J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION I

MATHEMATICS

2012

Weighting: 30% (HSC Assessment Mark) MAY TASK

NAME: ____________________

MARK: / 60

Time Allowed: 90 minutes

Topics: Inequalities, Graphs, Ratios, Integration, Polynomials, Trigonometry, Inverse Functions, Circle Geometry & Induction.

Directions:

- There are two sections:
  - Section 1 multiple choice 5 marks
  - Section 2 four questions worth 55 marks
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work

Section 1
Multiple Choice (5 marks) Use a SEPARATE writing booklet.

Marks

1. When the polynomial $P(x)=2x^3-x^2+px-1$ is divided by $(x-3)$, the remainder is 2. Find p. **1**

(a) $p=14$ (b) $p=-\frac{4}{3}$ (c) $p=-14$ (d) $p=-66$

---

2. Find $\lim_{x\to 0}\frac{\sin 3x}{5x}$. **1**

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) undefined (d) 1

---

3. If α, β and γ are the roots of the equation $x^3+2x^2-11x-12=0$, **1**

Find $\frac{2}{\alpha\beta}+\frac{2}{\alpha\gamma}+\frac{2}{\beta\gamma}$.

(a) 3 (b) $\frac{1}{3}$ (c) -3 (d) $-\frac{1}{3}$

Multiple Choice continued.

Marks

4. Solve $|3-5x| > 4$ for x. **1**

(a) $x > -\frac{1}{5}$ & $x < \frac{7}{5}$

(b) $x < -\frac{1}{5}$ & $x > \frac{7}{5}$

(c) $x < \frac{1}{5}$ & $x > -\frac{7}{5}$

(d) $x > \frac{1}{5}$ & $x < -\frac{7}{5}$

5. A point P divides the interval joining the points $A\ (-5, 6)$ and $B\ (1, 0)$ externally in the ratio 3 : 2. Find the coordinates of P. **1**

(a) $(13,-12)$ (b) $(7,-12)$ (c) $\left(\frac{-7}{5},\frac{12}{5}\right)$ (d) $\left(\frac{-13}{5},\frac{12}{5}\right)$

Please turn over and continue to section 2

Section II
Total marks – 55
Attempt Questions 1–4

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 *(15 marks)*

(a) Solve $\frac{4}{x-1} \geq 1$ **3**

(b) The angle between two lines $y = mx$ and $y = \frac{1}{3}x$ is $\frac{\pi}{4}$. **2**

Find the exact values of m.

(c) Find $\frac{d}{dx}\left(e^x \tan^{-1} x\right)$ **2**

(d) Use the table of standard integrals to find $\int \sec 2x \tan 2x\, dx$. **2**

(e) Evaluate $\int_0^{2\pi} \cos^2 2x\, dx$. **3**

(f) Prove that any number of the form $3^n + 7^n$ where n is an odd integer is divisible by 5, using mathematical induction. **3**

Question 2 *(13 marks)*

(a) Given the polynomial $P(x)=4x^3-8x^2-3x+9$, show $(x+1)$ is a factor and express $P(x)$ in a fully factorised form. **2**

(b) Evaluate $\int_0^{\frac{3}{2}}\sqrt{9-x^2}\,dx$ using the substitution $x=3\sin\theta$. **3**

(c) (i) Sketch the graph of the function $f(x)=e^x-2$. **1**

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$. **1**

(iii) State the equation of the function $f^{-1}(x)$. **1**

(iv) Explain why the coordinate of any point of intersection of the graphs $y=f(x)$ and $y=f^{-1}(x)$ satisfies the equation $e^x-x-2=0$. **1**

(v) One root of the equation $e^x-x-2=0$ lies between $x=1$ and $x=2$ **2**
Use one application of Newton's method, with a starting value of $x=1.5$, to approximate the root, to decimal places.

(d) Given $5-3x^2+2x^3\equiv a+bx+cx(x-1)+dx(x-1)(x+1)$ find the values of a, b, c & d. **2**

Question 3 *(15 marks)*

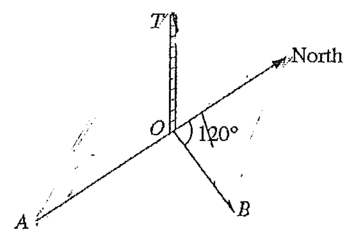
(a) Sketch the graph of $y=3\sin^{-1}2x$ showing clearly the domain and range of the function as well as any intercepts. **2**

(b) Sketch $y=4-x^2$ & $y=4x-x^2$ and find to the nearest degree the acute angle between the curves at the point of intersection. **3**

(c) Solve $2\sin^2\theta+3\cos\theta=0$ for $0\le\theta\le2\pi$. **2**

(d) Using the t results simplify $\cot\frac{\theta}{2}-2\cot\theta$. **2**

(e) From a point A due south of a tower, the angle of elevation of the top of the tower T is 23^0. From another point B, on a bearing of 120^0 from the tower, the angle of elevation is 32^0. The distance AB is 200 metres. **3**



(i) Copy or trace the diagram adding the given information to your diagram.

