J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION 2

HSC TASK #3

MATHEMATICS

2012

Weighting: 15% (HSC Assessment Mark)

NAME: ______________________

MARK: / 33

Time Allowed:	45 minutes
Topics:	Integration, Conics & Volumes

Directions:

- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks <u>may not</u> be awarded for careless or badly arranged work

Question 1 *(15 marks)* **Marks**

(a) By completing the square, find $\int \frac{2}{x^2+4x+13}dx$ **2**

(b) Use integration by parts to evaluate $\int 3xe^x\,dx$. **2**

(c) (i) Find real numbers a, b and c such that

$$\frac{7x+4}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$ **2**

(ii) Hence find $\int \frac{7x+4}{(x^2+1)(x+2)}dx$ **2**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + 2\sin\theta + 3}d\theta$ **3**

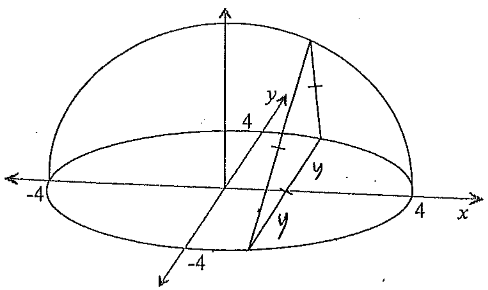
(e) (i) Let $I_n = \int_0^x \cos^n t\ dt$, where $0 \le x \le \frac{\pi}{2}$. **2**

Show that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$ with $n \ge 2$.

(ii) Hence, otherwise, find the exact value I_4. **2**

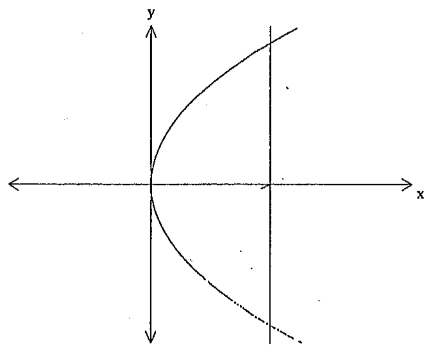
Question 2 *(18 marks)* **Marks**

(a)



The diagram above shows a solid which has the circle $x^2+y^2=16$ as its base. The cross-section perpendicular to the x-axis is an equilateral triangle. Calculate the volume of the solid. **4**

(b) A solid is formed by rotating the region enclosed by the parabola $y^2=4ax$, its vertex (0,0) and the line $x=a$, about the x-axis. **3**



Find the volume of this solid using the method of cylindrical shells.

(c) A hyperbola has equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(i) Verify that the point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola. **1**

(ii) The normal to the hyperbola at P cuts the x-axis at M and N is the foot of the perpendicular from P to the x-axis. Show that the equation of the normal at P is: **2**

$$ax\sin\theta+by=\left(a^2+b^2\right)\tan\theta$$

(iii) Show that $OM=e^2ON$, where O is the origin and e is the eccentricity of the hyperbola. **3**

(iv) Prove that $SM=e\times SP$, where S is the focus of the hyperbola **3**

(d) The point P lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where $a>b>0$. The tangent at P meets the tangents at the ends of the major axis at R and T.

Given the equation of the tangent at P is $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$

Show that RT subtends a right angle at the focus. **2**

13

(a) $2\int \frac{1}{(x+2)^2+9}$

$= \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$

b) $\int 3xe^x\,dx$

~~$du = e^x dx$~~

$u = 3x$ $\quad dv = e^x\,dx$

$du = 3dx$ $\quad v = e^x\,dx$

$I = 3xe^x - 3\int e^x\,dx$

$= 3xe^x - 3e^x + C$

c) i) $7x+4 = (ax+b)(x+2) + (x^2+1)c$

$= ax^2+bx+2ax+2b+cx^2+c$

$= (a+c)x^2 + (b+2a)x + (2b+c)$

equate $a+c=0 \quad b+2a=7 \quad 2b+c=4$

$a=-c \quad b-2c=7 \quad 14+4c+c=4$

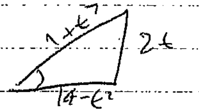
$b=7+2c \quad 5c=-10$

$a=2 \quad b=3 \quad c=-2$

ii) $\int \frac{2x+3}{x^2+1} - \frac{2}{x+2}\,dx$

$= \int \frac{2x}{x^2+1} + \frac{3}{x^2+1} - \frac{2}{x+2}\,dx$

$= \ln|x^2+1| + 3\tan^{-1}x - 2\ln|x+2| + C$



d) $t = \tan\frac{\theta}{2}$

$d\theta = \frac{2dt}{1+t^2}$

$\int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2(2t)}{1+t^2} + 3}$

$= \int_0^1 \frac{2dt}{1-t^2+4t+3+3t^2}$

$= \int_0^1 \frac{2}{2t^2+4t+4} = \int \frac{1}{t^2+2t+2}\,dt = \int_0^1 \frac{1}{(t+1)^2+1}\,dt$

