J.M.J.Ch.

MARCELLIN COLLEGE RANDWICK

YEAR 12 HSC

ASSESSMENT TASK # 2

MATHEMATICS

2007

Weighting: 30% of HSC Assessment Mark.

STUDENT NAME: ____________________ **MARK:** / 72

Time Allowed: 2 hours.

Directions:
- Answer all questions on the booklets provided.
- Show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Begin your answers to each new question in a new answer booklet.

Structure: 6 questions each worth 12 marks – Total 72 marks.

OUTCOMES TO BE ASSESSED:

- **H1** – seeks to apply mathematical techniques to problems in a wide range of practical contexts
- **H2** – constructs arguments to prove and justify results
- **H4** – expresses practical problems in mathematical terms based on simple given models
- **H5** – applies appropriate techniques from the study of calculus, geometry and series to solve problems
- **H6** – uses the derivative to determine the features of the graph of a function
- **H7** – uses the features of a graph to deduce information about the derivative
- **H8** – uses techniques of integration to calculate areas and volumes
- **H9** – communicates using mathematical language, notation, diagrams and graphs

Question 1

a. Solve $\frac{3t}{t-5}=\frac{2}{5}$. 2

b. Solve $|2-3x|\geq 1$. 3

c. Factorise $1-27a^3$. 1

d. If $\frac{5}{\sqrt{5}-2}=a+b\sqrt{5}$, find the value of a and b. 2

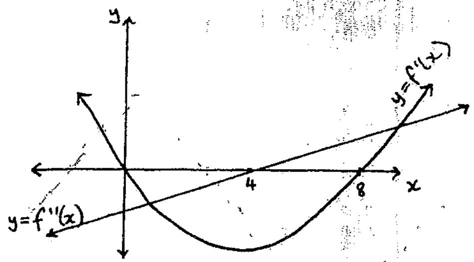
e. Evaluate $\sqrt{\frac{(3\cdot 4)^4}{15\cdot 6\times 12\cdot 8}}$, correct to 3 significant figures. 2

f. Find the primitive of x^3+5. 1

g. Find the derivative of $y=\frac{1}{3}x^3-\frac{1}{4}x^2+7\sqrt{x}-9$. 1

Question 2

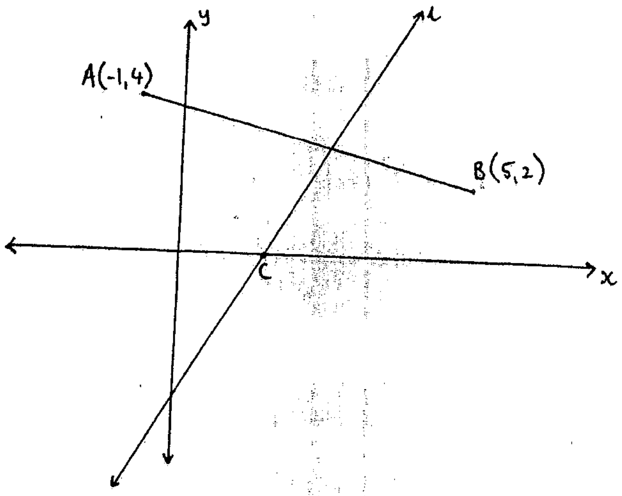
a.



The graphs of the first and second derivatives of the curve $y = f(x)$ are shown in the diagram. **4**

Write down the x-coordinates of the stationary points and determine their nature.

b. The diagram below shows the points A(-1,4) and B(5,2). The line l has the equation $3x - y - 3 = 0$ and cuts the x-axis at C.



i. Show that the length of AB is $2\sqrt{10}$ units. **1**

ii. Find the coordinates of M, the midpoint of AB. **1**

iii. Find the gradient of AB. **1**

iv. Show that the equation of AB is $x + 3y - 11 = 0$. **1**

v. Prove that l is the perpendicular bisector of AB. **2**

vi. Find the coordinates of C. **1**

vii. Write down the equation of the circle with AB as the diameter. **1**

Question 3

a. Find the derivative of $y = \dfrac{3x-5}{2x+3}$. **2**

b. Find the equation of the normal to the curve $y = x^2 - 4x$ at the point (1,-3). **3**

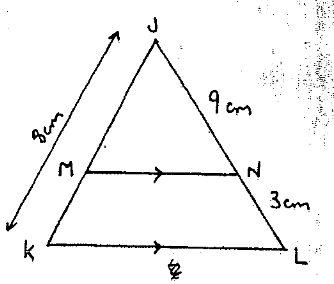
c. Find the primitive function of $3x^2 - \dfrac{2}{x^3} + 2$. **2**

d. Evaluate: i) $\int \sqrt{3-x}\,dx$ **2**

ii) $\displaystyle\int_{-1}^{2} \frac{x^4 + 3x^3 + x^2}{x}\,dx$ **3**

Question 4

a.



