בס"ד

MY ROCK AND סלעי ומצודתי MY FORTRESS

MATHEMATICS

YEAR 12
1999 HALF YEARLY EXAM

4 UNIT

Time allowed: Three hours
(Plus 5minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 10.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

Marks

QUESTION 1

(a) Find the modulus and argument of $Z = 3 + 4i$ (Express the argument in radians) **2**

(b) For any complex number Z where $Z = -\overline{Z}$ prove that Z must be purely imaginary **3**

(c) Find the square root of $Z = 5 - 12i$ **4**

(d) Draw a neat sketch to illustrate the following region of the Argand diagram **2**

$$-\frac{\pi}{6} \le \arg(Z-1) \le \frac{\pi}{6} \text{ and } |Z-1| \le 1$$

(e) If Z is a complex number such that **4**

$|Z-6| + |Z+6| = 60$ describe geometrically the locus of Z and find its Cartesian equation.

Marks

QUESTION 2

(a) If $1+i$ is a solution of $x^4-6x^3+5x^2+2x-10=0$ solve the equation over the field of real numbers. **3**

(b) If α, β, δ are the roots of $x^3-px+q=0$ find in terms of p and q a cubic equation with roots **4**

i. $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\delta}$

ii. $\alpha^3, \beta^3, \delta^3$

(c) If the cubic equation $2x^3-9x^2+12x+k=0$ has two equal roots. find the value of k **4**

(d) Find the condition (i.e. the relationship between a and b) that $x^4-3ax+b=0$ has a repeated root **4**

Marks

QUESTION 3

(a) Express z in the form $a+ib$ if **3**

cyk!

$$\arg(z+1)=\frac{\pi}{6} \text{ and } \arg(z-1)=\frac{2\pi}{3}$$

(b) If z is a complex number show that $z^2+(\bar{z})^2=2$ is a hyperbola and state its eccentricity **4**

(c) By writing each factor in the modulus-argument form, simplify **3**

$$(\sqrt{3}+i)^6 \div (1-i)^4$$

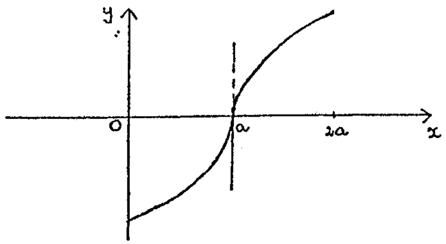
(d) i. Find the four complex roots of $z^4+4=0$ **5**

ii. Plot these roots on an Argand diagram

Marks

QUESTION 4

(a) Consider the graph of $y = f(x)$ for $0 \le x \le 2a$ — 3



The graph has point symmetry and a vertical tangent exists at $x = a$.

Sketch:

i. $y = f'(x)$

ii. $y = f''(x)$

iii. $y = \int_0^x f(t)\,dt$

(b) i. Given $F(x) = \dfrac{x^2-1}{x^2+1}$, sketch the following on separate axes — 12

1. $y = F(x)$

2. $[F(x)]^2 = \dfrac{x^2-1}{x^2+1}$

3. $y = [F(x)]^2$

4. $y = \log_e F(x)$

5. $y = \dfrac{|x+1|(x-1)}{x^2+1}$

ii. Use your graph in (5) to solve the inequality $x^2 + 1 > 2|x-1|(x-1)$

Marks

QUESTION 5

(a) i. Obtain the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ at the point $P(a,b)$ on the curve — 5

ii. This tangent meets the x and y axes at Q and R respectively. Show that $OQ + OR = c$ for all positions of P, where O is the origin

(b) i. Find the eccentricity, the equations of the directrices and the co-ordinates of the focii of the ellipse with equation $7x^2 + 16y^2 = 112$ — 10

ii. Sketch the ellipse showing the above information on your diagram. Also sketch the auxiliary circle on your diagram

iii. Set up the integrals that give:

1. The area of a quadrant of the circle with equation $x^2 + y^2 = 16$

2. The area of the quadrant of the ellipse $7x^2 + 16y^2 = 112$

iv. Show that the integral in (iii)2. ~~(i)2.~~ above is $\dfrac{b}{a}$ times the integral in (iii)1. ~~(i)1.~~ and deduce the area of the ellipse from the known area of the circle. Hence write down a general formula for the area of an ellipse

Marks

QUESTION 6

(a) The point $P(cp, \frac{c}{p})$ lies on the rectangular hyperbola $xy = c^2$ in the first quadrant. The tangent to the hyperbola at the point P, crosses the x axis at the point A and the y axis at the point B. **10**

i. Find the equation of the tangent to the hyperbola at the point P

ii. Show that the equation of the normal to the hyperbola at the point P is $p^3x - py = cp^4 - c$

iii. If the normal at P meets the other branch of the hyperbola at the point Q, determine the coordinates of Q

* when finding roots of quadratic → don't just use formula!
— sum of roots etc

iv Show that the area of the triangle ABQ is $c^2\left(p^2 + \frac{1}{p^2}\right)^2$

v. Prove that the area of the triangle is a minimum when $p = 1$

(b) If a, b and c are positive real numbers such that $a \neq b \neq c$, prove, **5**

