J.M.J.

MARCELLIN COLLEGE RANDWICK

# EXTENSION I

# MATHEMATICS

2012 – TASK 1

**Weighting: 40% (Preliminary Assessment Mark)**

NAME: ____________________

MARK: /33

**Time Allowed:** 45 minutes

**Topics:** Inequalities, Graphs, Ratios, Integration, Trigonometry, Circle Geometry & Induction.

**Directions:**

- There are two questions on this paper
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work

Marks

**Question 1** (15 marks) [START A NEW PAGE]

(a) Find the coordinates of the point P which divides the interval joining A (-2, 3) and B (3, -4) externally in the ratio 3 : 2 — 2

(b) Evaluate $\lim_{x \to 0}\left(\frac{\sin 5x}{2x}\right)$ — 2

(c) Solve $\frac{3}{2x-4} > -2$ — 3

(d) Evaluate $\int_{-1}^{1} \frac{-1}{\sqrt{2-x^2}}\, dx$ — 2

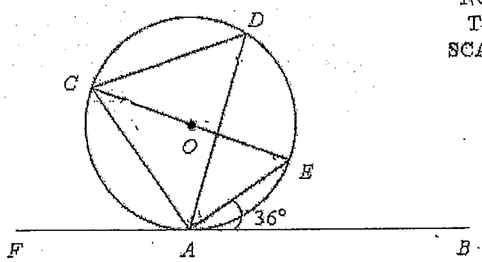
(e) Evaluate $\int_{0}^{\frac{\pi}{12}} 2\sin^2 4x\, dx$ — 3

(f) Use the substitution $u = x-1$ to evaluate $\int_{2}^{4} \frac{x}{(x-1)^2}\, dx$. — 3

Marks

**Question 2** (18 marks) [START A NEW PAGE]

(a)

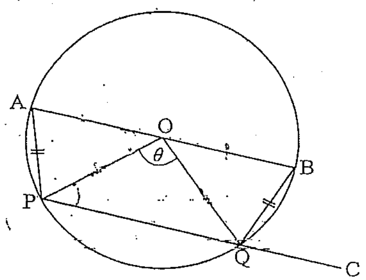



FB is a tangent meeting a circle at $A$. $CE$ is the diameter, $O$ is the centre and $D$ lies on the circumference $\angle BAE = 36^o$

(i) Find the size of $\angle ACE$, giving reasons. 1

(ii) Find the size of $\angle ADC$, giving reasons. 2

(b)




The points $A$, $B$, $P$ and $Q$ lie on the circle which has its centre at $O$. $AB$ is a diameter and $PC$ passes through the point $Q$. $AP$ is equal to $BQ$ and $\angle POQ = \theta$.

(i) Express $\angle AOP$ in terms of $\theta$. 2

(ii) Prove that $AB$ is parallel to $PC$. 2

(c) (i) Express $\sqrt{3}\sin t + \cos t$ in the form $R\sin(t + \alpha)$ where $\alpha$ is in radians. 2

(ii) Hence, or otherwise, find the solutions of the equation 2

$$\sqrt{3}\sin t + \cos t = \sqrt{3} \quad \text{for} \quad 0 \le t \le 2\pi.$$

(d) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. 4

(e) Use mathematical induction to prove that

$$1 + 6 + 15 + \ldots + n(2n-1) = \frac{1}{6}n(4n-1)(n+1)$$

for all positive integers $n$. 3

Question 1

a) $\left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$

$\left(\dfrac{3\cdot 3 - 2 \cdot -2}{-3}, \dfrac{3\cdot -4 \cdot 2 \cdot 2}{5}\right)$

$= \left(-\dfrac{13}{5}, -16\right)$

$(-2,3)\ (3,-4)$
$3:-2$

$= \left(\dfrac{-4+9}{3-2}, \dfrac{-6-12}{3-2}\right)$

$= (5, -18)$

b) $\lim_{x\to 0} \dfrac{\sin 5x}{2x} \times \dfrac{5}{2} = \dfrac{5}{2}$

c) $\dfrac{3}{2x-4} + 2 > 0$

$3 + 2(2x-4) > 0$

$3 + 4x - 8 > 0$

$4x > 5$

$x > \dfrac{5}{4}$

$\dfrac{3}{2x-4} > -2$

$3 < -2(2x-4)$

$3 < -4x + 4$

$4x < 1$

$x < \dfrac{1}{4}$

d) $\left[\cos^{-1}\left(\dfrac{x}{\sqrt{2}}\right)\right]_{-1}^{1}$

$= \left(\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) - \cos\left(-\dfrac{1}{\sqrt{2}}\right)\right) = \dfrac{\pi}{4} - \left(\pi - \dfrac{\pi}{4}\right)$