In the triangle JKL, $JK = 8$ cm, $JN = 9$ cm and $NL = 3$ cm.

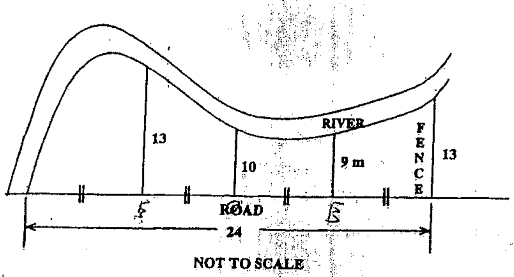
i) Prove that $\Delta JMN \parallel\!| \Delta JKL$. **2**

ii) Find the length of MK. **2**

b. For the function $f(x) = x^3 - 12x$, **6**

i) show that $f(x)$ is an odd function.
ii) where $y = f(x)$ crosses the x-axis.
iii) find the coordinates of the stationary points and determine their nature.
iv) find any points of inflexion.
v) sketch $y = f(x)$ showing all the above features.

c. **2**

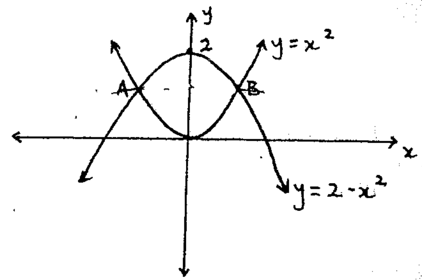


Wasteland bordering a river bank and a straight road was fenced off and used as a recreational park. Perpendicular distances from the road to the river bank are shown on the diagram. Use Simpson's rule with 5 function values to approximate the area of the recreational park.

Question 5

a. An arithmetic series has a third term of 14 and a seventh term of 30. Find the first term and common difference of the series. **3**

b. The curves $y = x^2$ and $y = 2 - x^2$ intersect at two points, A and B.



i. Find the coordinates of A and B. **2**

ii. Find the area bounded by the curves $y = x^2$ and $y = 2 - x^2$. **3**

c. Michael deposits \$1000 at the end of every year into a superannuation fund. The fund pays interest at a rate of 7% p.a. If he does this for 20 years, how much would be in his account after his final deposit? **4**

Question 6

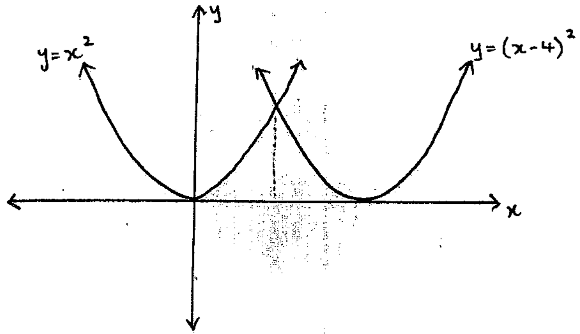
a. The population (P) of a coastal town is increasing at a decreasing rate. **2**

Comment on $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$ for this function.

b. i) Write $0.3\dot{4}$ as the sum of a geometric series. **1**

ii) Hence write $0.3\dot{4}$ as a fraction in simplest form. **2**

c. **4**



The area bounded by the curves $y = x^2$ and $y = (x-4)^2$ is rotated about the x-axis.

