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MARCELLIN COLLEGE RANDWICK

YEAR 12 HSC

ASSESSMENT TASK # 2

EXTENSION TWO

MATHEMATICS

2007

Weighting: 20% of HSC Assessment Mark.

STUDENT NAME: ____________________ **MARK:** ____________ / 43

PERCENTAGE: ____________ %

RANK ON THIS TASK: ________ / 10

Time Allowed: 1 ½ hours

Directions:
* Answer all questions on separate lined paper.
* Begin each question on a new page.
* Show all necessary working.
* Marks may not be awarded for careless or badly arranged work.

QUESTION ONE (16 MARKS) Begin answers on a new page

Marks

(a) i) Find $\int (\sec x + \tan x)^2 dx$ 2

ii) Find $\int \frac{1-x}{1-\sqrt{x}} dx$ 2

(b) Use the substitution $u = e^x + 1$ to find $\int \frac{e^{2x}}{(e^x+1)^2} dx$ 2

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2\sin x + 3} dx$ 4

giving the answer correct to three significant figures

(d) i) Find the exact value of $\int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx$ 2

ii) If $I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x^2} dx$ for $n = 0, 1, 2, \ldots$ show that 4

$$I_{n-2} - I_n = \frac{1}{(n-1)2^{n-1}}$$ for $n = 2, 3, 4, \ldots$ Hence find the exact value

of $\int_0^{\frac{1}{2}} \frac{x^4}{1-x^2} dx$

QUESTION TWO (12 MARKS) **Begin answers on a new page**

Marks

(a) The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at (0,0). Show that **3**

$\tan\theta\tan\phi = -\frac{a^2}{b^2}$

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

i) Show that the gradient of the tangent at P is $\frac{-b\cos\theta}{a\sin\theta}$ **1**

ii) Show that the equation of the tangent to the ellipse at P is given by: **1**

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

[You must show all essential working to gain the mark for this question]

iii) The tangent at P cuts the y axis at B and Y is the foot of the perpendicular from P to the y axis. **2**

Show that $OB\times OY = b^2$

(c) The point $P(x_o, y_o)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b >0

i) Write down the equations of the two asymptotes of the hyperbola **1**

ii) Show that the acute angle α between the two asymptotes satisfies **1**
$\tan\alpha = \frac{2ab}{a^2-b^2}$

iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that: **3**

$MP.NP = {a^2b^2}/{a^2+b^2}$

QUESTION THREE (15 MARKS) **Begin answers on a new page**

Marks

(a) Factorise the polynomial $P(z)=z^4-2z^2+12z-8$ fully over $\mathbb{C}$ given that $P(1+\sqrt{3}i)=0$ **4**

(b) Given that the quartic polynomial $p(x)=x^4-5x^3-9x^2+81x-108$ has a zero of multiplicity three, factorise the polynomial completely and find all its zeroes. **3**

(c) The cubic equation $x^3-x^2+4x-2=0$ has roots α,β and γ

i) Find the equation whose roots are α^2,β^2 and γ^2 **2**

ii) Find the value of $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$ **1**

(d) i) The equation $x^3+px^2+qx+r=0$ (where p, q, r are non zero) has roots α,β,γ such that $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. Show that $\beta=\frac{-3r}{q}$ **3**

ii) The equation $x^3-26x^2+216x-576=0$ has roots $\alpha,\beta, and\ \gamma$ such that $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. Find the values of $\alpha,\beta, and\ \gamma$ **2**

2007 HALF-YEARLY EXAM

ASSESSMENT TASK 2

EXTENSION 2 MATHS

Question One

(a) (i) $\int (\sec x + \tan x)^2 \, dx$

$= \int \sec^2 x \, dx + 2\int \sec x \tan x \, dx + \int \tan^2 x \, dx$ ← 1 mark

$= \tan x + 2\sec x + \int \sec^2 x - 1 \, dx$

$= \tan x + 2\sec x + \tan x - x + C$

$= 2(\sec x + \tan x) - x + C$ ← 1 mark

(ii) $\int \frac{1-x}{1-\sqrt{x}} \, dx = \int \frac{(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}} \, dx$ ← 1 mark

$= \int 1 + x^{\frac{1}{2}} \, dx$

$= x + \frac{2x^{3/2}}{3} + C$

$= x + \frac{2x\sqrt{x}}{3} + C$ ← 1 mark

cont...