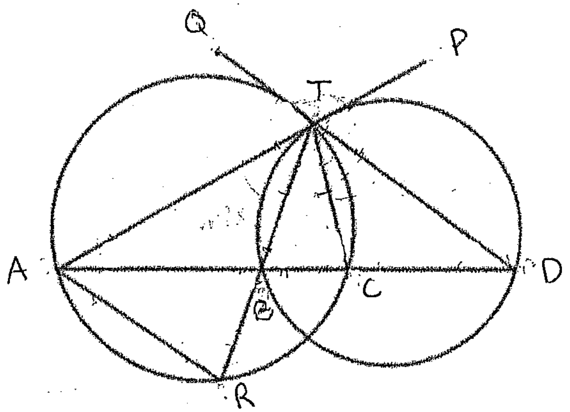
(ii) Hence find the height of the tower.

(d) $\int y\sqrt{y+1}$, using the substitution $u=y+1$. **3**

Question 4 *(12 marks)*

(a) Sketch the curve $y=\dfrac{x^2}{x^2-4}$. **2**

(b)



ATP is a tangent to the circle TBD. DTQ is a tangent to the circle TAC. TB produced meets the circle TAC at R.

(i) Explain why $\angle PTD = \angle TBD$. **1**

(ii) Explain why $\angle PTD = \angle QTA$. **1**

(iii) Deduce that $TB = TC$. **2**

(iv) Prove that ΔABR is isosceles. **2**

(c) (i) Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $R > 0$ and α is acute. **2**

(ii) Hence or otherwise solve $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2},\ for\ 0 \le \theta \le 2\pi$. **2**

SOLUTIONS

Sec 1 MC

4

1) $P(3) = 2$

$2 = 2(3)^3 - 9 + 3p - 1$

$= 54 - 10 + 3p$

$= 44 + 3p$

$-42 = 3p$

$p = -14$

(c)

2) $\frac{1}{5}\lim_{x\to 0}\frac{\sin 3x}{x} = \frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x} = \frac{3}{5}$ (b)

3) $x = \frac{2}{\alpha\beta}$

$\left(\frac{2}{x}\right)^3 + 2\left(\frac{2}{x}\right)^2 - 11\left(\frac{2}{x}\right) - 12 = 0$

$\frac{8}{x^3} + \frac{8}{x^2} - \frac{22}{x} - 12$

$8 + 8x - 22x^2 - 12x^3 = 0$

$\Sigma\alpha\beta = \frac{c}{a} = \frac{8}{-12} = -\frac{2}{3}$

(d)

4) $(3-5x) > 4$ $\quad -(3-5x) > 4$

$-1 > 5x$ $\quad 3 - 5x < -4$

$x < -\frac{1}{5}$ $\quad 7 < 5x$

$x > \frac{7}{5}$

(b)

5. $(-5, 6) \quad (1, 0)$

$3 : -2$

$\left(\frac{10+3}{1}, \frac{-12+0}{1}\right)$

$(13, -12)$

a)

Q1

12

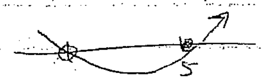
a) $\frac{4(x-1)^2}{x+1} \geq (x-1)^2$ $\quad x \neq 1$

$4(x-1) \geq x^2 - 2x + 1$

$4x - 4 \geq x^2 - 2x + 1$

$0 \geq x^2 - 6x + 5$

$\geq (x-5)(x-1)$



$x < 1 \quad x \geq 5$ $\quad 1 < x \leq 5$

b) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\frac{\pi}{4} = \left|\frac{m_1 - \frac{1}{3}}{1 + m_1 \cdot \frac{1}{3}}\right|$

$\pm 1 = \frac{m_1 - \frac{1}{3}}{1 + m_1\frac{1}{3}}$

$(1 + m_1\frac{1}{3}) = m_1 - \frac{1}{3}$ $\quad -(1 + m_1\frac{1}{3}) = m_1 - \frac{1}{3}$