Find the volume of the solid formed.

d. Consider the function $f(x) = \frac{2x}{(x^2+1)^2}$.

i) Find $\frac{d}{dx}\left(\frac{1}{x^2+1}\right)$. **1**

ii) Use your answer from part i, to find the exact value of $\int_1^3 \frac{2x}{(x^2+1)^2}dx$. **2**

Yr 12 Half yearly solutions 2007

Question 1

a) $\frac{3t}{t-5} = \frac{2}{5}$

$15t = 2t - 10$

$13t = -10$

$t = -\frac{10}{13}$

(2 marks for correct answer)
1 mark for writing without denom.
1 mark for correct answer

b) $|2-3x| \geq 1$

$2-3x \geq 1 \qquad 2-3x \leq -1$

$1 \geq 3x \qquad 3 \leq 3x$

$\frac{1}{3} \geq x \qquad 1 \leq x$

(3 marks for correct answers)
1 mark for correct statements
1 mark for each correct answer

c) $1 - 27a^3$

$= 1 - (3a)^3$

$= (1-3a)(1+3a+9a^2)$

1 mark for correct answer

d) $\frac{5}{\sqrt{5}-2}$

$= \frac{5}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$

$= \frac{5\sqrt{5}+10}{5-4}$

$= 5\sqrt{5} + 10$

$\therefore a = 10, \; b = 5$

1 mark for each correct answer

e) $\sqrt{\frac{(3.4)^4}{15.6 \times 12.8}} = \sqrt{0.66928878}$

$= 0.818$

2 marks for correct answer
1 mark if not correct to 3 sig. fig

f) $x^3 + 5$

$= \frac{x^4}{4} + 5x + c$

1 mark for correct answer

g) $y = \frac{1}{3}x^3 - \frac{1}{4}x^2 + 7\sqrt{x} - 9$

$y = \frac{1}{3}x^3 - \frac{1}{4}x^2 + 7x^{1/2} - 9$

$\frac{dy}{dx} = x^2 - \frac{1}{2}x + \frac{7}{2}x^{-1/2}$

$= x^2 - \frac{1}{2}x + \frac{7}{2\sqrt{x}}$

1 mark for correct answer

uestion 2

Stationary pts occur

t $x = 0$ (maximum turn. pt.)

d $x = 8$ (minimum turn. pt.)

1 mark each for correct x-value
1 mark each for correct concavity

i) $A(-1,4)$ $B(5,2)$

$$d = \sqrt{(5+1)^2 + (2-4)^2}$$
$$= \sqrt{36 + 4}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10} \text{ units}$$

1 mark for correctly showing distance

) $M = \frac{-1+5}{2}, \frac{4+2}{2}$

$M = (2, 3)$

1 mark for finding midpoint

ii) $m = \frac{2-4}{5+1}$

$= \frac{-2}{6}$

$m_1 = -\frac{1}{3}$

1 mark for finding gradient

iv) $m = -\frac{1}{3}$ $A(-1,4)$

$y - 4 = -\frac{1}{3}(x+1)$

$-3y + 12 = x + 1$

$0 = x + 3y - 11$

1 mark for correctly showing equation

1) equation of l is $3x - y - 3 = 0$

m_2 of $l = 3$

$\therefore m_1 \times m_2 = -1$

$-\frac{1}{3} \times 3 = -1$

$\therefore l$ is $\perp$ AB

test $(2,3)$ lies on l

LHS $= 3(2) - (3) - 3$

$= 0$

RHS $= 0$

$\therefore (2,3)$ is point of intersection

$\therefore l$ is perp. bisector of AB

1 mark for finding gradient of line l

1 mark for proving bisector

) $3x - y - 3 = 0$

let $y = 0$ to find x-intercept

i.e. $3x - (0) - 3 = 0$

$x = 1$

when $x = 1$ $y = 0$ $\therefore (1,0)$

1 mark for showing coordinate

iii) centre $(2,3)$ radius $= \sqrt{10}$

$(x-2)^2 + (y-3)^2 = 10$

1 mark for finding equation

Question 3

a) $y = \frac{3x-5}{2x+3}$ $\begin{matrix} u \\ v \end{matrix}$

$$\frac{dy}{dx} = \frac{3(2x+3) - 2(3x-5)}{(2x+3)^2}$$
$$= \frac{6x + 9 - 6x + 10}{(2x+3)^2}$$
$$= \frac{19}{(2x+3)^2}$$

(2 marks for correct answer)
1 mark for correct substitution into rule

1 mark for correct answer

b) $y = x^2 - 4x$ $(1,-3)$

$\frac{dy}{dx} = 2x - 4$

at $x = 1$ $m = 2(1) - 4$

$m_1 = -2$

$m_2 \perp = \frac{1}{2}$

$y + 3 = \frac{1}{2}(x-1)$

$2y + 6 = x - 1$

$0 = x - 2y - 7$

1 mark for derivative

1 mark for correct gradient

1 mark for correct answer

c) $3x^2 - \frac{2}{x^3} + 2$

$= 3x^2 - 2x^{-3} + 2$

$= x^3 + x^{-2} + 2x + c$

$= x^3 + \frac{1}{x^2} + 2x + c$

(2 marks for correct answer)
1 mark for adjusting fraction

1 mark for correct answer

d i) $\int \sqrt{3-x}\ dx$

$= \int (3-x)^{1/2}\ dx$

$= \frac{(3-x)^{3/2}}{3/2\ (-1)} + c$

$= \frac{-2(3-x)^{3/2}}{3} + c$

(2 marks for correct answer)