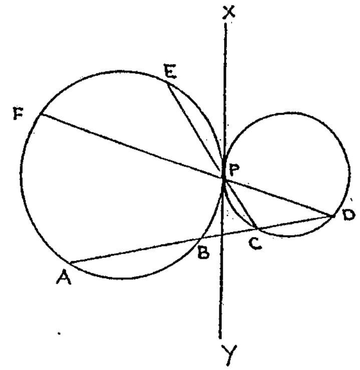
i. $\frac{a}{b} + \frac{b}{a} > 2$

ii. $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) > 9$

Marks

QUESTION 7

(a) Two circles touch externally at point P. The line ABCD cuts the first circle at A and B and the second circle at C and D. The lines CPE and DPF meet the first circle at E and F respectively. XPY is the common tangent. **7**



Copy the diagram onto your answer paper.
Prove that:

i. FE||AD

ii. $\angle FPA = \angle BPC$

iii. $\Delta FPA ||| \Delta BPC$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of a hyperbola with eccentricity e. **8**

i. Prove the perpendicular from the focus $S(ae, 0)$ to the asymptote $y = \frac{b}{a}x$ meets it on the directrix.

ii. Prove that the angle between the two asymptotes is $2\tan^{-1}\sqrt{e^2 - 1}$

Marks

QUESTION 8

(a) Find, as a relation between k, l, and m, the condition for the quadratic equation in x, **3**

$$(k^2+l^2)x^2+2l(k+m)x+(l^2+m^2)=0$$

to have real roots. Simplify your answer as far as possible.

(b) If $|a|>2|b|$, prove $2|a-b|>|a|$ **3**

(c) i. Show that $\int_0^{\frac{\pi}{2}}\cos^4 x\,dx=\frac{3\pi}{16}$ **9**

ii. Prove $3(\cos^4 x+\sin^4 x)-2(\cos^6 x+\sin^6 x)=1$

iii. Without attempting to evaluate any integrals, explain why:

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx\text{, for all positive integers } n$$

iv By integrating the identity in part (ii), and using parts (i) and (iii),

find $\int_0^{\frac{\pi}{2}}\cos^6 x\,dx$

v. Without attempting to evaluate any integrals. explain why:

$$\int_0^{\frac{\pi}{2}}\sin^{n+1} x\,dx<\int_0^{\frac{\pi}{2}}\sin^n x\,dx\text{. for all positive integers } n$$

4 unit Solutions

1 a $3+4i = 5\left(\frac{3}{5}+\frac{4i}{5}\right)$

$\therefore \cos\theta = \frac{3}{5}$

$\theta = \cos^{-1}\frac{3}{5}$

$= \cdot 927295218^c$

$= \cdot 93^c$

$\therefore 3+4i = 5\,\text{cis}\,(\cdot 93)$

$\therefore$ modulus $=5$, argument $=0.93^c$

b) let $z = x+iy$

if $z = -\bar{z}$

$x+iy = -(x-iy)$

i.e $2x = 0$

$x = 0$

Hence $z = 0+iy$

$= iy$

which is purely imaginary

c) let $x+iy = \sqrt{5-12i}$

ie $(x+iy)^2 = 5-12i$

$x^2-y^2+2xyi = 5-12i$

Comparing real and imaginary parts

$x^2-y^2 = 5 \quad \ldots 1)$

$2xy = -12 \quad \ldots 2)$

from 2) $\quad y = \frac{-6}{x} \quad \ldots 3)$

Substitute 3) into 1)

$x^2 - \frac{36}{x^2} = 5$

$x^4-5x^2-36=0$

$(x^2+4)(x^2-9)=0$

Hence $x = \pm 3$ (x is real)

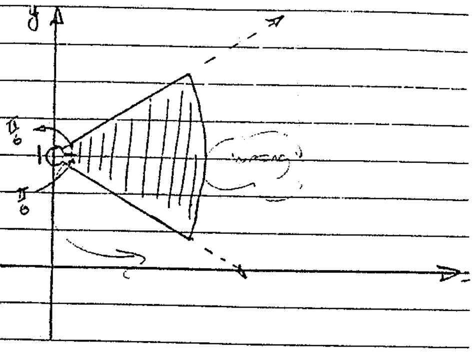
When $x=3 \quad y=-2$

$x=-3 \quad y=2$

$\therefore \sqrt{5-12i} = \pm(3-2i)$

d) $-\frac{\pi}{6} \leq \arg(z-1) \leq \frac{\pi}{6}$

and $|z-1| \leq 1$



e) $|z-6|+|z+6| = 60$

The locus of z is an ellipse, with foci at $(6,0)$ and $(-6,0)$. The length of the major axis is 60.

ie. $(a=30)$

$\therefore a = 30$

$ae = 6$

$e = \frac{1}{5}$

Now

$b^2 = a^2(1-e^2)$

$= 900\left(1-\frac{1}{25}\right)$

$= 864$

Hence the equation is

$\frac{x^2}{900}+\frac{y^2}{864} = 1$

Question 2.