$= \dfrac{\pi}{4} - \pi + \dfrac{\pi}{4}$

$= -\dfrac{\pi}{2}$

e) $\cos 8x = \cos^2 4x - \sin^2 4x$

$= 1 - 2\sin^2 4x$

$2\sin^2 4x = 1 - \cos 8x$

$\int_0^{\frac{\pi}{12}} (1 - \cos 8x)\,dx = \left[x - \dfrac{1}{8}\sin 8x\right]_0^{\frac{\pi}{12}} \longrightarrow \left[\dfrac{\pi}{12} - \dfrac{1}{8}\cdot\dfrac{\sqrt{3}}{2}\right] - 0$

$= \left[\dfrac{\pi}{12} - \dfrac{1}{8}\sin\dfrac{\pi}{6}\right] - (0 - 0)$

$= \dfrac{\pi}{12} - \dfrac{1}{4} = \dfrac{\pi - 3}{12}$

$= \left(\dfrac{\pi}{12} - \dfrac{\sqrt{3}}{16}\right)$

$= \dfrac{16\pi - 12\sqrt{3}}{192} = \dfrac{\pi}{12} - \dfrac{\sqrt{3}}{16}$

f) $\int_2^4 \dfrac{x}{(x-1)^2}\,dx$

$u = x - 1 \quad x = u + 1$

$du = 1\,dx$

$\int_1^3 \dfrac{u+1}{u^2}\,du$

$\int_1^3 \dfrac{1}{u}\,du + \int_1^3 \dfrac{1}{u^2}\,du$

$= [\ln u]_1^3 + \left[-u^{-1}\right]_1^3$

$= \ln 3 - \left(\dfrac{1}{3} - 1\right)$

$= \ln 3 + \dfrac{2}{3}$

Question 2.

a i) $\angle ACE = \angle EAB$ ($\angle$s = to alt segment)

$= 36°$

ii) $\angle CAE = 90$ ($\angle$ in semi circle)

$\angle AEC = 180 - (90 + 36)$

$= 54°$

$\angle CDA = 54°$ ($\angle$ subtended on same arc are equal)

b i) $\triangle AOP \parallel \triangle OBQ$ (ISO $\triangle$, as sides are equal radii and $BQ = \dots$ given)

$\therefore \angle AOP = \angle BOQ = \alpha$

$\therefore \alpha = \angle AOP = \dfrac{180 - \theta}{2}$ ($\angle$-sum of straight line AB)

$= 90 - \dfrac{\theta}{2}$

ii) $\triangle OPQ$ is ISO

$\therefore \angle OPQ = \dfrac{180 - \theta}{2}$ ($\angle$-sum of $\triangle OPQ$)

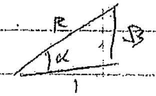
$= 90 - \dfrac{\theta}{2}$

$= \angle AOP$

$\therefore AB \parallel PQ$ (alt angles) c i) $\sqrt{3}\sin t + \cos t = R(\sin t \cos\alpha + \cos t \sin\alpha)$

$R\sin\alpha = 1$

$R\cos\alpha = \sqrt{3}$




$R = 2 \quad \alpha = \frac{\pi}{6}$

$2\sin(t+\frac{\pi}{6})$

ii) $2\sin(t+\frac{\pi}{6}) = \sqrt{3}$

$\sin(t+\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$(t+\frac{\pi}{6}) = \frac{\pi}{3}, \frac{2\pi}{3}$ $\qquad \frac{\pi}{6} \le t+\frac{\pi}{6} \le 2\pi+\frac{\pi}{6}$

$\therefore t = \frac{\pi}{6}, \frac{\pi}{2}$

d) $\sin 2\theta = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$

$\cos\theta(2\sin\theta - \sqrt{2}) = 0$

$\cos\theta = 0$ or $2\sin\theta = \sqrt{2}$

$\sin\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\frac{\pi}{4}, \frac{3\pi}{4}$

e) $1+6+15+\dots+n(2n-1) = \frac{1}{6}n(4n-1)(n+1)$

Prove true for $n=1$

LHS $= 1(2-1) = 1$ $\qquad$ RHS $= \frac{1}{6}(3)(2) = 1$

LHS = RHS

$\therefore$ true for $n=1$

Assume true for $n=k$

$1+6+15+\dots+k(2k-1) = \frac{1}{6}k(4k-1)(k+1)$

next →

continued

Prove true for $n=k+1$

i.e. $1+6+15+\dots+k(2k-1) + (k+1)(2k+1) = \frac{1}{6}(k+1)(4(k+1)-1)(k+2)$

$= \frac{1}{6}(k+1)(4k+3)(k+2)$

LHS $= \frac{1}{6}k(4k-1)(k+1) + (k+1)(2k+1)$ from assumption

$= (k+1)\left[\frac{1}{6}k(4k-1) + (2k+1)\right]$

$= (k+1)(\frac{2}{3}k^2 - \frac{1}{6}k + 2k + 1)$

$= (k+1)(\frac{2}{3}k^2 + \frac{11}{6}k + 1)$

$= (k+1)(\frac{1}{6})(4k^2+11k+6)$

$= (k+1)(\frac{1}{6})(4k+3)(k+2)$

= RHS

$(4k+3)(k+2)$

$(4k^2+8k+3k+6)$

$\therefore$ By mathematical induction, statement true for all positive integers of $x$.