(b) $\int \frac{e^{2x}}{(e^x+1)^2} \, dx$

Let $u = e^x + 1$

$\therefore \frac{du}{dx} = e^x$

$\therefore dx = \frac{du}{e^x}$

$= \int \frac{u-1}{u^2} \, du$ ← 1 mark

$= \int \frac{1}{u} - \frac{1}{u^2} \, du$

$= \ln|u| + u^{-1} + C$

$= \ln|e^x+1| + \frac{1}{e^x+1} + C$ ← 1 mark

(c) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2\sin x + 3} \, dx$

Let $t = \tan \frac{x}{2}$

$\therefore \frac{x}{2} = \tan^{-1} t$

$\therefore x = 2\tan^{-1} t$

$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$

$\therefore dx = \frac{2dt}{1+t^2}$

when $x = \frac{\pi}{2}$, $t = 1$

when $x = 0$, $t = 0$

Now $\cos x + 2\sin x + 3$

$= \frac{1-t^2 + 4t + 3(1+t^2)}{1+t^2}$

$= \frac{2(t^2+2t+2)}{1+t^2}$

↑ 1 mark

$\int_0^1 \frac{1+t^2}{2(t^2+2t+2)} \cdot \frac{2dt}{1+t^2}$

$\int_0^1 \frac{1}{t^2+2t+2} \, dt$ ← 1 mark

$\int_0^1 \frac{1}{1+(t+1)^2} \, dt$

$\left[\tan^{-1}(t+1)\right]_0^1$ ← 1 mark

$= \tan^{-1} 2 - \tan^{-1} 1$

$= \tan^{-1} 2 - \frac{\pi}{4}$ ← 1 mark

$= 0.322$ (3 sf)

1 cont...

(d)(i) $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx$

$= \int_0^{\frac{1}{2}} \frac{1}{(1-x)(1+x)}\,dx$

Let $\frac{1}{(1-x)(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1+x)}$

$\therefore 1 = A(1+x)$ and $B(1-x)$

Let $x = 1 \quad \therefore A = \frac{1}{2}$

Let $x = -1 \quad \therefore B = \frac{1}{2}$

$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{1-x} + \frac{1}{1+x}\,dx$

$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-1}{1-x}\,dx$ — 1 mark

$= \frac{1}{2}\left[\ln(1+x)\right]_0^{\frac{1}{2}} - \frac{1}{2}\left[\ln(1-x)\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left[\ln\frac{3}{2}\right] - \frac{1}{2}\left[\ln\frac{1}{2}\right]$

$= \frac{1}{2}\ln 3 \quad \longleftarrow$ 1 mark

(ii) $I_{n-2} - I_n$

$= \int_0^{\frac{1}{2}} \frac{x^{n-2}}{1-x^2}\,dx - \int_0^{\frac{1}{2}} \frac{x^n}{1-x^2}\,dx$

$= \int_0^{\frac{1}{2}} \frac{x^{n-2} - x^n}{1-x^2}\,dx$

$= \int_0^{\frac{1}{2}} \frac{x^{n-2}(1-x^2)}{1-x^2}\,dx$

$= \int_0^{\frac{1}{2}} x^{n-2}\,dx \quad \longleftarrow$ 1 mark

$= \left[\frac{x^{n-1}}{n-1}\right]_0^{\frac{1}{2}}$

$= \frac{\left(\frac{1}{2}\right)^{n-1}}{n-1}$

$= \frac{1}{(n-1)2^{n-1}}$

1 mark

Q1 (d) cont...