a) Let $f(x) = x^4-6x^3+5x^2+2x-10$

$f(x)$ has real coefficients

$\therefore$ if $1+i$ is a root $1-i$ is also

~~and $(1+i)(1-i)$~~

and $(x-1-i)(x-1+i)$ is a factor.

ie. x^2-2x+2 is a factor

$$\begin{array}{r}
x^2-4x-5 \\
x^2-2x+2\ \overline{)\,x^4-6x^3+5x^2+2x-10} \\
\underline{x^4-2x^3+2x^2} \\
-4x^3+3x^2+2x \\
\underline{-4x^3+8x^2-8x} \\
-5x^2+10x-10 \\
\underline{-5x^2+10x-10} \\
0
\end{array}$$

$\therefore f(x) = (x^2-2x+2)(x^2-4x-5)$

$= (x^2-2x+2)(x+1)(x-5)$

Hence the real roots are $5, -1$

b) Let $f(x) = x^3-px+q$

i) Put $u = \frac{1}{x} \quad \therefore x = \frac{1}{u}$

Hence $\left(\frac{1}{u}\right)^3 - \frac{p}{u}+q=0$ has roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

$\frac{1}{u^3}-\frac{p}{u}+q=0$

$1-pu^2+qu^3=0$

ie $qu^3-pu^2+1=0$

So the required function is

$qx^3-px^2+1=0$

ii) Put $u=x^3 \Rightarrow x = u^{1/3}$

$(u^{1/3})^3 - pu^{1/3}+q=0$ has roots $\alpha^3, \beta^3, \gamma^3$

$u-pu^{1/3}+q=0$

$u+q = pu^{1/3}$

$(u+q)^3 = (pu^{1/3})^3$

$u^3+3u^2q+3uq^2+q^3 = p^3u$

$u^3+3u^2q+(3q^2-p^3)u+q^3=0$

ie

$x^3+3x^2q+(3q^2-p^3)x+q^3=0$

c) $f(x) = 2x^3-9x^2+12x+k=0$

$f'(x) = 6x^2-18x+12$

If X is a double root then $f'(X)=0$ and $f(X)=0$

$f'(x)=0$ when $6(x^2-3x+2)=0$

ie $(x-1)(x-2)=0$

$x=1,2$

$f(1) = 2-9+12+k=0 \Rightarrow k=-5$

$f(2) = 16-36+24+k=0 \Rightarrow k=-4$

$\therefore$ the equation has equal roots when $k=-4, -5$.

Question 2.

a) $f(x) = x^4 - 3ax + b \quad \text{----} 1)$

$f'(x) = 4x^3 - 3a \quad \text{----} 2)$

If there is a repeated root $f'(x) = f(x) = 0$

$2) \times x \qquad 4x^4 - 3ax = 0 \quad \text{---} 3)$

$x^4 - 3ax + b = 0 \quad \text{----} 1)$

$3) - 1) \qquad 3x^4 - b = 0$

$x^4 = \frac{b}{3} \quad \text{----} 4)$

Since $4x^3 - 3a = 0$

$x^3 = \frac{3a}{4} \quad \text{-----} 5)$

from 4) $\quad x^{12} = \left(\frac{b}{3}\right)^3 = \frac{b^3}{27}$

from 5) $\quad x^{12} = \left(\frac{3a}{4}\right)^4 = \frac{81a^4}{256}$

$\therefore \frac{b^3}{27} = \frac{81a^4}{256}$

ie $256b^3 = 2187a^4$

Question 3

a) $\arg(z-1) - \arg(z+1) = \frac{2\pi}{3} - \frac{\pi}{6}$

$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

ie the locus is the top semi-circle of $x^2+y^2=1 \quad x \neq \pm 1$

ie $y = \sqrt{1-x^2} \quad x \neq \pm 1$

ie $|z| = 1$

Hence $z = x + \left(\sqrt{1-x^2}\right)i$

b) Let $z = x+iy$, $\bar{z} = x - iy$

Hence $z^2 + \bar{z}^2 = 2$

$\Rightarrow$

$(x+iy)^2 + (x-iy)^2 = 2$

$x^2 + 2ixy + i^2y^2 + x^2 - 2ixy + i^2y^2 = 2$

$2x^2 - 2y^2 = 2$

$x^2 - y^2 = 1$

which is a hyperbola

Now $b^2 = a^2(e^2-1) \qquad a = 1, \ b = 1$

$\therefore e^2 = 2$

$e = \sqrt{2}$

So we have a hyperbola with eccentricity $\sqrt{2}$

c) Let $z_1 = \sqrt{3} + i \qquad |z_1| = \sqrt{3+1} = 2$

$\arg z_1 = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore z_1 = 2 \operatorname{cis} \frac{\pi}{6}$

