J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION I

MATHEMATICS

HSC TASK 2

2012

Weighting: 20% (Assessment Mark)

NAME:

MARK: / 35

Time Allowed: 45 minutes

Topics: Parametric Equations and Applications of Calculus to the Physical World

Directions:

- There are two questions on this paper
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work

Marks

Question 1 (17 marks) Use a SEPARATE writing booklet.

(a) A piece of hot metal is placed in a room with a surrounding air temperature of 20°C and allowed to cool. It loses heat according to Newton's law of cooling, $\frac{dT}{dt} = -k(T - A)$ where T is the temperature of the metal in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant. After 6 minutes the temperature of the metal is 80°C, and after a further 2 minutes it is 50°C.

(i) Verify that $T = A + Be^{kt}$ satisfies the above equation. **1**

(ii) Show that $k = \frac{\log_e 2}{2}$. **3**

(iii) What is the value of B? **1**

(iv) Determine the initial temperature of the metal. **1**

(b) A particle is moving such that its displacement x metres at time t seconds is given by $x = 4\cos(3t - 1)$.

(i) Show that the motion is simple harmonic. **2**

(ii) Find the centre and the period of the motion. **2**

(iii) Find the speed of the particle when $x = 2$, correct to 3 significant figures. **1**

(c) A particle moves in a straight line with acceleration given by $\frac{d^2x}{dt^2} = 9(x-2)$ where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of O, so that $x = 4$, and has velocity $v = -6$.

(i) Show that $v^2 = 9(x-2)^2$. **2**

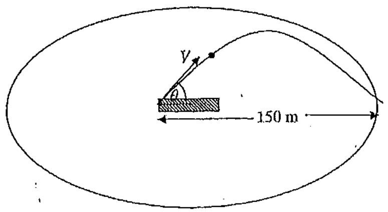
(ii) Find an expression for v and hence find x as a function of t. **2**

(iii) Explain whether the velocity of the particle is ever zero. **2**

Marks

Question 2 (18 marks) Use a SEPARATE writing booklet.

(a) A batsmen stands at the crease which is at the centre of a circular cricket ground of radius $150m$. He hits the ball at angle of elevation of θ with a speed of V *metres / second*. (Take $g = 10m/s^2$).



(i) Assuming the origin is at the point at which the ball is hit, show That the equations of motion are given by: **2**

$$x = Vt\cos\theta \text{ and } y = -\frac{gt^2}{2} + Vt\sin\theta.$$

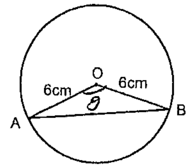
(ii) A batsmen hits the ball at an angle of elevation of 40^0 with a velocity of 36 m/s. What are the maximum height and the horizontal range of the path of this ball? (Answer to two decimal places.) **2**

(iii) A second batsmen hits the ball at an angle of elevation of 60^0. At what speed must the ball be hit in order to clear the boundary of 150 metres. (Answer to two decimal places) **2**

Marks

Question 2 continued

(b) O is the centre of a circle with radius 6cm and $\angle AOB = \theta$ radians. θ is increasing at a rate of 0.2 radians/second.



(i) Find the rate of change of the area of ΔAOB **3**

(ii) Find the rate of change of the area of the minor segment formed by AB when $\angle AOB = \frac{2\pi}{3}$. **3**

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

(i) If the chord PQ passes through the point $R(2a, 3a)$, show that $pq = p + q - 3$. **2**

(ii) If M is the midpoint of PQ, show that the coordinates of M are **2**

$$\left[a(pq+3), \frac{a}{2}\left\{(pq+3)^2 - 2pq\right\}\right].$$

(iii) Hence, find the locus of M. **2**

Q1: (a) (i) $T = A + Be^{-kt} \Rightarrow \frac{dT}{dt} = -k.Be^{-kt}$ where k is a positive constant

$= -k\left[Be^{-kt} + A - A\right]$

$= -k\left[T - A\right]$ as reqd.