Now $I_0 - I_2 = \frac{1}{2}$ and $I_2 - I_4 = \frac{1}{24}$

$\therefore I_0 - I_2 + (I_2 - I_4) = \frac{1}{2} + \frac{1}{24}$

$\therefore I_0 - I_4 = \frac{13}{24}$

$\therefore I_4 = I_0 - \frac{13}{24} \quad \longleftarrow$ 1 mark

$= \frac{1}{2}\ln 3 - \frac{13}{24}$

from part (i)

$\therefore \int_0^{\frac{1}{2}} \frac{x^4}{1-x^2}\,dx = \frac{1}{2}\ln 3 - \frac{13}{24} \quad \longleftarrow$ 1 mark

Alternatively:

$I_n - I_{n-2} = \frac{-1}{(n-1)2^{n-1}}$

$\therefore I_n = \frac{-1}{(n-1)2^{n-1}} + I_{n-2} \quad \longleftarrow$ 1 mark

$\therefore I_4 = -\frac{1}{24} + I_2$

$= -\frac{1}{24} + \left[-\frac{1}{2} + I_0\right]$

$= -\frac{1}{24} - \frac{1}{2} + \frac{1}{2}\ln 3 \quad \longleftarrow$ from part (i)

$= \frac{1}{2}\ln 3 - \frac{13}{24} \quad \longleftarrow$ 1 mark

Question Two

(a) m of $OP = \dfrac{a\cos\theta}{b\sin\theta}$

and m of $OQ = \dfrac{a\cos\phi}{b\sin\phi}$ — 1 mark

Now $OP \perp OQ \therefore m$ of $OP \times m$ of $OQ = -1$

ie. $\dfrac{a^2\cos\theta\cos\phi}{b^2\sin\theta\sin\phi} = -1$ ← 1 mark

$\therefore \dfrac{b^2\sin\theta\sin\phi}{a^2\cos\theta\cos\phi} = -1$

$\therefore \dfrac{b^2}{a^2}\tan\theta\tan\phi = -1$

$\therefore \tan\theta\tan\phi = -\dfrac{a^2}{b^2}$ — 1 mark

(b)(i) $x = a\cos\theta$ and $y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta \qquad \dfrac{dy}{d\theta} = b\cos\theta$

Now $\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}$

$= \dfrac{-b\cos\theta}{a\sin\theta}$ — 1 mark

Q2(b) continued ...

(ii) Eqn of T at P:

$y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\therefore ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$\therefore ay\sin\theta + bx\cos\theta = ab\sin^2\theta + ab\cos^2\theta$

$= ab(\sin^2\theta + \cos^2\theta)$

$= ab$

$\therefore \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ — 1 mark for correctly deriving

(iii) Let $x = 0 \therefore y = b/\sin\theta \quad \therefore$ coords of B are $\left(0, \dfrac{b}{\sin\theta}\right)$ ← 1 mark

$\therefore OB = b/\sin\theta$

Also, $OY = b\sin\theta$

$\therefore OB \cdot OY = b^2$ — 1 mark for correctly deriving

(c)(i) $y = \dfrac{bx}{a}$ and $y = -\dfrac{bx}{a}$ ← 1 mark

(ii) $\tan\alpha = \left|\dfrac{\frac{b}{a} + \frac{b}{a}}{1 - b^2/a^2}\right|$

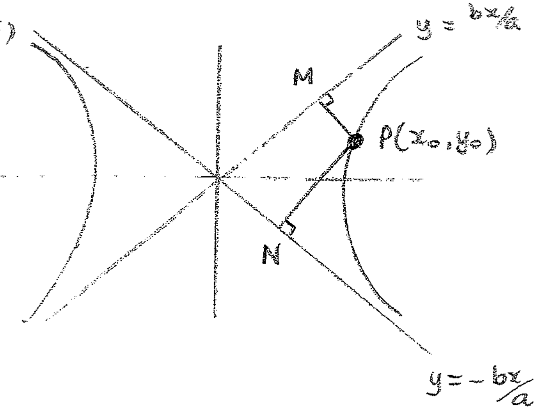
$= \left|\dfrac{2b/a}{\frac{a^2-b^2}{a^2}}\right|$

$= \dfrac{2ab}{a^2-b^2}$ — 1 mark

NB. We can leave off absolute value signs as $a > b$ (defn) and a and $b > 0$ (defn)

Q2(c) cont...