$z_2 = 1 - i \qquad |z_2| = \sqrt{1+1} = \sqrt{2}$

$\arg z_2 = -\tan^{-1} 1 = -\frac{\pi}{4}$

$\therefore z_2 = \sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$

$(\sqrt{3}+i)^6 \div (1-i)^4 = \left(2 \operatorname{cis} \frac{\pi}{6}\right)^6 \div \left(\sqrt{2} \operatorname{cis} \frac{-\pi}{4}\right)^4$

$= 2^6 \operatorname{cis} \pi \div 2^2 \operatorname{cis}(-\pi)$

$= 2^4 \operatorname{cis}(\pi - (-\pi))$

$= 2^4 \operatorname{cis} 2\pi$

$= 2^4 \cos 2\pi + i \sin 2\pi$

$= 2^4$

$= 16$

d) $z^4 + 4 = 0$

i) $z^4 = -4$

$\therefore r^4(\cos\theta + i\sin\theta)^4 = -4$

$r^4(\cos 4\theta + i \sin 4\theta) = -4$

$\therefore r^4 = 4 \qquad \cos 4\theta + i\sin 4\theta = -1$

$r = \sqrt{2} \qquad \cos 4\theta = -1$ and $\sin 4\theta = 0$

$4\theta = \pi, 3\pi, 5\pi, 7\pi$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

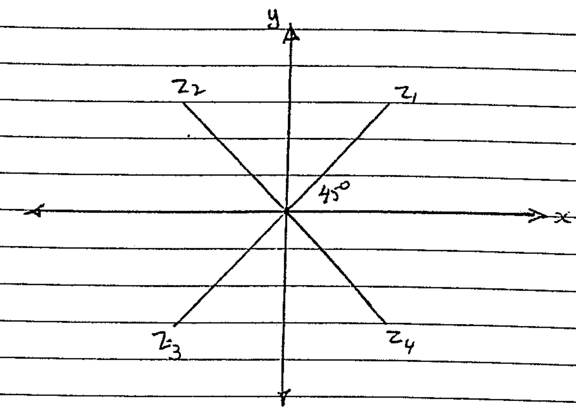
Roots $\quad z_1 = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

$z_2 = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$

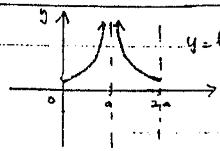
$z_3 = \sqrt{2} \operatorname{cis} \frac{5\pi}{4}$

$z_4 = \sqrt{2} \operatorname{cis} \frac{7\pi}{4}$

ii)

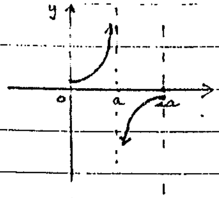


4 (a). i. $y = f'(x)$



$-\frac{1}{2}$ each mistake
Check for:
Symmetry
Asymptote
Always +ve/shape
Start above $y=0$

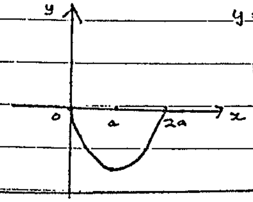
ii. $y = f''(x)$ $-\frac{1}{2}$ each mistake



Check for
point symmetry
asymptote
Shape
$x<a$, $f''>0$
$x>a$, $f''<0$

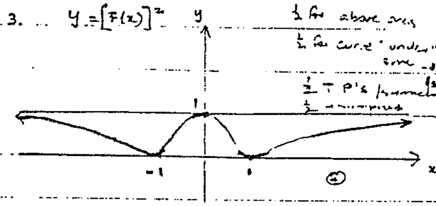
(3)

iii. $y = \int_0^x f(t)\,dt$ $-\frac{1}{2}$ each mistake



Check for.
Symmetry
+ P at $x=a$
0 at $x=0$, $x=2a$

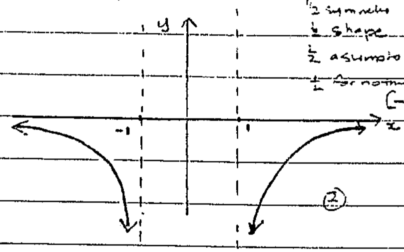
3. $y = [F(x)]^2$



$\frac{1}{2}$ for above axis
$\frac{1}{2}$ for curve "under" $F(x)$ since $-1<F(x)<1$
$\frac{1}{2}$ TP's (symmetry) (shape)
$\frac{1}{2}$ ...

(2)

4. $y = \log_e F(x)$

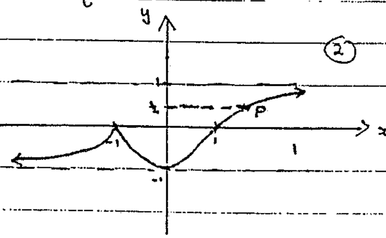


$\frac{1}{2}$ symmetric
$\frac{1}{2}$ shape
$\frac{1}{2}$ asymptotes
$\frac{1}{2}$ for no ... $(-1, 1)$

(2)

5. $y = \dfrac{|x+1|(x-1)}{x^2+1}$

$$= \begin{cases} F(x) & \text{if } x \geq -1 \\ -F(x) & \text{if } x < -1 \end{cases}$$ 1



1

(2)