III) $T = 20 + Be^{t} \times \sqrt{2}$

@ $t = 8$ $T = 50$

$50 = 20 + B\sqrt{2}\,e^{8}$

$B = \frac{30}{\sqrt{2}e^{8}}$ ✗

$k = \frac{1}{2}\ln 2$

$= \ln\sqrt{2}$

IV) @ $t = 0$

$T = 20 + \frac{30e^{t}}{e^{8}}$

$= 20 + \frac{30}{e^{8}}$

$\approx 20.01\,^\circ C$ ✓

(ii) $T = 20 + Be^{-kt}$

$t = 6$, $T = 80^\circ C$

$80 = 20 + Be^{-6k} \dots (I)$

$t = 8$, $T = 50^\circ C$

$50 = 20 + Be^{-8k}$

$\therefore$ From (I) $60 = Be^{-6k} \dots (III)$

From (II) $30 = Be^{-8k} \dots (IV)$

$\therefore \frac{III}{IV}$ ~~$2 = Be^{6k}$~~

$2 = e^{2k}$

$\ln 2 = \ln e^{2k}$

$\ln 2 = 2k$

$\frac{1}{2}\ln 2 = k$ as reqd.

(iii) Sub $k = \frac{1}{2}\ln 2$ into IV

$30 = Be^{-4\ln 2}$

$= Be^{\ln 2^{-4}}$

$30 = \frac{B}{16}$

$\underline{480 = B}$

b I) $x = 4\cos(3t-1)$

$\dot{x} = -12\sin(3t-1)$ ✓

$\ddot{x} = -36\cos(3t-1)$ ✓

$= -9x.$

II) ~~C.O.S @ x=0~~

$T = \frac{2\pi}{n}$ C.O.S @ $x = 0$ ✓

$= \frac{2\pi}{3}$ ✓

III) ~~$\frac{1}{2}v^2 = -\frac{9}{2}x$~~

~~$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -9x$~~

~~$\frac{1}{2}v^2 = -\frac{9}{2}x^2 + c$~~

~~$v^2 = -9x^2$~~

~~@ $x = 2$ $v^2 = 36$~~

$$\boxed{\begin{aligned} &v^2 = n^2(a^2 - x^2) \\ &@x{=}2\ \ v^2 = 9(16-4) \\ &v = 10.4\,ms^{-1} \end{aligned}}$$ ✓

