin^2\theta) - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ ①

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$ ①

$\therefore \tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$

$= \frac{\frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta - 3\cos\theta\sin^2\theta}{\cos^3\theta}}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$ ①

(c) (i) let $z = x + iy$

$$\frac{1}{\bar{z} - i} = \frac{1}{(x - iy) - i}$$

$$= \frac{1}{x - (y+1)i} \times \frac{[x + (y+1)i]}{[x + (y+1)i]}$$

$$= \frac{x + (y+1)i}{x^2 + (y+1)^2}$$

(1) (1)

(ii) $$\frac{y+1}{x^2 + (y+1)^2} = 2$$

$$(y+1) = 2x^2 + 2(y+1)^2$$

$$2x^2 + 2y^2 + 4y + 2 - y - 1 = 0$$

$$2x^2 + 2y^2 + 3y + 1 = 0$$ (1)

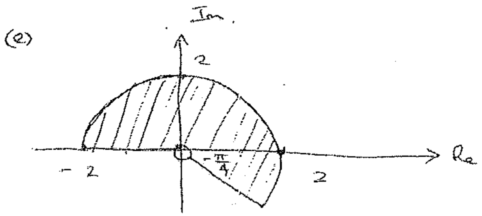
$$2x^2 + 2\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{1}{8}$$

$$x^2 + \left(y + \frac{3}{4}\right)^2 = \frac{1}{16}$$

centre $\left(0, -\frac{3i}{4}\right)$ (1)

radius $= \frac{1}{4}$ unit.

(e)



(3)

(f) (i) $\angle COA = \frac{4\pi}{5}$ & $|z_2| = 2$. (1)

$\therefore z_2 = 2\operatorname{cis}\frac{4\pi}{5}$ (1)

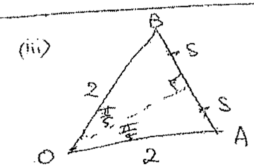
(ii) $$(z_2)^5 = 2^5\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)^5$$ (1)

$$= 32(\cos 4\pi + i\sin 4\pi)$$

$$= 32(1 + 0)$$

$$= 32.$$ (1)

(iii)



$S = 2\sin\frac{\pi}{5}$ (1)

$\therefore AB = 4\sin\frac{\pi}{5}$

$\therefore$ perimeter $= 5\left[4\sin\frac{\pi}{5}\right]$ (1)

$= 20\sin\frac{\pi}{5}$

(a) (i) $2x - \left(x \cdot \frac{dy}{dx} + y\right) + 2y \cdot \frac{dy}{dx} = 0$ (1)

$2x - y - \frac{dy}{dx}(x - 2y) = 0$

$\frac{dy}{dx}(x - 2y) = 2x - y$ (1)

$\frac{dy}{dx} = \frac{2x - y}{x - 2y}$.

(ii) $\frac{dy}{dx} = 0$

$\therefore 2x - y = 0$

$y = 2x$ (1)

sub $y = 2x$

$x^2 - 2x^2 + 4x^2 = 3$

$3x^2 = 3$

$x = \pm 1$ (1)

stat pts $(1, 2)$ & $(-1, -2)$

(iii) $x - 2y = 0$

$x = 2y$

sub $y = \frac{x}{2}$

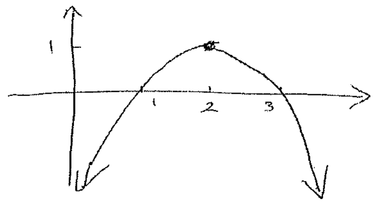
$x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 3$

$x^2 = 4$ (1)

$\therefore x = 2 \quad x = -2$

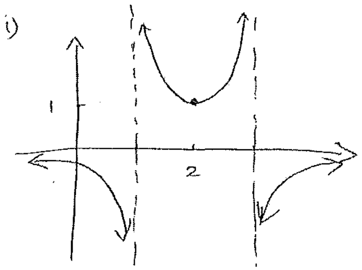
Question 2.

(a)



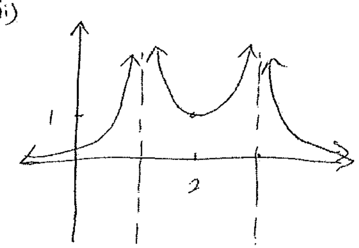
(1)

(i)



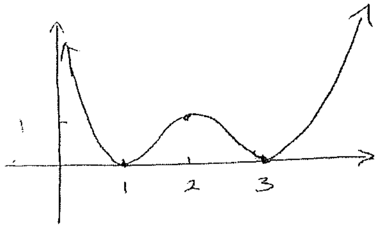
(1) (1)

(ii)



(2)

(iii)



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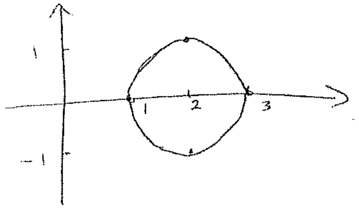
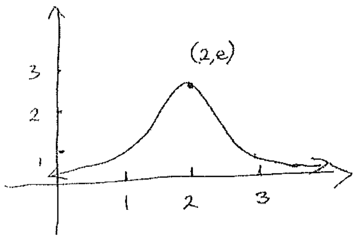
(iv) $y^2 = f(x)$



2

(v) $y = e^{f(x)}$



1

(b) (i) $TD = TF$ given

$\therefore \angle TFD = \angle TDF$ (base angles of isos Δ are equal)

$\angle TDF = \angle CAD$ ($\angle$ between tangent and chord equals the $\angle$ in the alternate segment)

$\therefore \angle TFD = \angle CAD$

$\therefore$ AEFD is cyclic quad (exterior $\angle$ = interior opposite $\angle$)

(ii) $\angle PEA = \angle ADF$ (exterior $\angle$ of cyclic quad AEFD = interior opposite $\angle$)

$\angle PBA = \angle ADF$ (exterior $\angle$ of cyclic quad ABCD equals interior opposite $\angle$).