1 mark for integration

1 mark for correct answer

ii) $\int_{-1}^{2} \frac{x^4 + 3x^3 + x^2}{x}\ dx$

$= \int_{-1}^{2} x^3 + 3x^2 + x\ dx$

$= \left[\frac{x^4}{4} + x^3 + \frac{x^2}{2}\right]_{-1}^{2}$

$= (4 + 8 + 2) - (\frac{1}{4} - 1 + \frac{1}{2})$

$= 14 + \frac{1}{4}$

$= 14\frac{1}{4}$

(3 marks for correct answer)
1 mark for simplifying

1 mark for correct substitution into integral

1 mark for correct answer

Question 4

a) i) $\angle JMN = \angle JKL$ (corresponding $MN \parallel KL$)
$\angle JNM = \angle JLK$ (corresponding $MN \parallel KL$)
$\angle J$ is common
$\therefore \triangle JMN \,|||\, \triangle JKL$ (equiangular)

1 mark for two reasons

1 mark for final statement

ii) $\frac{JN}{JL} = \frac{JM}{JK}$

$\frac{9}{12} = \frac{JM}{8}$

$72 = 12JM$

$6 = JM$

If $JM = 6$ and $JK = 8$

$\therefore MK = 2cm$

(2 marks for correct answer)
1 mark for correct ratio

1 mark for correct answer

b) $f(x) = x^3 - 12x$

i) $f(a) = (a)^3 - 12(a)$

$f(-a) = (-a)^3 - 12(-a)$
$= -a^3 + 12a$

$-[f(-a)] = -[a^3 + 12a]$
$= -a^3 - 12a$

$\therefore f(a) = -[f(-a)]$
$\therefore$ odd function

1 mark for correctly proving odd function

ii) x intercepts, let $y=0$
i.e $0 = x^3 - 12x$
$0 = x(x^2 - 12)$
$\therefore x = 0 \quad x^2 = 12$
$x = 0 \quad x = \pm 2\sqrt{3}$

1 mark for finding x-intercep

iii) $f'(x) = 3x^2 - 12$
let $f'(x) = 0$ to find stat. pts
$0 = 3(x^2 - 4)$
$\therefore x = \pm 2$
test $f''(x) = 6x$
at $x = 2 \quad f''(x) = 12$
$\therefore f''(x) > 0$
$\therefore$ at $(2, -16)$ a min. turn pt exists
at $x = -2 \quad f''(x) = -12$
$\therefore f''(x) < 0$
$\therefore$ at $(-2, 16)$ a max. turn pt exists

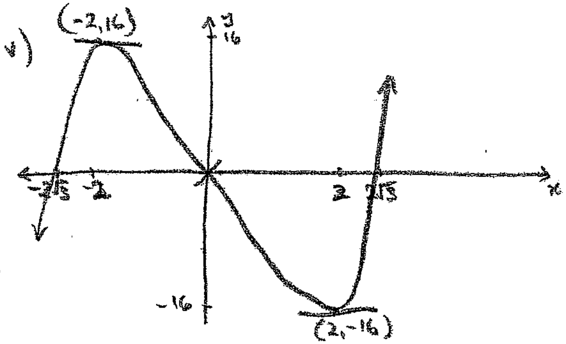
1 mark each for finding stationary point and determining nature.