(b) i. $F(x) = \dfrac{x^2-1}{x^2+1}$

1. even function, $F(x) = 1 - \dfrac{2}{x^2+1}$

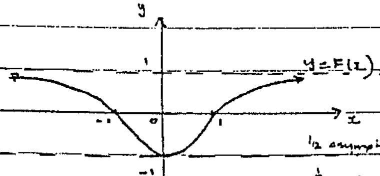
$F(x)=0$, $x=\pm1$; $x=0$, $F(x)=-1$ (min. value)

$-1 \leq F(x) < 1$

As $x \to \infty$, $F(x) \to 1$

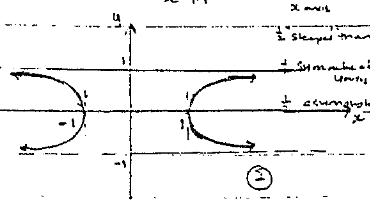
As $x \to -\infty$, $F(x) \to 1$

(2)



$\frac{1}{2}$ asymptotes
$\frac{1}{2}$ TP
$\frac{1}{2}$ symmetry/shape
$\frac{1}{2}$ x, y intercepts

2. $[F(x)]^2 = \dfrac{x^2-1}{x^2+1}$



$\frac{1}{2}$ symmetry about the x axis
$\frac{1}{2}$ steeper than $\frac{1}{2}$ shape
$\frac{1}{2}$ symmetric about ... limits
$\frac{1}{2}$ asymptotes

(2)

(ii) $x^2+1 > 2|x+1|(x-1)$

$\dfrac{|x+1|(x-1)}{x^2+1} < \frac{1}{2}$ $\frac{1}{2}$

To find P,

Solve $F(x) = \frac{1}{2}$ (since $x \geq -1$)

$2(x^2-1) = x^2+1$

$x^2 - 3 = 0$

$x = \sqrt{3}$ $(x>0)$ 1

$\therefore x^2+1 > 2|x+1|(x-1)$ for all $x < \sqrt{3}$

(from graph) $\frac{1}{2}$

(2)

5 (a) (i) $\sqrt{x} + \sqrt{y} = \sqrt{c}$

Diff. wrt x: $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$

$\therefore$ At $P(a,b)$, $\dfrac{dy}{dx} = -\sqrt{\dfrac{b}{a}}$ 1

$\therefore$ Eqn. of tangent is

$y - b = -\sqrt{\dfrac{b}{a}}(x-a)$

$\sqrt{a}y - b\sqrt{a} = -\sqrt{b}x + a\sqrt{b}$

ie. $\sqrt{b}x + \sqrt{a}y = a\sqrt{b} + b\sqrt{a}$ 1

(5)

(ii) when $y=0$, $x = \dfrac{a\sqrt{b}+b\sqrt{a}}{\sqrt{b}}$

$= a + \sqrt{ab}$ $\therefore Q(a+\sqrt{ab}, 0)$

$x=0$, $y = \dfrac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}}$ 1

$= \sqrt{ab} + b$ $\therefore R(0, b+\sqrt{ab})$

$\therefore OQ = a + \sqrt{ab}$ units

$OR = b + \sqrt{ab}$ units 1

$OQ + OR = a + b + 2\sqrt{ab}$

$= (\sqrt{a}+\sqrt{b})^2$ 1

$= (\sqrt{c})^2 = c$ (since a, b satisfy eqn of curve).

(b) i) $7x^2 + 16y^2 = 112$

ie $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$ $\therefore a=4, b=\sqrt{7}$ 1

$b^2 = a^2(1-e^2)$

$7 = 16(1-e^2)$

$e^2 = 1 - \dfrac{7}{16} = \dfrac{9}{16}$

$\therefore e = \dfrac{3}{4}$ $(e>0)$ 1

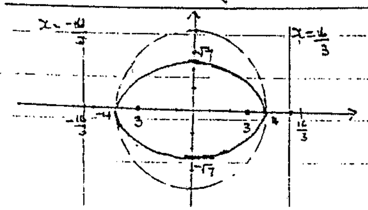
$\therefore$ eccentricity $= \dfrac{3}{4}$

Directrices are $x = \pm\dfrac{16}{3}$ 1

Foci are $(3,0)$ and $(-3,0)$ 1

(4)

ii) $x = -\dfrac{16}{3}$ $x = \dfrac{16}{3}$



(1)

(iii) 1. $\int_0^4 \sqrt{16-x^2}\,dx$

2. $\int_0^4 \sqrt{\dfrac{112-7x^2}{16}}\,dx$

$= \dfrac{1}{4}\int_0^4 \sqrt{112-7x^2}\,dx$

(2)

(iv) $\dfrac{1}{4}\int_0^4 \sqrt{112-7x^2}\,dx$

$= \dfrac{1}{4}\int_0^4 \sqrt{7}.\sqrt{16-x^2}\,dx$

$= \dfrac{\sqrt{7}}{4}\int_0^4 \sqrt{16-x^2}\,dx$

(3)

$= \dfrac{b}{a}\int_0^4 \sqrt{16-x^2}\,dx$. 1

$\therefore$ Area of ellipse $= \dfrac{\sqrt{7}}{4} \times \dfrac{\pi r^2}{4} \times 4$ where $r=4$.