(iii)



$y = \frac{bx}{a} \rightarrow bx - ay = 0$

$y = -\frac{bx}{a} \rightarrow bx + ay = 0$

$$\text{dist } MP = \frac{|b(x_0) - a(y_0)|}{\sqrt{b^2 + a^2}} \quad \text{and dist } NP = \frac{|b(x_0) + a(y_0)|}{\sqrt{b^2 + a^2}}$$

1 mark for correct dist. formulae

$$\text{Now } MP.NP = \frac{|b^2(x_0)^2 - a^2(y_0)^2|}{b^2 + a^2}$$

1 mark

Now (x_0, y_0) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

ie. $\frac{(x_0)^2}{a^2} - \frac{(y_0)^2}{b^2} = 1$

$\therefore b^2(x_0)^2 - a^2(y_0)^2 = a^2 b^2$

1 mark

$$\therefore MP.NP = \frac{|a^2 b^2|}{b^2 + a^2}$$

But $a^2b^2 > 0 \quad \therefore MP.NP = \frac{a^2 b^2}{a^2 + b^2}$

Question Three

(a) If $1 + \sqrt{3}i$ is a root of $P(z)$, and all coeff's of $P(z)$ are real, $1 - \sqrt{3}i$ is also a root (as roots occur in conjugate pairs)

1 mark

Now $(z - \alpha)(z - \beta)$

$= z^2 - (\alpha + \beta)z + \alpha\beta$

$= z^2 - 2z + 4 \quad \leftarrow$ 1 mark

By long division or otherwise:

$P(z) = (z^2 - 2z + 4)(z^2 + 2z - 2) \quad \leftarrow$ 1 mark

$= (z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i)\left[(z+1)^2 - 3\right]$

$= (z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i)(z + 1 + \sqrt{3})(z + 1 - \sqrt{3})$

1 mark

(b) If $P(x)$ has a zero of multiplicity 3, this zero must also satisfy $P''(x)$

Now $P'(x) = 4x^3 - 15x^2 - 18x + 81$

and $P''(x) = 12x^2 - 30x - 18$

$= 6(2x^2 - 5x - 3)$

$= 6(2x + 1)(x - 3)$

1 mark

$\therefore$ zero of multiplicity 3 is $x = 3$

1 mark

$\therefore P(x) = (x - 3)^3(x + 4)$

and zeroes are $3, 3, 3, -4$

1 mark

Q3 cont...

c)(i) Roots of required equation are of the form

$x = \alpha^2$, $x = \beta^2$ and $x = \gamma^2$

ie. $\alpha = \sqrt{x}$, $\beta = \sqrt{x}$ and $\gamma = \sqrt{x}$

But α solves $x^3 - x^2 + 4x - 2 = 0$

$\therefore (\sqrt{x})^3 - (\sqrt{x})^2 + 4\sqrt{x} - 2 = 0$

$\therefore x\sqrt{x} + 4\sqrt{x} = x + 2$ ← 1 mark for correct to this line.

$\therefore \sqrt{x}(x+4) = x+2$

$\therefore x(x+4)^2 = (x+2)^2$

$\therefore x^3 + 8x^2 + 16x = x^2 + 4x + 4$

$\therefore x^3 + 7x^2 + 12x - 4 = 0$ will have roots of the form α^2, β^2 and γ^2 ← 1 mark for correct equation

(ii) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$

$= \Sigma \alpha\beta$

$= c/a$

$= 12$ ← 1 mark

Q3. cont...

(d)(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= -q/r$ (1) ← 1 mark for correct relationship

But $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are consecutive terms of an AP

$\therefore \frac{1}{\beta} - \frac{1}{\alpha} = \frac{1}{\gamma} - \frac{1}{\beta}$

$\therefore \frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$ (2) ← 1 mark for correct relationship

Using (1) and (2):

$\frac{2}{\beta} + \frac{1}{\beta} = -\frac{q}{r}$

$\therefore \frac{3}{\beta} = -\frac{q}{r}$

$\therefore \beta = -3r/q$

1 mark for correctly deriving

Q3. cont...

(ii) Using $\beta = -\frac{3r}{q}$ from part (i):

$\beta = 8$ ⟵ 1 mark for correct value of β

Now $\alpha + \beta + \gamma = 26$ and $\alpha\beta\gamma = 576$

$\therefore \alpha + \gamma = 18$ $\therefore \alpha\gamma = 72$

Solving simultaneously:

$\alpha = 6$ and $\gamma = 12$

or $\alpha = 12$ and $\gamma = 6$

$\therefore$ zeros are $6, 8, 12$ ⟵ 1 mark for correct zeros