$\therefore \angle PEA = \angle PBA$

$\therefore$ PBEA is cyclic since two $\angle$'s stand on same arc are equal at the circumference.

Question 3

(a) (i) $\alpha\beta + \alpha r + \beta r = q$ (1)

(ii) $\alpha^2 + \beta^2 + r^2 = (\alpha+\beta+r)^2 - 2(\alpha\beta + \alpha r + \beta r)$

$= (-p)^2 - 2(q)$

$= p^2 - 2q.$ (1)

(iii) $2\alpha + 2\beta + 2r$ are zeros then

$\left(\frac{y}{2}\right)^3 + p\left(\frac{y}{2}\right)^2 + q\left(\frac{y}{2}\right) + r$ (1)

$y^3 + 2py^2 + 4qy + 8r$ (1)

(b) (i) Let $P(x) = (x-r)^2 . Q(x)$, where $Q(x)$ is a polynomial (1)

$P'(x) = 2(x-r).Q(x) + (x-r)^2 . Q'(x)$

$= (x-r)\left[2.Q(x) + (x-r).Q'(x)\right]$ (1)

$\therefore (x-r)$ is a factor of $P'(x)$

(ii) $P(3) = 0$

$81 + 27B + 9C - 72 + 36 = 0$

$27B + 9C = -45$

$3B + C = -5$ (1) (1)

$P'(3) = 0$

$4(3)^3 + 3B(3)^2 + 2C(3) - 24 = 0$

$108 + 27B + 6C - 24 = 0$ (1)

$9B + 2C = -28$ (2)

$B = -6$ & $C = 13$ (1)

$(x-3)^2$ is a factor.

$$\begin{array}{r|l} & x^2 + 4 \\ x^2 - 6x + 9 & x^4 - 6x^3 + 13x^2 - 24x + 36 \\ & x^4 - 6x^3 + 9x^2 \\ \hline & 4x^2 - 24x + 36 \\ & 4x^2 - 24x + 36 \\ \hline & 0 \end{array}$$

(1)

$\therefore P(x) = (x-3)^2(x-2i)(x+2i)$ (1)

(c) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

$= 6(2x^2 + x - 1)$

$= 6(2x-1)(x+1)$

$\therefore\ x = \frac{1}{2}$ or $x = -1$ (1)

$P'(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) - 5$

$= 0$

$\therefore -1$ is a zero of multiplicity 3 of $P(x)$

$P(x) = (x+1)^3(ax+b)$

Hence $a = 1$ & $b = -2$

$\therefore$ zeros $-1, -1, -1$ & 2. (1)

(d) let $y = \frac{1}{\sqrt{x}} \Rightarrow x = \frac{1}{y^2}$

sub into $x^3 - 2x^2 - 5x - 1 = 0$

$\left(\frac{1}{y^2}\right)^3 - 2\left(\frac{1}{y^2}\right)^2 - 5\left(\frac{1}{y^2}\right) - 1 = 0$ (1)

$\frac{1}{y^6} - \frac{2}{y^4} - \frac{5}{y^2} - 1 = 0$

$1 - 2y^2 - 5y^4 - y^6 = 0$

$\therefore\ y^6 + 5y^4 + 2y^2 - 1 = 0$. (1)

(e) (i) $w = z + \frac{1}{z}$

$w^3 - 2w = z^3 + 3z + 3\left(\frac{1}{z}\right) + \frac{1}{z^3} - 2z - 2\left(\frac{1}{z}\right)$

$= \left(z^3 + \frac{1}{z^3}\right) + \left(z + \frac{1}{z}\right)$ (1)

$w^2 - 2 = z^2 + 2 + \frac{1}{z^2} - 2$

$= z^2 + \frac{1}{z^2}$ (1)

$\therefore\ w^3 + w^2 - 2w - 2$

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$

(ii) $\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$

$2\cos\alpha + 2\cos 2\alpha + 2\cos 3\alpha = 0$

$\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right) = 0$

$w^3 + w^2 - 2w - 2 = 0$

$(w+1)(w^2 - 2) = 0$.

$\therefore\ w = -1$ & $\pm\sqrt{2}$ (1)

$2\cos\alpha = w = z + \frac{1}{z}$

$\therefore\ 2\cos\alpha = -1,\ \sqrt{2}$ or $-\sqrt{2}$

$\cos\alpha = -\frac{1}{2},\ \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$ (1)

$\cos\alpha = -\frac{1}{2}$ $\qquad$ $\cos\alpha = \frac{1}{\sqrt{2}}$

$\alpha = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ $\qquad$ $\alpha = \frac{\pi}{4}$ or $\frac{7\pi}{3}$

$\cos\alpha = -\frac{1}{\sqrt{2}}$

$\alpha = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

1

$\therefore \alpha = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}$ & $\frac{7\pi}{4}$.