$= \left[\tan^{-1}(t+1)\right]_0^1$

$= \tan^{-1}2 - \tan^{-1}1 \Rightarrow \frac{\pi}{4}$

e) i) $I_n = \int_0^x \cos^n t\,dt$

$= \int_0^x \cos t \,.\, \cos^{n-1}t\,dt$

$u = (\cos t)^{n-1} \quad dv = \cos t\,dt$

$du = (n-1)(-\sin t)(\cos t)^{n-2} \quad v = \sin t$

$I_n = \sin t\cos^{n-1}t + (n-1)\int(1-\cos^2 t)(\cos^{n-2}t)\,dt$

$= \sin t\cos^{n-1}t + (n-1)\int \cos^{n-2}t - \cos^n t\,dt$

$= \sin t\cos^{n-1}t + (n-1)\left[I_{n-2} - I_n\right]$

$\left[(n-1)+1\right]I_n = \sin t\,\cancel{\cos^{n-1}t} + I_{n-2}(n-1)$

$nI_n = I_{n-2}(n-1)$

$\therefore I_n = \frac{n-1}{n}I_{n-2}$

ii) $I_4 = \left(\frac{3}{4}\right)I_2$

$I_2 = \frac{1}{2}I_0$

$I_0 = \int_0^x 1\,dt$

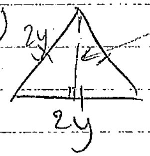
$= x$

$I_4 = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)x \qquad x = \frac{\pi}{2}$

$= \frac{3}{8}x$

(12)

Q2

a)


$\sqrt{4y^2 - y^2}$

$= \sqrt{3y^2}$

$= y\sqrt{3}$

$\frac{1}{2} AB \sin C$

$\frac{1}{2} \times 4y^2 \frac{\sqrt{3}}{2}$.

$A(x) = y^2\sqrt{3} = (16 - x^2)\sqrt{3}$ ✓

$\delta V = \lim_{\Delta x \to 0} \sum_{x=0}^{4} A(x)\ \Delta x$

$V = \int_0^4 16\sqrt{3} - x^2\sqrt{3}\ dx$

$= \left[(16\sqrt{3})x - \frac{\sqrt{3}}{3}x^3\right]_0^4$ ✗

$= \left[(16\sqrt{3})4 - \frac{64\sqrt{3}}{3}\right]$ ✓

$= 64\sqrt{3} - \frac{64\sqrt{3}}{3}$

$= 64\sqrt{3}\left(1 - \frac{1}{3}\right)$

$V = \frac{128\sqrt{3}}{3}\ u^3$

b) $V = 2\pi \int rh\ dx$

$= 2\pi \int x(4ax)\ dx$

$= \cancel{2\pi \int 8a}$

$= 8a\pi \int_0^a x^2\ dx$ ✗

$= 8a\pi \left[\frac{1}{3}x^3\right]_0^a$

$= 8a\pi \left[\frac{a^3}{3}\right]$

$= \frac{8a^4\pi}{3}$

c) I) @ $P \Rightarrow H$

$LHS = \frac{a^2\sec^2\theta}{a^2} - \frac{b^2\tan^2\theta}{b^2}$ ✓

$= 1$

$= RHS \therefore P$ lies on H

II) $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{b^2 x}{a^2 y}$

$m = -\frac{y a^2}{x b^2}$ ✓

@ $P = -\frac{a\tan\theta}{b\sec\theta}$

$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$ ✓

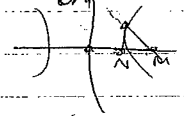
$by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$= -ax\sin\theta + a^2\tan\theta$