$= \dfrac{\sqrt{7}}{4} \times 16\pi$ 1

$= 4\sqrt{7}\pi$ units^2

$\therefore$ General form for area of ellipse is

$A = \dfrac{b}{a} \times \pi a^2$

$= \pi ab$ units^2 1

6 (i) $xy = c^2$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

$\therefore$ At P, $m = -\dfrac{1}{p^2}$ 1

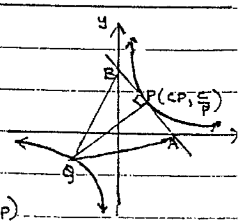
$\therefore$ Eqn of tangent at P is

$y - \dfrac{c}{p} = -\dfrac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

ie $x + p^2y = 2cp$ 1

(2)



(ii) Eqn of normal is

$y - \dfrac{c}{p} = p^2(x - cp)$

$py - c = p^3x - cp^4$

ie $p^3x - py = cp^4 - c$

(1)

Alternatively, θ must be of form $(cq, \frac{c}{q})$
subs. into ②
$c p^3 q - \frac{cp}{q} = cp^4 - c$
which leads to
$q = -\frac{1}{p^3}$

(iii) Solve $xy = c^2$ — ①

$p^3 x - py = cp^4 - c$ — ② simultaneously.

From ① $y = \frac{c^2}{x}$, sub. into ②

$p^3 x - \frac{pc^2}{x} = cp^4 - c$

$p^3 x^2 - pc^2 = (cp^4 - c)x$

$p^3 x^2 - (cp^4 - c)x - pc^2 = 0$ 1

We know $x = cp$ is one solution $\therefore$ factorise:

$(x - cp)(p^3 x + c) = 0$

$\therefore x = cp$ or $x = -\frac{c}{p^3}$

$x = cp$ corr. to 1st quad. ②

$x = -\frac{c}{p^3}$ is the x value at Q.

Sub. into ②

$y = \frac{c^2 x - p^3}{c}$ 1

$= -cp^3$

$\therefore Q\left(-\frac{c}{p^3}, -cp^3\right)$

(iv) Now $A(2cp, 0)$ and $B\left(0, \frac{2c}{p}\right)$

$\therefore d_{AB} = \sqrt{4c^2p^2 + \frac{4c^2}{p^2}}$

$= \frac{2c}{p}\sqrt{p^4 + 1}$ 1

$d_{PQ} = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)^2}$ ③

$= \frac{c}{p^3}\sqrt{(p^4+1)^2 + p^2(1+p^4)^2}$

$= \frac{c(p^4+1)}{p^3}\sqrt{1+p^2}$ 1

$\therefore$ Area of $\Delta ABQ = \frac{1}{2} AB.PQ$

$= \frac{1}{2} \times \frac{2c}{p}\sqrt{p^4+1} \times \frac{c(p^4+1)}{p^3}\sqrt{p^4+1}$

$= \frac{c^2}{p^4}(p^4+1)^2$

$= c^2\left(p^2 + \frac{1}{p^2}\right)^2$ 1

(v) $A = c^2\left(p^2 + \frac{1}{p^2}\right)^2$

$= c^2\left[p^4 + 2 + \frac{1}{p^4}\right]$

$\frac{dA}{dp} = c^2\left(4p^3 - \frac{4}{p^5}\right]$ 1

For min., $\frac{dA}{dp} = 0$

$4p^3 = \frac{4}{p^5}$

$p^8 = 1$ ②

$\therefore p = 1.$ 1

$\frac{d^2A}{dp^2} = 4c^2\left[3p^2 + \frac{5}{p^6}\right]$

$= 4c^2[8]$

> 0 when $p = 1$,

$\therefore p = 1$ corresponds to a minimum.

[ie when P is at the point (c, c)]

⑥

(b)(i) To prove: $\frac{a}{b} + \frac{b}{a} > 2$ $a \neq b$

a, b real pos. int.

Proof: $\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 > 0$ 1 $a, b > 0$

$\therefore \frac{a}{b} + \frac{b}{a} - 2\sqrt{\frac{a}{b}}\sqrt{\frac{b}{a}} > 0$ ②

$\therefore \frac{a}{b} + \frac{b}{a} > 2$ 1

(ii) To prove: $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$

Proof: $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

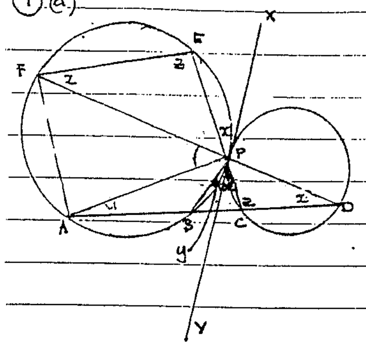
$= 1 + 1 + 1 + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}$

$= 3 + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}$

$> 3 + 2 + 2 + 2 = 9$ 1

(from (i), $\frac{a}{c} + \frac{c}{a} > 2$ etc). 1

⑦(a)