$3 + m_1 = 3m_1 - 1$ $\quad (1 + m_1\frac{1}{3}) = -m_1 + \frac{1}{3}$

$4 = 2m_1$ $\quad 3 + m_1 = -3m_1 + 1$

$m_1 = 2$ or $\quad 4m_1 = -2$

$m_1 = -\frac{1}{2}$

c) $\tan^{-1}x\, e^x + e^x \cdot \frac{1}{1+x^2}$

$= e^x\left(\tan^{-1}(x) + \frac{1}{1+x^2}\right)$

d) $\frac{1}{2}\sec 2x + C$

e) $\cos 4x = \cos^2 2x - \sin^2 2x$

$= 2\cos^2 2x - 1$

$\frac{1}{2}(\cos 4x + 1) = \cos^2 2x$

$\frac{1}{2}\int_0^{2\pi}(\cos 4x + 1)\,dx$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_0^{2\pi}$

$= \frac{1}{8}\left[\sin 4x + 4x\right]_0^{2\pi}$

$= \frac{1}{8}\left[(\sin 8\pi + 8\pi) - (0)\right]$

Q1

f) $3^n + 7^n$ odd divisible by 5

~~prove true~~

Prove true for $n=1$

$3^1 + 7^1 = 10$

divisible by 5

$\therefore$ true for $n=1$

Assume true for $n=k$ (where k is odd)

$3^k + 7^k = 5M \qquad (M \in Q)$

$3^k = 5M - 7^k$

R.T.P. true for $n = k+2$ (for odd integers)

$\therefore \; 3^{k+2} + 7^{k+2} = 5N$ (where $N \in Q$)

Proof

$LHS = 3^k . 3^2 + 7^k . 7^2$

$= 9(5M - 7^k) + 49 . 7^k$

$= 5 \times 9M - 9.7^k + 49.7^k$

$= 5 \times 9M + 40.7^k$

$= 5(9M + 8.7^k) = 5N$ which is divisible by 5.

$\therefore$ by mathematical induction, ~~all integers~~ True for all odd positive integers

Q2

11

a) $P(-1) = 4(-1)^3 - 8(-1)^2 + 3 + 9$

$= 0$

$\therefore (x+1)$ is a factor

$$\begin{array}{r} 4x^2 - 12x + 9 \\ x+1 \enclose{longdiv}{4x^3 - 8x^2 - 3x + 9} \\ 4x^3 + 4x^2 \\ \hline -12x^2 - 3x \\ -12x^2 - 12x \\ \hline 9x + 9 \\ 9x + 9 \end{array}$$

$(4x^2 - 12x + 9)$

$\frac{(4x-6)(4x-6)}{4}$

$(2x-3)^2$

$(x+1)(4x^2 - 12x + 9)$

$(x+1)(2x-3)^2$ ~~ans~~

(b) $\int_0^{\frac{3}{2}} \sqrt{9 - x^2}\, dx \qquad$ Using $x = 3\sin\theta$

$\therefore \frac{dx}{d\theta} = 3\cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2\theta} \, . 3\cos\theta \, d\theta$ | When $x = \frac{3}{2}$, $\theta = \frac{\pi}{2}$; $x = 0$, $\theta = 0$

$= \int_0^{\frac{\pi}{2}} 9\cos^2\theta \, d\theta \qquad$ but $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$

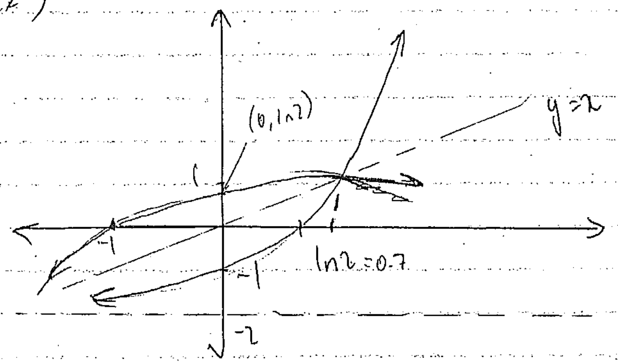
$= \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

$= \frac{9}{2}\left[0 + \frac{\pi}{2}\right]$

$= \frac{9\pi}{4}$

Q2 continued

c) ii)



$y = e^x - 2$

$y + 2 = e^x$

$\ln(y+2) = x$

$\therefore \ln(x+2) = y$

iii) $y = \ln(x+2)$

iv) both int. at $y = x$

$y = e^x - 2,\ y = x$

$x = e^x - 2$

$e^x - x - 2 = 0$

v) $x_0 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\qquad f'(x) = e^x - 1$