c i) $\ddot{x} = 9(x-2)$ $\quad 9x - 18$

~~$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right)$~~

$\frac{1}{2}v^2 = \frac{9}{2}x^2 - 18x + C$

@ $t = 0$ $x = 4$ $v = -6$

$18 = 72 - 18 \times 4 + C$

$C = 18$ ✓

$v^2 = 9x^2 - 36x + 36$

$= 9(x^2 - 4x + 4)$

$= 9(x-2)^2$ ✓

II) $v = 3(x-2)$ ✗ $x = 4$ $v = 6$

$\frac{dx}{dt} = 3(x-2)$

$\frac{dt}{dx} = \frac{1}{3} \times \frac{1}{x-2}$

@ $t = 0$ $x = 4$

$0 = \frac{1}{3}\ln(2) + c$

$c = -\frac{1}{3}\ln 2$

$t = \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln 2$

$3t + \ln 2 = \ln(x-2)$ ✓

$2e^{3t} = x - 2$

$x = 2e^{3t} + 2$

III) @ $v = 0$ $0 = (x-2)^2$

$x = 2$ ✓

@ $x = 2$

$2 = 2e^{3t} + 2$

$e^{3t} \neq 0$ ✓

$\therefore v \neq 0$

@ $t =$

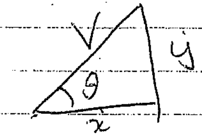
14

Question 2

qi) $\ddot{x}=0$ $\quad \ddot{y}=-g$

$\dot{x}=c$ $\quad \dot{y}=-gt+c$

@ $t=0$



$\dot{x}=V\cos\theta$ $\quad \dot{y}=-gt+V\sin\theta$

$x=Vt\cos\theta+c$ $\quad y=-\frac{1}{2}gt^2+Vt\sin\theta+c$

@ $t=0$ $\quad x=0$ $\quad y=0$

$x=vt\cos\theta$ $\quad y=-\frac{1}{2}gt^2+Vt\sin\theta$

ii) $\theta=40^\circ$ $\quad V=36$

$0=-5t^2+36t\sin 40$

$=t(-5t+36\sin 40)$

$5t=36\sin 40$

$t_1=4.63$

@ $\frac{1}{2}t=\frac{4.63}{2}$

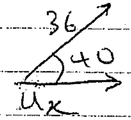
$y=-5t^2+t\,36\sin 40$

$=26.77m$

~~$x=Vt\cos\theta$~~

~~$=36\times 4.63$~~

~~$=167m$~~



$u_x=36\cos 40$

$x=u_x\times t$

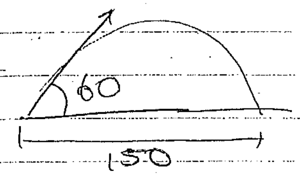
$=36\cos 40\times 4.63$

$=127.68m$

iii) $\theta=60$ $\quad x=Vt\cos\theta$

$x\geq 150$ $\quad$ ~~$150=V$~~

$t=$



~~$150=Vt\cos 60$~~

$0=-5t^2+Vt\sin 60$

$=-5t^2+\frac{\sqrt{3}}{2}Vt$

$x=Vt\cos\theta$

$150=Vt\cos\theta \Rightarrow t=\frac{150}{V\cdot\frac{1}{2}}=\frac{300}{V}$

$t=\frac{x}{V\cos\theta}$

$=\frac{300}{V}$

$0=-5\left(\frac{300}{V}\right)^2+\left(\frac{300}{V}\right)V\sin 60$

$=-5\left(\frac{300}{V}\right)^2+150\sqrt{3}$

$\left(\frac{300}{V}\right)^2=150\sqrt{3}$

$\frac{300}{V}=\sqrt{150\sqrt{3}}$

$V=\frac{300}{\sqrt{150\sqrt{3}}}$

$=53.73ms^{-1}$

$\frac{90000}{V^2}=150\sqrt{3}$

$\therefore V^2=\frac{9000}{15\sqrt{3}}$

$=\frac{600\sqrt{3}}{3}$

$=200\sqrt{3}$

$\therefore V=18.61$ m/s

(to 2d.p)

Question 2 Continued

b i) $\frac{d\theta}{dt} = 0.2$

$\frac{dA}{dt} = \frac{d\theta}{dt} \times \frac{dA}{d\theta}$

$A = \frac{1}{2} \times 36 \times \sin\theta$

$= 18\sin\theta$

$\frac{dA_1}{d\theta} = 18\cos\theta$

$\frac{dA_1}{dt} = 0.2 \times 18\cos\theta = 3.6\cos\theta$

II) $A = \cancel{6\pi} \times \frac{\theta}{2\pi}$

$\cancel{A = 18\theta}$

$A = \pi r^2 \times \frac{\theta}{2\pi}$

$= 18\theta$

$\frac{dA_2}{d\theta} = 18$

$\frac{dA_2}{d\theta} - \frac{dA_1}{d\theta} = $ Minor segment

$= 18 - 1.6\cos\frac{2\pi}{3}$

$= 18.8 \text{ cm}^2/\text{deg}$

$A_s = A_{segment} = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \cdot 36 (\theta - \sin\theta)$

$\frac{dA_s}{d\theta} = 1 - \cos\theta$

$\therefore \frac{dA_s}{dt} = \frac{dA_s}{d\theta} \times \frac{d\theta}{dt}$

$= (1 - \cos\theta) \cdot 0.2$

$= (1 - \cos\frac{2\pi}{3}) \cdot 0.2$

$= (1 + \frac{1}{2})\ 0.2$

$= \frac{3}{2} \times \frac{1}{5}$

$= \frac{3}{10}$ units^2/s

c I) $m = \frac{a(p^2 - q^2)}{2a(p-q)}$

$= \frac{p+q}{2}$

$y - ap^2 = \frac{p+q}{2}(x - 2ap)$

@ $(2a, 3a)$

$3a - ap^2 = \frac{p+q}{2}(2a - 2ap)$

$6a - \cancel{2ap^2} = 2a(p+q) - 2ap(p+q)$

$= 2ap + 2aq - \cancel{2ap^2} - 2apq$

$6a = 2p + 2q - 2pq$

$3 = p + q - pq$

$pq = p + q - 3$

II) $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(\frac{2a(p+q)}{2}, \frac{a(p^2+q^2)}{2}\right)$

$= \left(a(pq+3), \frac{a}{2}\left[(pq+3)^2 - 2pq\right]\right)$

$= \left(a(pq+3), \frac{a}{2}\left[(pq+3)^2 - 2pq\right]\right)$

III) $x = a(pq+3)$

$\frac{x}{a} = pq + 3 \longrightarrow$

$y = \frac{a}{2}\left[(pq+3)^2 - 2pq\right]$

$y = \frac{a}{2}\left(\left(\frac{x}{a}\right)^2 - 2\left(\frac{x}{a} - 3\right)\right)$

$= \frac{a}{2}\left(\frac{x^2}{a^2} - \frac{2x}{a} + 6\right)$

$= \frac{x^2}{2a} - x + 3a$

locus of M is a parabola