(i) To Prove: $FE \parallel AD$

Proof: Let $\angle EPX = x$

then $\angle PFE = x$ (angle bet. chord & tangent equals $\angle$ in alt. segment). ½

Also $\angle YPC = x$ (vertically opp. $\angle$'s) ½

$\therefore \angle CDP = x$ ($\angle$ bet. chord & tangent equals $\angle$ in alt. segment). ½ ②

ie $\angle EFP = \angle CDP$ ½

$\therefore FE \parallel AD$ (alternate $\angle$'s equal)

(ii) To prove: $\angle FPA = \angle BPC$

Proof: Let $\angle BPY = y$.

then $\angle BPC = x + y$.

Let $\angle FEP = z$ ③

½

Then $\angle DCP = z$ (alternate $\angle$'s, $FE \parallel AD$)

$\therefore \angle PBC = z - (x+y)$ (exterior $\angle$ of ΔPBC) ½

Also, $\angle PAB = y$ ($\angle$ bet chord & tangent equals $\angle$ in alt segment) ½

$\therefore \angle BPA = z - (x+y) - y$ (ext $\angle$ of ΔPBA)

$= z - x - 2y$ ½

And $\angle EPF = 180° - (x+z)$ ($\angle$ sum ΔFEP) ½

$\therefore \angle FPA = 180° - [180° - (x+z)] - [z - x - 2y]$

$- [x+y]$ (EPC is straight $\angle$) ½

$= 180 - 180 + x + z - z + x + 2y - x - y$

$= x + y = \angle BPC$

(iii) To prove: $\Delta FPA \parallel\!\parallel \Delta BPC$

Proof: In ΔFPA and ΔBPC,

$\angle FPA = \angle BPC$ (from (ii) above) ½

$\angle FAP = 180° - z$ (opp. $\angle$'s of a cyclic quad. supp.) ½ ②

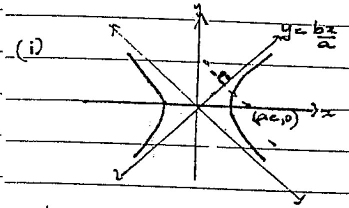
And $\angle BCP = 180° - z$ ($ABCD$ straight $\angle$)

$\therefore \angle FAP = \angle BCP$ ½

$\therefore \Delta FPA \parallel\!\parallel \Delta BPC$ (equiangular). ½

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i)



Line $\perp$ to $y = \frac{bx}{a}$ has gradient $-\frac{a}{b}$

$\therefore$ Eqn of perp. through focus is,

$y - 0 = -\frac{a}{b}(x - ae)$

$by = -ax + a^2e$

ie $ax + by = a^2e$. 1

This meets the line $y = \frac{b}{a}x$ when

$ax + b\left(\frac{b}{a}x\right) = a^2e$

$a^2x + b^2x = a^3e$

$x = \frac{a^3e}{a^2+b^2}$ 1 ③

But $b^2 = a^2(e^2 - 1)$

so $a^2 + b^2 = a^2e^2$ 1

$\therefore x = \frac{a^3e}{a^2e^2} = \frac{a}{e}$ which is a pt. on the directrix.

$\therefore$ the perp. meets the asymptote on the directrix.

(ii) Angle between the asymptotes is

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$ where $m_1 = \frac{b}{a}$

$m_2 = -\frac{b}{a}$

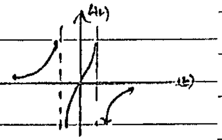
$\therefore$ If $t = \tan\frac{\theta}{2}$,

$\frac{2t}{1-t^2} = \frac{\frac{b}{a}+\frac{b}{a}}{1-\frac{b^2}{a^2}}$

$\therefore \frac{2t}{1-t^2} = \frac{2\left(\frac{b}{a}\right)}{1-\left(\frac{b}{a}\right)^2}$ 1

$\therefore t = \frac{b}{a}$ 1

(Since $f(t) = \frac{2t}{1-t^2}$ is a function, - only one value of f(t) for each t.)



But $b^2 = a^2(e^2-1)$

$\therefore \frac{b}{a} = \sqrt{e^2-1}$ $\quad\left(\frac{b}{a} > 0\right)$

5

$\therefore \tan\frac{\theta}{2} = \sqrt{e^2-1}$ 1

$\therefore \frac{\theta}{2} = \tan^{-1}\sqrt{e^2-1}$

$\therefore \theta = 2\tan^{-1}\sqrt{e^2-1}$. 1

8\. (a) $(k^2+l^2)x^2 + 2l(k+m)x + (l^2+m^2) = 0$ is to have real roots. $\therefore \Delta \geq 0$