Q4b continued

iv) $f''(x) = 6x$
let $f''(x) = 0$ to find pt of inflexion
$0 = 6x$
$\therefore x = 0$
since a max and min turn pt exist, a point of inflexion must exist at $(0,0)$

1 mark for finding point of inflexion

v)



1 mark for correctly drawing curve

c)

x	$f(x)$
0	0
6	13
12	10
18	9
24	13

$A \doteqdot \frac{6}{3}\{0 + 13 + 4(13+9) + 2(10)\}$
$\doteqdot 2\{13 + 88 + 20\}$
$\doteqdot 2 \times 121$
$A \doteqdot 242 \, m^2$

1 mark for correct substitution into simpson's rule

1 mark for correct answer

Question 5

$T_3: \quad a + 2d = 14$

$T_7: \quad a + 6d = 30$

$4d = 16$

$d = 4$

if $d = 4 \quad a + 2(4) = 14$

$a = 6$

1 mark for set up of simultaneous equation

1 mark for correct values for a and d

b) i) $y = x^2 \qquad y = 2 - x^2$

$x^2 = 2 - x^2$

$2x^2 = 2$

$x^2 = 1$

$\therefore \quad x = \pm 1$

$\therefore A(-1, 1) \qquad B(1, 1)$

1 mark for solving simultaneously

1 mark for correct answer (coordinates)

ii) $\int_{-1}^{1} 2 - x^2 \, dx - \int_{-1}^{1} x^2 \, dx$

$= \left[2x - \frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^3}{3}\right]_{-1}^{1}$

$= \left[\left(2 - \frac{1}{3}\right) - \left(-2 + \frac{1}{3}\right)\right] - \left[\frac{1}{3} + \frac{1}{3}\right]$

$= 1\frac{2}{3} + 1\frac{2}{3} - \frac{2}{3}$

$= 2\frac{2}{3} u^2$

1 mark for correctly showing integral statement

1 mark for correct subst.

1 mark for correct answer

c) $A_1 = 1000(1.07)^{19}$

$A_2 = 1000(1.07)^{18}$

$A_3 = 1000(1.07)^{17}$

$\vdots$

$A_{19} = 1000(1.07)^1$

$A_{20} = 1000$

1 mark showing yearly statements

$S_{20} = 1000 + 1000(1.07) + 1000(1.07)^2 + \ldots + (1.07)^{19}$

$S_{20} = 1000\left[1 + 1.07 + 1.07^2 + \ldots + 1.07^{19}\right]$

$= 1000\left[\frac{1.07^{20} - 1}{0.07}\right]$

$= 1000 \times 40.99549$

$S_{20} = \$40995.49$

1 mark for showing sum of series

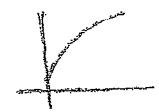
1 mark for correct subst.

1 mark for answer.

Question 6

a) $\frac{dP}{dt} > 0$

$\frac{d^2P}{dt^2} < 0$



1 mark for each statement

b) i) $0.\dot{3}\dot{4} = 0.34 + 0.0034 + 0.000034 + \ldots$

1 mark for showing sum

ii) $a = 0.34 \quad r = 0.01$

$S_\infty = \frac{a}{1-r}$

$= \frac{0.34}{1 - 0.01}$

$= \frac{0.34}{0.99}$

$= \frac{34}{99}$

1 mark for correct substituting into limiting sum formula

1 mark for correct answer

c) $y = x^2 \qquad y = (x-4)^2$

$x^2 = x^2 - 8x + 16$

$8x = 16$

$\therefore x = 2$

1 mark for finding x-intercept

$V = \pi \int_0^2 (x^2)^2 \, dx + \pi \int_2^4 \left[(x-4)^2\right]^2 dx$

1 mark for correct statement

$= \pi \int_0^2 x^4 \, dx + \pi \int_2^4 (x-4)^4 \, dx$

$= \pi \left\{ \left[\frac{x^5}{5}\right]_0^2 + \left[\frac{(x-4)^5}{5}\right]_2^4 \right\}$

1 mark for correct statement

$= \pi \left\{ \frac{32}{5} + \left[0 + \frac{32}{5}\right] \right\}$

$= \frac{64\pi}{5} u^3$

1 mark for correct answer

d) i) $\frac{d}{dx}\left(\frac{1}{x^2+1}\right)$

$\frac{d}{dx} = -(x^2+1)^{-2} \times 2x$

$= -2x(x^2+1)^{-2}$

$= \frac{-2x}{(x^2+1)^2}$

1 mark for correct answer

ii) $\int_1^3 \frac{2x}{(x^2+1)^2} \, dx$

$= -\int_1^3 \frac{-2x}{(x^2+1)^2} \, dx$

1 mark for correct substitution

$= -\left[\frac{1}{x^2+1}\right]_1^3$

$= -\left[\frac{1}{10} - \frac{1}{2}\right]$

$= \frac{1}{2} - \frac{1}{10}$

$= \frac{4}{10}$

1 mark for correct answer