$ax\sin\theta + by = (a^2 + b^2)(\tan\theta)$

III) @ $y=0$ $x = \frac{(a^2+b^2)\tan\theta}{a\sin\theta}$

RTP: $\frac{OM}{ON} = e^2$



$= \frac{(a^2+b^2)}{a}\sec\theta$

$M\left(\frac{(a^2+b^2)}{a}\sec\theta, 0\right)$; ✓

$N(a\sec\theta, 0)$

$OM = \frac{a^2+b^2}{a}\sec\theta$ ✓

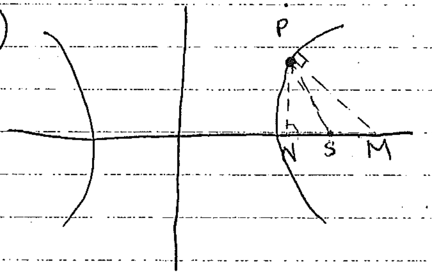
$ON = a\sec\theta$ $b^2 = a^2(e^2 - 1)$

$\frac{OM}{ON} = \frac{a^2+b^2}{a^2}$

$= \frac{a^2 + a^2(e^2-1)}{a^2}$ ✓

$= 1 + e^2 - 1$

~~IV~~)



$\frac{SM}{SP} = e \quad \frac{SM^2}{SP^2} = e^2$

$S(ae, 0)$; $P(a\sec\theta, b\tan\theta)$

$SM = \frac{(a^2+b^2)}{a}(\sec\theta) - ae$

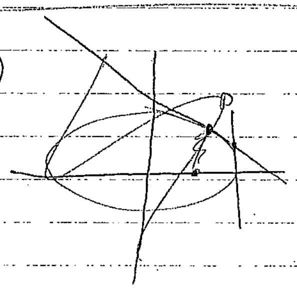
$ = \frac{(a^2+b^2)\sec\theta - a^2 e}{a}$ ✓

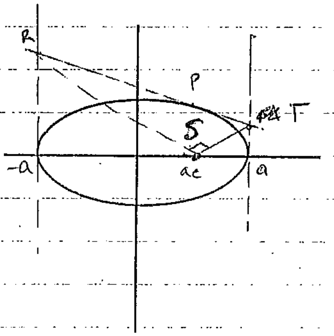
$SP = \sqrt{(b\tan\theta)^2 - (a\sec\theta - ae)^2}$

$ = \sqrt{b^2\tan^2\theta - a^2\sec^2\theta - 2a^2 e\sec\theta}$

(continued on extra sheet)

d)





$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

@ $x = a$ (for T) $\cos\theta + \frac{y}{b}\sin\theta = 1$

$y = \frac{b(1-\cos\theta)}{\sin\theta}$

@ $x = -a$ (for R) $-\cos\theta + \frac{y}{b}\sin\theta = 1$

$y = \frac{b(1+\cos\theta)}{\sin\theta}$ ✓

$T\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$; $R\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$

$M_{TS} = \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a - ae}$ $\quad M_{RS} = \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a - ae}$ ✓

$M_{TS} \times M_{RS}$

$\frac{\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}}{(a-ae+a)(-ae-a)}$ $\quad b^2 = a^2(1-e^2)$

$= \frac{a^2(1-e^2)}{a^2e^2 - a^2} = -1$ $\therefore$ RT subtends S at 90°

~~IV~~ Q2 C ~~IV~~ Question 2c IV

$\frac{(SM)^2}{(SP)^2} = e^2$

LHS $= \frac{\left[\frac{(a^2+b^2)\sec\theta - a^2 e}{a}\right]^2}{b^2\tan^2\theta - a^2\sec^2\theta - 2a^2 e\sec\theta - a^2e^2}$ $\quad b^2 = a^2(e^2-1)$

$= \frac{(a^2+b^2)^2\sec^2\theta - 2(a^2+b^2)a^2 e + a^4e^2}{a^2(b^2\tan^2\theta - a^2\sec^2\theta - 2a^2e\sec\theta - a^2e^2)}$ $\quad a^2 + a^2(e^2-1)$

$a^2(\cancel{1}$

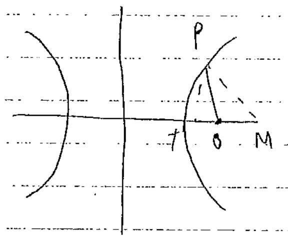
a^2e^2

$= \frac{(\cancel{a^2 - a^2(e^2-1)})^2\sec^2\theta - 2(a^2e^2)a^2e + a^4e^2}{a^2(b^2\tan^2\theta - a^2\sec^2\theta - 2a^2e\sec\theta - a^2e^2)}$ ✓

$= \frac{e^4\sec^2\theta - 2a^2e^3 + a^2e^2}{(b^2\tan^2\theta - a^2\sec^2\theta - 2a^2e\sec\theta - a^2e^2)}$

$= e^2(e^2\sec^2\theta - 2a^2e + a^2)$

✗



$\frac{SI}{SP} = e$