$\Delta = 4l^2(k+m)^2 - 4(l^2+m^2)(k^2+l^2)$

$= 4l^2(k^2+2km+m^2)$ $\quad \frac{1}{2}$

$- 4(l^2k^2 + l^4 + m^2k^2 + m^2l^2)$

$= 4l^2k^2 + 8l^2km + \cancel{4l^2m^2}$

$- 4\cancel{l^2k^2} - 4l^4 - 4m^2k^2 - \cancel{4m^2l^2}$ $\frac{1}{2}$

$= -4(l^4 - 2l^2km + m^2k^2)$

$= -4(l^2-mk)^2$ $\quad \frac{1}{2}$

If $\Delta \geq 0$, $-4(l^2-mk)^2 \geq 0$ $\frac{1}{2}$

$\therefore (l^2-mk)^2 \leq 0$

But $(l^2-mk)^2 \geq 0$ $\frac{1}{2}$ as it is a square. $\therefore$ Only possible value is 3

$l^2 - mk = 0$ ie $l^2 = mk$

(b) To prove: $2|a-b| > |a|$ if $|a| > 2|b|$

Proof: $2|a-b| = 2\left|a\left(1-\frac{b}{a}\right)\right|$ $\frac{1}{2}$

$= 2|a|\left|1-\frac{b}{a}\right|$ $\frac{1}{2}$

$\geq 2|a|\left[1-\left|\frac{b}{a}\right|\right]$ $\frac{1}{2}$

(Since $|a-b| \geq |a|-|b|$) $\frac{1}{2}$ $a \neq b$.

$\therefore 2|a-b| \geq 2|a|\left[1-\frac{1}{2}\right]$ $\frac{1}{2}$

(Since $\left|\frac{b}{a}\right| < \frac{1}{2}$)

$= |a|$ $\frac{1}{2}$

ie $2|a-b| > |a|$.

(c) i) $\cos^2 x = \frac{\cos 2x + 1}{2}$

$\cos^4 x = \frac{1}{4}(1+\cos 2x)^2$ 1

$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}\left[1 + 2\cos 2x + \frac{\cos 4x + 1}{2}\right]$ 1

$\therefore \int_0^{\frac{\pi}{4}} \cos^4 x\,dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$ 3

$= \left[\frac{3x}{8}\right]_0^{\frac{\pi}{2}} + 0 + 0$ (these graphs are symmetrical about the x-axis for $0 \leq x \leq \frac{\pi}{2}$)

$= \frac{3\pi}{16}$ 1

ii) To prove:

$3(\cos^4 x + \sin^4 x) - 2(\cos^6 x + \sin^6 x) = 1$

L.S $= \cos^4 x + \sin^4 x + 2\cos^4 x + 2\sin^4 x$

$- 2\cos^6 x - 2\sin^6 x$

$= \cos^4 x + \sin^4 x + 2\cos^4 x(1-\cos^2 x)$

$+ 2\sin^4 x(1-\sin^2 x)$

$= \cos^4 x + \sin^4 x + 2\cos^4 x\sin^2 x$ 1

$+ 2\sin^4 x\cos^2 x$ 2

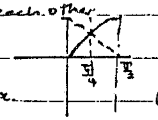
$= \cos^4 x + \sin^4 x + 2\sin^2 x\cos^2 x(\cos^2 x + \sin^2 x)$

$= \cos^4 x + \sin^4 x + 2\sin^2 x\cos^2 x$

$= (\cos^2 x + \sin^2 x)^2 = 1 = R.S.$

(P.S. There are other methods) 1

(iii) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$. $y = \cos x$ and $y = \sin x$ are reflections of each other in the line $x = \frac{\pi}{4}$



$\therefore \int_0^{\frac{\pi}{2}} \cos x\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx$. 1

Similarly $y = \cos^n x$ and $y = \sin^n x$ must be reflections of each other in the line $y = \frac{\pi}{4}$

$\therefore \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

(iv) $3\int_0^{\frac{\pi}{2}} (\cos^4 x + \sin^4 x)dx - 2\int_0^{\frac{\pi}{2}} (\cos^6 x + \sin^6 x)dx = \frac{\pi}{2}$

But $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \cos^4 x\,dx$ and similarly for $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$

$\therefore 3\int_0^{\frac{\pi}{2}} 2\cos^4 x\,dx - 2\int_0^{\frac{\pi}{2}} 2\cos^6 x\,dx = \frac{\pi}{2}$. 1

$\therefore$ From (i)

$6\left(\frac{3\pi}{16}\right) - 4\int_0^{\frac{\pi}{2}} \cos^6 x\,dx = \frac{\pi}{2}$

$\therefore 4\int_0^{\frac{\pi}{2}} \cos^6 x\,dx = \frac{9\pi}{8} - \frac{\pi}{2}$ 2

$= \frac{5\pi}{8}$

$\therefore \int_0^{\frac{\pi}{2}} \cos^6 x\,dx = \frac{5\pi}{32}$. 1

(v) Since $0 \leq \sin x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$

then $\sin^n x \times \sin x \leq \sin^n x$.

ie $\sin^{n+1} x \leq \sin^n x$ for all $0 \leq x \leq \frac{\pi}{2}$ 1

(for all +ve integers n).

$\therefore \int_0^{\frac{\pi}{2}} \sin^{n+1} x\,dx < \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

for all +ve integers n.