$x_0 = 1.5 - \frac{0.982}{3.482}$

$= 1.22$

d) $2x^3 - 3x^2 + 5 \equiv a + bx + cx^2 - cx + dx(x^2-1)$

$\equiv a + bx + cx^2 - cx + dx^3 - dx$

$\equiv dx^3 + cx^2 + x(b - c - d) + a$

equate $2 = d$

$-3 = c$

$b - c - d = 0 \qquad b + 3 - 2 = 0$

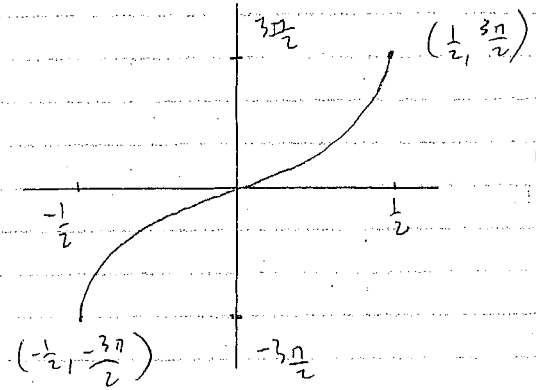
$a = 5$

$b = -1$

Q3

14

a)

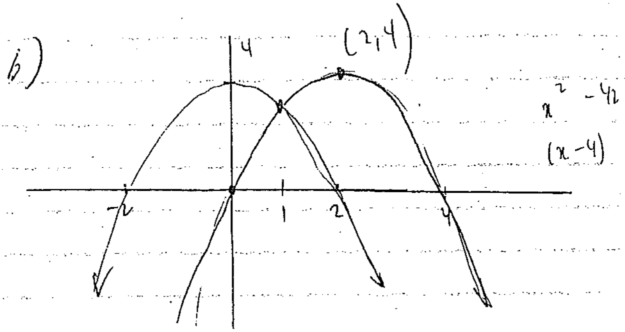


State the

$D: -\frac{1}{2} \le x \le \frac{1}{2}$

& $R: -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

b)



$x^2 - 4x$

$(x-4)$

$4x - x^2 = 4 - x^2$

$x = 1$

$y = 3 \qquad (1, 3)$

$f_1(x) = 4 - x^2$

$f_1'(x) = -2x$

m_1 at $x = 1$

$m_1 = -2$

$f_2'(x) = 4 - 2x$

m_2 at $x = 1$

$m_2 = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-4}{-3}\right|$

$= \frac{4}{3}$

Q3 cont.

c) $\cos 2x = \cancel{2x} 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\cancel{1-} \cos 2x + 3\cos x = 0$

$1 - (\cos$

$2(1-\cos^2 x) + 3\cos x = 0$

$2 - 2\cos^2 x + 3\cos x = 0$

$2\cos^2 x - 3\cos x - 2 = 0$

$\cos x = \dfrac{3 \pm \sqrt{9+16}}{4}$

$= \dfrac{3 \pm 5}{4}$

$= 2, -\frac{1}{2}$

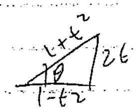
$\cos x = -\frac{1}{2}$

$x = 120, 240$

$= \cancel{\frac{\pi}{3}, \frac{2\pi}{3}}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$

d) $\cancel{\tan\theta}$ $\tan\theta = \dfrac{2t}{1-t^2}$



$\dfrac{1}{t} - \dfrac{2}{\frac{2t}{1-t^2}}$

$= \dfrac{1}{t} - \dfrac{1-t^2}{t}$

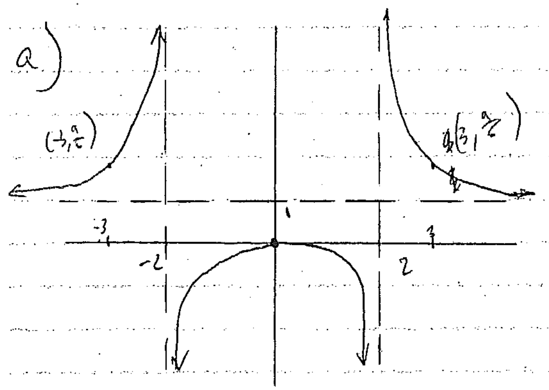
$= \dfrac{t^2}{t}$

$= t$

Q4

11

a)



$x \neq \pm 2$

b) I) ∠ in opp segment is equal to ∠ tangent

II) Vertically opp

III) ~~∠PTD = ∠~~

~~let ∠PTD = α~~

let ∠QTR = α

~~∠QTR~~ ∠TRA = α (∠ in opp seg is equal to tangent ∠)

∠TRA = ∠TCA = α (∠ sub by same arc are equal)

∴ ∠PTD = α (II)

∠TBD = α (I)

∴ △TBC is an iso △ (base ∠ equal)

∴ TB = TC

~~IV) (to prove ∠RTD = α)~~

~~∠ATB = (180 − 2α)~~

~~∠ATB = ∠ATR + ∠RTD~~

~~= (180 − 2α) + ∠RTD~~

~~∠ATB = 180 − 2α (∠ sum of △)~~

~~∠RTD = 180 − [(180 − 2α) + α] (supp ∠)~~

~~= α~~

~~∠RTD = α~~

~~∴ ∠TAR = α~~

~~∴ △TAR~~

IV) ∠TBC = ∠ABK = α
(vert opp)

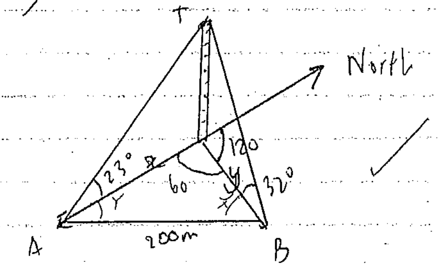
∠TCA = ∠TRA
(∠ sub by same arc)

∠ABR = ∠TKA

∴ △ABK is iso
(base ∠s are equal)

Q3 cont.

e)



$200^2 = x^2 + y^2 - 2xy\cos 60$

$40000 = x^2 + y^2 - \cancel{2}xy$

$= x^2 \cancel{60x} + y^2 - xy$

$\frac{1}{2}$

$\cancel{\tan 32} = \cancel{TO}$

$\tan 32 = \frac{TO}{y}$

$\tan 23 = \frac{TO}{x}$

$\cancel{\frac{\sin 60}{200}} = \frac{\sin X}{x}$

$x = \cancel{\frac{\sin X}{400}}$

$x = \frac{TO}{\tan 23}$

$y = \frac{\cancel{\tan}\, TO}{\tan 32}$

$40{,}000 = \left(\frac{TO}{\tan 23}\right)^2 + \left(\frac{TO}{\tan 32}\right)^2 - \left(\frac{TO}{\tan 23}\right)\left(\frac{TO}{\tan 32}\right)$

$= \frac{TO^2}{\tan^2 23} + \frac{TO^2}{\tan^2 32} - \frac{TO^2}{\tan 23 \tan 32}$

$= \cancel{TO^2 \, 2\left(\frac{1}{\tan^2 23}\right)}$

$= TO^2(\cot^2 23 + \cot^2 32 - \cot 23 \cot 32)$

$TO^2 = \frac{40{,}000}{\cot^2 23 + \cot^2 32 - \cot 23 \cot 32}$

$\cot 23 = 2.4$

$\cot 32 = 1.6$

$= \frac{40{,}000}{\cancel{1025.42}\; 4.34\ldots}$

$\cancel{TO^2 = 38.99}$

$TO = \cancel{6.2m}$ $\quad TO > 0$

96 m (to nearest metre)

d) $\int y\sqrt{y+1}\, dy$ $\qquad du = dy \qquad y = u-1$

$\int (u-1)\sqrt{u}\; du$

$= \int u\sqrt{u} - \sqrt{u}\; du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\; du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$

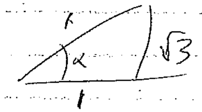
sub $u = y+1$

Q4 cont.

c) i) $R\cos(\theta - \alpha) = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

~~$R\cos\theta\cos$~~

$R\cos\alpha = 1 \quad R\sin\alpha = \sqrt{3}$



$R = 2 \quad \alpha = \frac{\pi}{3}$

$2\cos\left(\theta - \frac{\pi}{3}\right)$

$-\frac{\pi}{4} + \frac{\pi}{3}$

$\frac{-3\pi + 4\pi}{12}$

ii) $2\cos\left(\theta - \frac{\pi}{3}\right) = \sqrt{2}$

$\cos\left(\theta - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

$\left(\theta - \frac{\pi}{3}\right) = \frac{\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}$ $\qquad -\frac{\pi}{3} \le \theta - \frac{\pi}{3} \le 2\pi - \frac{\pi}{3}$

$\theta = \frac{7\pi}{12}, \frac{25\pi}{12}, \frac{\pi}{12}$

$= \frac{7\pi}{12}, \frac{\pi}{12}$